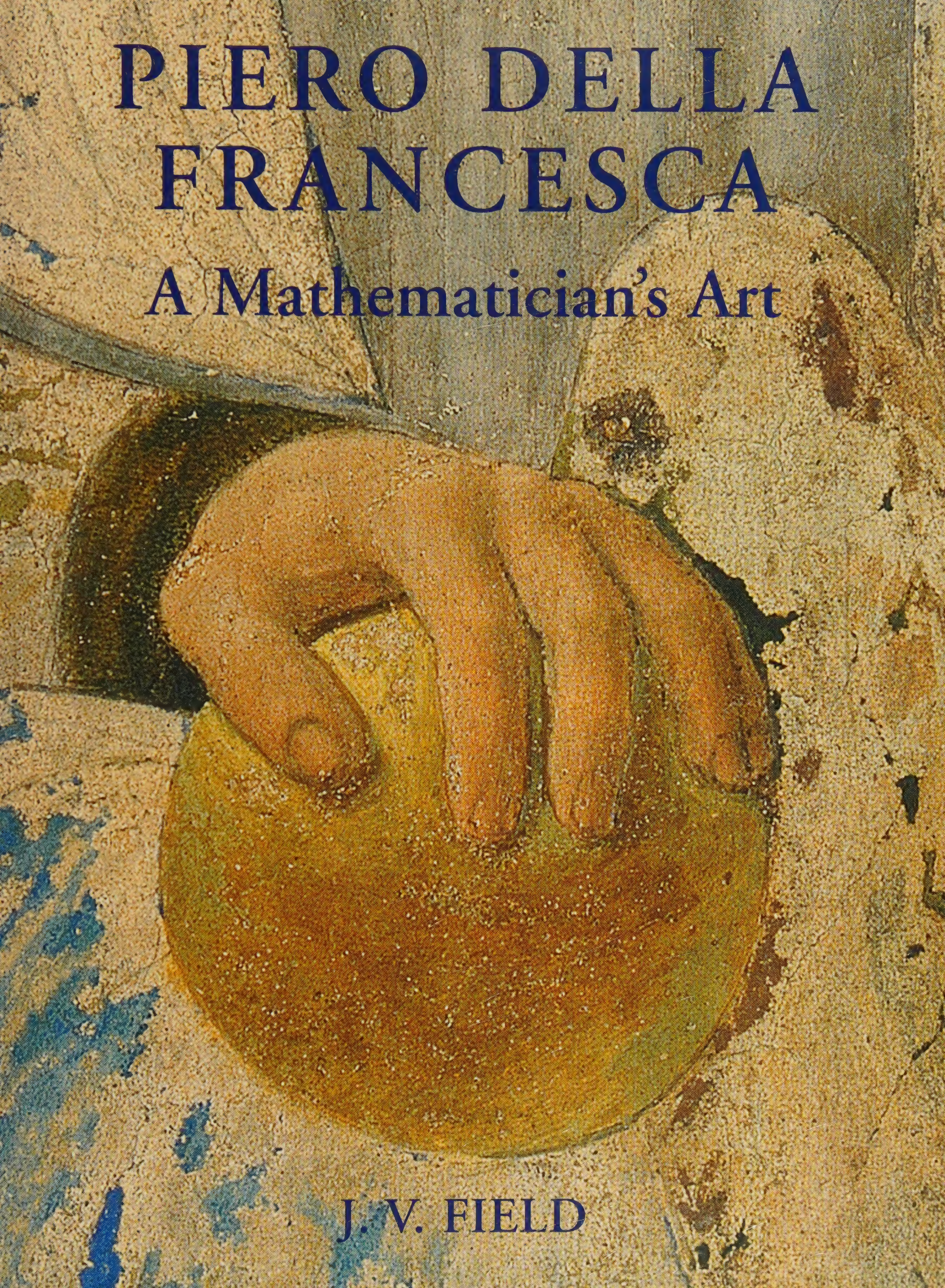
PIERO DELLA FRANCESCA

A Mathematician's Art

J. V. FIELD



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Piero della Francesca, one of the greatest painters of the fifteenth century, was also an accomplished mathematician. This book – the first combined study of Piero's work as a mathematician and as a painter – explores the connections between these two sides of his activity and thus enhances our understanding of both his paintings and his writings.

J. V. Field begins by describing Piero's education, family background and training as a painter. We then examine the strong sense of three-dimensional form shown in his art and the abstract solid geometry discussed in his writings. Field next considers Piero's treatise on perspective and paintings that exemplify the prescriptions it provides, and then assesses the optical or pictorial 'rules' Piero followed as a painter. Piero is identified as an exemplar of a learned craft tradition. The book concludes by considering the historical significance of that tradition and its connections with the extensive changes in natural philosophy that began to appear in the next century (the changes that historians of science know as the Scientific Revolution).

Piero's mathematics is seen to be highly accomplished. He emerges as a figure of considerable intellectual weight. We can see that his art, as a painter and as a mathematician, exemplifies some of the important contributions that the Renaissance made to the development of the sciences as well as to that of the visual arts.

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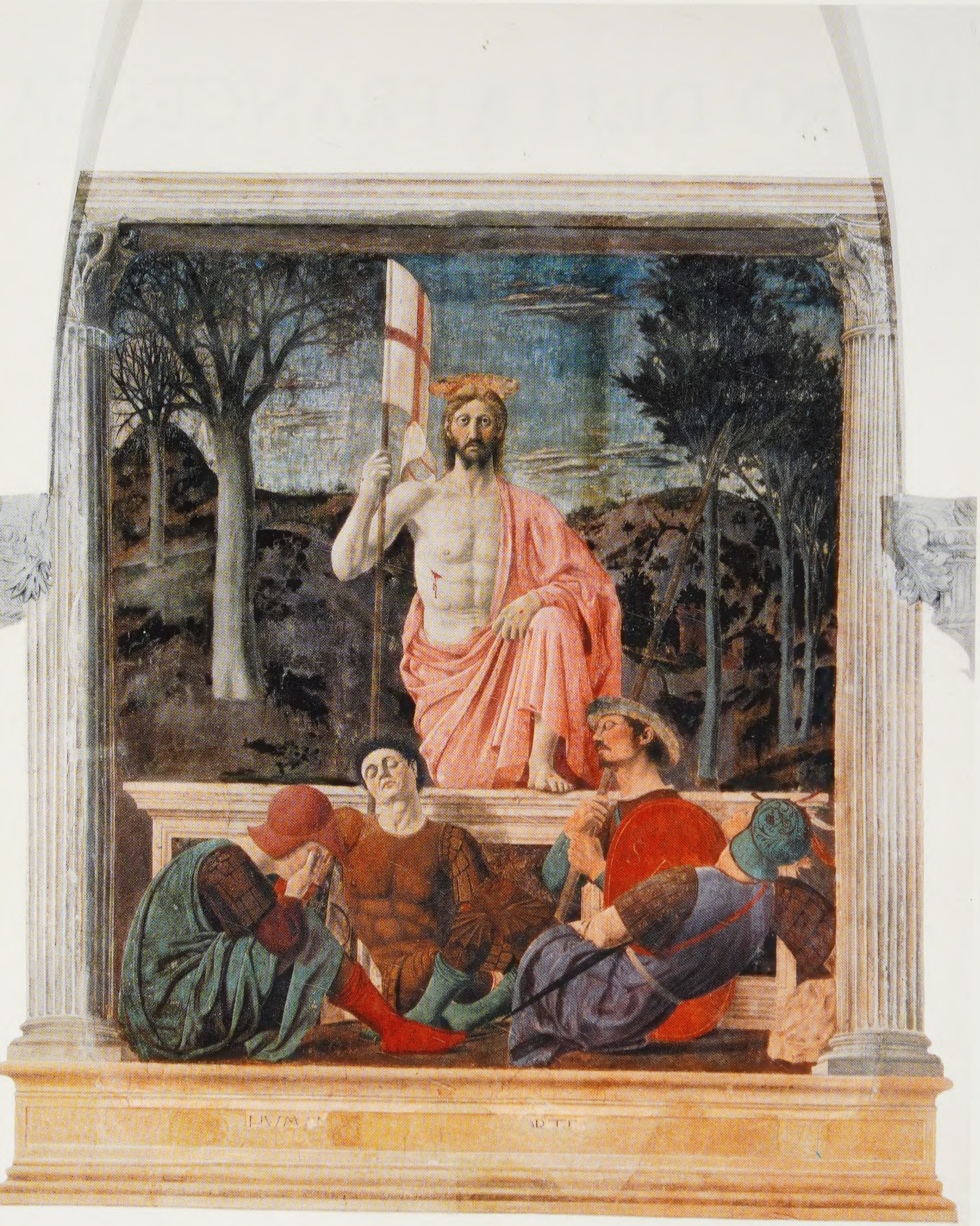

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Frontispiece: Piero della Francesca (*c.*1412–1492), *The Resurrection*, tempera and fresco, 225 × 200 cm, Museo Civico, Sansepolcro.

To

Arthur Beer

(1900–1980)

Contents

Preface

For as long as I can remember, I have intended to write a book about Piero della Francesca. Research for the work that follows was well under way before I began to write *The Invention of Infinity: Mathematics and Art in the Renaissance* (1997). That book was conceived as providing a wider historical context for Piero's work, or at least for the part of it that has connections with perspective. Chapters 1 to 6 and part of Chapter 7 of the present book were written by mid-1996, but for various reasons writing then slowed down. One reason was that the book was taking me into regions I had not previously explored, a development I regarded as entirely healthy.

I am grateful to the Leverhulme Trust for a grant that enabled me to join the History of Art Department at Birkbeck College, University of London, for the years 1993–6, and I am grateful to the department for its continuing hospitality to me as an honorary visiting research fellow. Several Birkbeck colleagues have been helpful, but particular thanks are due to Francis Ames-Lewis, who gave me much useful advice and patiently read and commented on each chapter as it was written, and to Peter Draper, who not only answered rather numerous miscellaneous queries about architecture but also kindly allowed me to use his office while he was away on sabbatical.

Many other colleagues, at Birkbeck and elsewhere, have also helped me. In particular, my research benefited a great deal from discussions with Martin Kemp. Rupert Hall kindly read and commented on the Introduction and Chapters 7 and 8. Informal discussions with Stephen Clucas, A. E. L. Davis and Muriel Seltman have helped clear my mind on matters of the history of science; I am particularly indebted to Dr Davis for reading drafts of Chapter 8 and for guiding me through some of the intricacies of the notoriously complicated calculations that led Johannes Kepler to his elliptical orbit for Mars. Some of the encumbering administrative business associated with my continued activity as a historian of science brought me into contact with practising mathematicians. I think of myself as an ex-mathematician but, on occasion, my attempts to explain to some of these very clever people what was interesting about Renaissance mathematics gave rise to serious discussions. As these debts do not appear in footnotes, I should like to take this opportunity to thank Whitfield Diffie and James A. Reeds for their unplanned contributions to some of the thoughts behind Chapters 7 and 8. I am also grateful to Karen Reeds for reading and commenting on an earlier draft of the Introduction. Finally, I am grateful to Michael Baxandall for his very helpful comments on an earlier draft of the complete text, which he read on behalf of the publishers, and to M. E. W. Williams for help with proofreading. I hope that all my other specific debts are recorded in the footnotes.

The paper by James R. Banker, 'Contributi alla cronologia della vita e delle opere di Piero della Francesca', *Arte Cristiana*, 92 (204) 248–58, reached me too late for me to take account of what it says. Two matters, from opposite ends of Piero's life, are of significance here, though neither affects any argument I have put forward. First, it seems that there is hope of a work by Antonio d'Anghiari being published in the near future. Second, the earliest date for Piero's becoming blind is now 1490.

JVF
London, December 2003

Abbreviations

Battisti: Eugenio Battisti, *Piero della Francesca*, 2nd edn, 2 vols, edited by M. Dalai Emiliani, Milan, 1992.

BL MS: *Incipit* 'Petrus pictor burgenisis' [Piero della Francesca, De prospectiva pingendi], British Library, London, Add. MS 10366 [Latin].

BML MS: Piero della Francesca, 'Trattato d'abaco', Biblioteca Medicea Laurenziana, Florence, Codex Ashburnham 280 (359*.291*).

KGW: *Johannes Kepler gesammelte Werke*, edited by M. Caspar et al., Munich: Beck, 1938–.

Parma MS: Piero della Francesca, 'De prospectiva pingendi', Parma, Biblioteca Palatina, MS no. 1576 [vernacular, autograph].

Piero ed. Arrighi: Piero della Francesca, *Trattato d'abaco: Dal Codice Ashburnhamiano 280 (359*.291*) della Biblioteca Medicea Laurenziana di Firenze*, edited by G. Arrighi, Pisa, 1970.

Piero ed. Mancini: Piero della Francesca, 'L'Opera "De corporibus regularibus" di Pietro dei Franceschi detto della Francesca, usurpata da Fra' Luca Pacioli', ed. G. Mancini, *Memorie della R. Accademia dei Lincei*, series 5, 14.8B, Rome, 1916, pp.441–580.

Piero ed. Nicco Fasola: Piero della Francesca, *De prospectiva pingendi*, edited by G. Nicco Fasola, Florence, 1942. Reprint, Florence, 1984.

General notes

1 Copies of diagrams from Renaissance sources follow the convention of that time in employing Roman lettering, sometimes in lower case. When new diagrams have been supplied they follow today's convention of having lettering in italic.
2 All translations are by J. V. Field unless otherwise attributed.
3 All dates are A.D. (A.C.E.) unless otherwise indicated.

Introduction

Piero della Francesca (*c.*1412–1492) is now best remembered as a painter. He has a secure position in the history of Italian Renaissance art. However, in his own lifetime, and for some time thereafter, he was also known as a mathematician. In fifteenth-century Italy, such a combination of interests arose quite naturally from the fact that prospective craftsmen were commonly taught a fair amount of 'practical' mathematics. Piero is exceptional chiefly in having been unusually talented both as a mathematician and as a painter. In recent years there has been an increasing recognition of his duality, at least on the part of art historians. Indeed, in his detailed scholarly monograph on Piero's painting, Eugenio Battisti went so far as to say that there should be a further volume to consider Piero's activity as a mathematician.[1]

This plan was never carried out. In fact, although Piero's mathematical work is undoubtedly of sufficient interest to merit detailed study in its own right, it seems a trifle perverse to consider it separately from his painting. Some of the mathematics, most notably the treatise on perspective, *De prospectiva pingendi*, has obvious relevance to Piero's work as a painter. By neglecting Piero's writings, we are surely ignoring a possible source of evidence for interpreting his paintings. There are repeated references to the lack of contemporary documentation regarding Piero's activity as a painter, so it is rather strange that historians of art have not paid more attention to Piero's own writings. Since the mathematician and the painter were one person, it seems clear that the mathematics and the painting should be taken together, to see what one may have to tell us about the other. On the other hand, historians of science are thoroughly accustomed to having very little biographical information about the authors they study, yet like historians of art they too have been narrow in their use of sources. They have apparently concentrated solely on Piero's mathematics, without reference to his painting, often even omitting any consideration of his perspective treatise. This style of historiography was more acceptable in the nineteenth century, when Piero was held in low regard as a painter. Indeed, when the British Museum acquired its manuscript of Piero's perspective treatise in 1836, it was catalogued under its incipit, the name of the author presumably having proved impossible to trace.[2] But treating Piero as only a mathematician seems less understandable a century later, and still less understandable today. Although to the art historian Piero della Francesca is a painter about whom we

1 Battisti. Carlo Bertelli, *Piero della Francesca*, trans. E. Farelly, New Haven and London: Yale University Press, 1992, also expresses some interest in Piero's mathematics, but does not take it very far.

2 BL MS. The incipit is 'Petrus pictor burgensis', which is now recognizable as a habitual form of Piero's signature.

know rather little, to the historian of science he is a mathematician about whom we know unusually much.

As the title of the present study indicates, I am inclined to think Piero's mathematics does have a recognizable relationship with his painting. In examining the two together I am attempting to reintegrate Piero della Francesca in rather the same way that some modern studies have reintegrated another artist whose activity fails to observe today's disciplinary boundaries: Piero's younger contemporary Leonardo da Vinci (1452–1519).[3] The personality that emerges from this study of Piero's work as a whole is one of some historical stature. Piero's mathematics was both up to date and highly accomplished, within the limitations of the style of its time and social context, which is to say that it was, in its own way, as 'state of the art' as his painting. He was a learned craftsman whose work showed the beginnings of what was later to become a historically important process of interaction between the practical tradition and the scholarly one, that is between vernacular and Latin learning. Modern investigations of this interaction usually refer back to the generalized suggestions put forward in a classic paper by Edgar Zilsel, published in 1942; more recent work, concentrating on the sixteenth and seventeenth centuries, has tended to make the relationship appear more complicated than Zilsel seems to have hoped.[4]

In any case, this widening of Piero's intellectual context also, I hope, leads to a deeper understanding of his painting, for, while I have no wish to belittle his achievements as a mathematician, there is no avoiding the recognition that Piero is likely to remain a more significant figure in the history of art than in the history of science. This is partly because painting, much of which is designed as mass communication, continues to work over time in a way that most mathematics, providing useful techniques for a relatively small number of practitioners, simply cannot rival. This difference makes for some awkwardness in dealing with mathematics and painting together, since the former is liable to require a bit more than the latter by way of technical explanation.

As it happens, the relevant parts of Piero's mathematics are rather elementary, so this difficulty hardly arises in the present case. There is nonetheless an inevitable shift in style between the historical study of art and that of mathematics. In mathematics the course of history can reasonably be considered in terms of progress. In most sciences this is a dangerous notion because there are changes in what is required of a theory. *Mutatis mutandis* a similar caution is, of course, habitually exercised in writing the history of art. However, at least since the time of Euclid (active *c.*300 B.C.), there has been very little change in what is required of a theorem. In Europe and in the Islamic world a more general theorem, say one that applies to all triangles and not merely to ones that have two equal sides, is regarded as stronger and therefore better than a less general one. The concept of mathematical beauty is much harder to define than generality is, but that does not really matter because in mathematics it is generality that is the goal, pleasurable though it may be to come upon a result

3 For instance M. J. Kemp, *Leonardo da Vinci: The Marvellous Works of Nature and Man*, London: Dent, 1981.

4 Edgar Zilsel, 'The Sociological Roots of Science', *American Journal of Sociology* 47, 1942, pp. 544–62, reprinted in Edgar Zilsel, *The Social Origins of Modern Science*, Boston Studies in the Philosophy of Science, vol.20, ed. Diederick Raven, Wolfgang Krohn and Robert S. Cohen, Dordrecht: Kluwer Academic Publisher, 2000, pp. 4–21; A. R. Hall, 'The Scholar and the Craftsman in the Scientific Revolution', in *Critical Problems in the History of Science*, ed. M. Clagett, Madison: University of Wisconsin Press, 1959 (a lecture given in 1957); Paolo Rossi, *I filosofi e le macchine (1400–1700)*, Milan, 1962 (partly based on work Rossi published in 1955–7).

that provides an aesthetic thrill. Thus historians of mathematics are at liberty to make rather tidy judgements about the works they study.

Historians of other sciences, or of science in general, have to be more circumspect, and they may have something to learn from art historians' recognition of the autonomy of individual works, that is the advisability, in order to avoid talking nonsense, of assessing a work, in the first instance, in terms of its maker's intentions. Such historical sensitivity is certainly necessary in dealing with the science of the fifteenth century, for this was not a period in which it is easy to point to great 'advances'. In particular, the practical tradition, within which Piero was writing, shows a continuing slow development – though this was to accelerate considerably in the years following Piero's death. In Piero's time there was an active tradition that had social consequences, such as a rising general level of mathematical education, rather than one that bore immediate fruit in the form of powerful new theorems. There is, in fact, a theorem in Piero's perspective treatise that I think should carry his name,[5] but it still seems inevitable, and entirely reasonable, that Piero's paintings, with their strongly individual style, will continue to dominate historians' assessment of his character. On the other hand, social and intellectual historians do seem to have been unduly neglectful of scientific aspects of fifteenth-century culture, and the present study is partly an attempt to redress the balance a little. I hope this may also have the effect of making Leonardo da Vinci look less of an isolated figure, though his work is certainly unusually multifarious even for the fifteenth century. Further context for Piero and for Leonardo might also be provided by examining the scientific interests of Paolo Uccello (1397–1475), which, though less intense and wide-ranging than Leonardo's, also included natural history as well as mathematics.

Historical studies of Piero's mathematics began in the nineteenth century, but his 'rediscovery' as a painter – that is, the gradual establishment of his present position in the history of art – began only in the early twentieth century. In this, the greatest debt is owed to the pioneering studies of Roberto Longhi, whose work, from 1914 onwards, did much to set the style for future generations of scholarship directed to setting Piero in his proper place in the history of art.[6] In the last few decades, the emphasis has shifted a little and there have been detailed analyses of particular pictures, and several general studies that relate Piero's paintings more closely to their religious and social context.[7] The later progress of Piero's art in general esteem in the twentieth century seems to have been partly mediated by an appreciation of the 'abstract' qualities of his works, such as a balance in their compositions that can be expressed in mathematical terms.

Mathematics in the interpretation of paintings

Since Piero was known to have written on the mathematics of perspective, art historians have generally been ready to describe his pictures as in some sense 'mathematical', though

5 Piero della Francesca, *De prospectiva pingendi*, Book 1, Section 8; see Chapter 5.

6 Roberto Longhi, *Piero della Francesca*, Rome, 1927.

7 A complete survey of the literature is impractical here. Most of the works in my bibiography have bibliographies of their own. Here I should like to mention only two books I have found particularly useful, though they may not receive repeated reference in my text. They are M. A. Lavin, *Piero della Francesca: The Flagellation of Christ*, New York: Viking Press, 1972, reprint, with additional bibliography, Chicago: University of Chicago Press, 1990, for its detailed consideration of the structure of the picture; and R. Lightbown, *Piero della Francesca*, London: Abbeville Press, 1992, particularly for its exploration of the religious context. I am, of course, also deeply indebted to the second edition of the detailed study by Battisti referred to in note 1 above.

the quality most often remarked upon is a 'stillness' that is rather hard to define in any precise way. The assertion that Piero's pictures are 'mathematical' is usually so vague that it is understandable that some art historians have preferred largely to ignore it. However, there has also been a most unfortunate fashion for drawing lines over Piero's pictures with the purpose of exposing their alleged underlying geometrical structure.[8]

The commonest target, or victim, of this form of interpretation is the *Baptism of Christ* (National Gallery, London) (Fig. 4.2), the painting being shown overlaid by a regular star pentagon and various accessory geometrical figures. The presence of the star pentagon among the superimposed figures is a clue to the historical origins of this form of interpretation. As readers of Euclid's *Elements* know, the sides of the star pentagon, which are diagonals of the convex regular pentagon, cut one another in what Euclid called 'extreme and mean proportion' (*Elements*, Book 13, Proposition 8), a proportion dubbed 'the golden section' in the nineteenth century. This proportion, which has some interesting mathematical properties, gained considerable mystical significance in the late nineteenth and early twentieth century. In the manner of occultists (the same habit may be observed in astrologers at all periods), enthusiasts for the 'golden section' laid claim to a respectably long ancestry for their opinions, in this case tracing them back to the Renaissance, and in particular to Renaissance art.

On the whole, historians – mindful of the difficulty of proving a negative – have allowed these ideas to wither away rather than attempting a direct refutation. For the most part such ideas have indeed withered, under the combined onslaught of silence, change of fashion, and the occasional dismissive comment to the effect that if one draws so many lines and makes them thick enough some of them are bound to go through specific features in a photograph of, say, the façade of the Porte Saint-Denis in Paris. However, perhaps because of his reputation as a mathematician, this generally efficacious method of rubbish disposal seems not to have worked very well in regard to pictures by Piero della Francesca.

There is perhaps a deeper difficulty here. It is, after all, accepted practice to trace lines in Piero's compositions in order to show their perspective schemes, sometimes with spectacular success, as in the case of *The Flagellation of Christ* (Galleria Nazionale delle Marche, Urbino) (Fig. 5.28). There can be no reasonable doubt about the legitimacy of the reconstructions of the ground plan and elevations of the scene shown in the *Flagellation*. The lines that are used in the reconstruction are relatively long, and all of them are clearly visible in the painting. However, exceedingly few pictures provide enough evidence to allow such mathematical reconstruction. This is true of Renaissance art in general, and it also applies to Piero's work in particular. It appears that, in this respect, his mathematical skill did not lead him to use perspective very differently from the majority of his contemporaries. In most cases, trying to find perspective schemes – that is, trying to extract from pictures the mathematics that went into their construction – seems to me to be like trying to extract the sunbeams from cucumbers.[9] Mathematics may have contributed to the nature of a picture, but it can prove to be so thoroughly absorbed into the final structure as to be inextricable from it.

8 For a discussion of this practice in general see Judi Loach, 'Le Corbusier and the Creative Use of Mathematics', in *Science and the Visual*, ed. J. V. Field and F. A. J. L. James, *British Journal for the History of Science* 31(2), 1998, pp. 185–215.

9 Jonathan Swift, *Gulliver's Travels*, London, 1726, Part III, Chapter 5. Swift's notion was picked up by W. S. Gilbert and used in the Major General's song in *The Pirates of Penzance*.

This applies to perspective schemes as well as to any other use of mathematics. Indeed, there can often be serious doubt whether it is reasonable to claim that the painter constructed the spatial relationships in the picture in mathematical terms. However, since Piero della Francesca wrote a treatise on perspective, it is at least certain that he was capable of carrying out the required mathematical constructions. In practice it is not difficult to decide, simply by looking at the number and length of available straight lines, whether there is a fighting chance of making reasonably secure deductions about the perspective scheme of a picture. Although there are exceptions, on the whole the longer and the more numerous the available straight lines, the more legitimate the deductions. Unfortunately, scholars sometimes seem to be unwilling to recognize that this is not an area in which it is permissable to apply the proverb that where there's a will there's a way. Nevertheless, although failure must sometimes be accepted as a final outcome, it is clear that the search for a perspective scheme in Piero's paintings is inherently a legitimate exercise. It is known that Piero was interested in perspective and his writing on the subject provides a substantial amount of guidance in the search.

This point may perhaps need some emphasis in the present context because it will be necessary to use mathematics, that is today's mathematical understanding, as a neutral tool in investigating Piero's use of his mathematics. What must be settled first, in every case and with a suitable degree of historical sensitivity, is that the use of particular tools is appropriate. There is, moreover, the additional difficulty that Piero's use of perspective has attracted attention from a wide range of scholars, some of whom have made things unnecessarily complicated for themselves and have thus got out of their mathematical depth. Unfortunately, some published work contains mathematics that is simply incorrect. Such errors could, of course, be analysed and explained in detail, but in the present study I have generally preferred to run the risk of being considered unscholarly in neglecting apparently relevant literature rather than stir my own and my readers' unhappy memories of maths homework that came back covered with a teacher's comments in red. There is, of course, also some work that, while excellent in its day, has now been overtaken by an increased knowledge of fifteenth-century mathematics.

In the case of the geometrical figures so frequently associated with Piero's *Baptism of Christ*, I consider that the method of investigation is inappropriate. It can be contrasted with the mathematical investigation of the perspective of the *Flagellation*. First, whereas the lines used in investigating the perspective of the *Flagellation* are all clearly visible, drawn by Piero as part of the scene, the lines that make up the postulated pentagon in the *Baptism* are not shown as such in the picture. This is not to deny that there are more or less exact alignments – as would be expected in an orderly composition – but the lines in question are a matter of inference. Moreover, whereas in the *Flagellation* the lines used are exact when the picture itself, or a natural-size reproduction, is used, the lines postulated for the *Baptism* look a great deal more convincing when shown on a small photograph rather than when a straight edge is held up between the viewer's eye and the actual painting. The effect of using a small photograph is, of course, that the fine lines superimposed on the image correspond to very much thicker lines on the true scale of things.

It may be noted also that much of the geometrical apparatus suggested for the *Baptism* consists of elaborate figures being used to achieve simple geometrical results such as making Christ's two arms mirror images of one another, or making the centre line of His body coincide with the centre line of the painting. So the appropriate instrument for disposing of much of the alleged construction apparatus is the principle of economy commonly known as Occam's razor. It must, of course, be admitted that vulnerability to Occam's razor is not rigorously equivalent to falsity. All the same, it should give a scholar pause. It surely throws the onus back onto the proposer of such a theory to provide reasons for taking it seriously rather than, as is often the case, simply daring opponents to prove the theory untrue.

Whatever one's feelings about extreme and mean proportion – and there is no direct evidence as to Piero's – the pentagon is unconvincing as a major component of a Christian religious painting because of its association with Islam. Moreover, it is not clear in what way these geometrical figures might have become part of Piero's picture. The preliminary drawings discussed in Piero's perspective treatise do not use geometrical figures in this way. The only evidence for the use of preliminary drawings for the *Baptism* relates to part of the drapery of one of the angels.[10]

Piero and Pacioli

One explanation for some scholars' tendency to suppose Piero della Francesca was interested in the pentagon and in extreme and mean proportion is that it was at one time believed that Piero had been the teacher of Luca Pacioli (*c.* 1445–1517), whose book on the properties of this proportion, *De divina proportione* (Venice, 1509), seems to have enjoyed considerable success (no doubt helped by its woodcut illustrations of polyhedra, which were based on drawings supplied by Leonardo da Vinci). Pacioli, a Franciscan friar who had a fairly successful career as a teacher of mathematics, is now best remembered by historians of science for his bulky and successful volume *Summa de arithmetica, geometria, proportioni e proportionalità* (Venice, 1494), which was the first printed book to deal with 'commercial arithmetic' and its offshoot, algebra. Many of Pacioli's algebra problems seem to be derived from Piero della Francesca's *Trattato d'abaco*. However, Pacioli's solutions tend to be slightly different from Piero's, so it is more reasonable to regard the borrowing as part of the practical tradition – few if any of Piero's problems are original to him either – rather than discussing the matter as a possible case of plagiarism, which is an essentially anachronistic notion.

The evidence does, however, establish beyond reasonable doubt that Pacioli knew Piero's *Trattato d'abaco*. In the past this knowledge was generally interpreted as an indication that Pacioli had been Piero's pupil. What is now known about the lives of both Piero and Pacioli shows that it is exceedingly unlikely that Pacioli was taught mathematics by Piero. Pacioli was indeed, like Piero, born in Borgo San Sepolcro, and he several times refers to Piero as his fellow-countryman (*conterraneo*), but he never refers to him as his teacher, and the dates, as they are now known, show that Piero had left Borgo San Sepolcro before Pacioli was born. Piero did repeatedly return to the town, but there is no evidence that he ever taught mathematics, either there or anywhere else. Pacioli's familiarity with Piero's mathematical writings seems to date from the 1490s, which suggests that he may have come into possession of manuscripts after Piero's death.

10 D. Bomford, ed., *Art in the Making: Underdrawings in Renaissance Paintings*, London: National Gallery Publications, 2002.

Pacioli's overt concern with the visual arts may reflect contacts with Piero, but it must surely also be taken into account that Pacioli became friendly with Leonardo da Vinci. (They shared a house, which probably goes some way to explaining how it happened that Leonardo actually finished the set of illustrations for *De divina proportione*.) Pacioli made extensive use of Piero's mathematical works, but the style of the first part of *De divina proportione*, the part from which the work takes its title, is very different from that of anything known to have been written by Piero. The text is essentially a series of extracts from Euclid's *Elements*; no proofs are supplied but precise references are given to where the result is proved by Euclid. The style is thus more or less that of a formal, though rather simple, commentary on Euclid. In those of his writings that are now known, Piero never tackled his subject matter in this way. His mathematical treatises are all didactic in form and they all belong to the practical tradition, which is to say that, except in connection with perspective, the work on plane geometry is largely concerned with problems that have direct relevance to surveying, and that the subject is presented by means of a series of worked examples, set out in numerical form.

All the same, abstraction does have a way of creeping in. Having dealt at some length with triangles, and more briefly with squares, Piero provides a few problems on pentagons. From his work on solid geometry it is clear that he is familiar with *Elements*, Book 13, and he thus certainly knew about the connection of extreme and mean proportion with the diagonals of the regular pentagon. However, although his problems on the pentagon do mention the proportion between the side and the diagonal of the figure, this proportion is not given a name and its properties are not investigated. The method of Piero's exposition of geometry is inimical to a presentation of extreme and mean proportion because problems are set, and solved, in numerical form. As in any didactic text, Piero frequently arranges for the answer to be a whole number, but fractions do sometimes appear and in some more complicated problems the answer is given in a form that involves roots. Handling extreme and mean proportion numerically would involve using the numbers $\frac{1}{2}(\sqrt{5} \pm 1)$ (or some scaled version of them). It is easy to see why, other things being equal, Piero might have considered it advisable to avoid this kind of calculation. It thus appears his estimate of the importance of extreme and mean proportion was not so great as to cause him to override such didactic considerations. Euclid deals with the matter geometrically, which is much more elegant (at least to today's taste).

Piero's mathematics

This explanation of the absence of evidence of Piero's having any interest in extreme and mean proportion does not, of course, prove that he was not interested in it. But in the absence of evidence that he was, one should surely first look elsewhere in seeking to identify traces of mathematics in Piero's pictures.

The present study will accordingly look much more carefully at Piero's mathematics than previous studies have done. The nature of the practical tradition will be considered in Chapter 1 and its relationship with the learned tradition of university mathematics will be explored in Chapter 7, which attempts to assess Piero's place in the intellectual life of his time. The relationship of Piero's visibly practical writings to the learned tradition is somewhat ambiguous, though most of the work itself is relatively unremarkable for its time and place, a fact that no doubt partly explains why there have been no monographs on Piero's mathematics. However, there are two elements that are exceptional: first that Piero wrote on perspective; and second that he paid far more than the customary degree of attention to solid geometry. As Piero's painting is exceptional in his mastery of composition in three dimensions as well as in the plane, this concern with solid forms is not unexpected. However, seeing it in his mathematics gives additional legitimacy to examining its manifestations in his paintings, and the two aspects will accordingly be examined together in Chapter 4. The same kind of consequences follow from a study of various other parts of Piero's mathematical writings. Thus a detailed study of Piero's perspective treatise (see Chapter 5) allows one to proceed with some confidence in making some general comments on Piero's use of perspective (in Chapters 5 and 6).

From all this, I believe it is possible to offer an explanation of that famous 'stillness'. It is not possible to give mathematical proof, but from my failure to find serious deviations from mathematically correct perspective – except in cases where this is flagrantly deliberate – I strongly suspect that the reason why Piero's pictures look mathematically correct is because they are indeed correct. That is, I suspect everything susceptible of calculation has been calculated. Thus everything shown is represented as seen from the single ideal viewpoint, though, thanks to the tolerance of the eye, it will of course look convincing even when seen from points at a considerable distance from the ideal viewpoint. So the viewer really has the sense of seeing the scene, as fifteenth-century theory prescribes, in a single immobile glance, directed from the ideal viewpoint to the centric point of the perspective. Piero freezes the scene the way a photograph does. This is not a method used by most of his contemporaries, though it may be that Paolo Uccello aspired to it and, largely because of the high visibility of his perspective devices – such as scattered lances making too neat a pattern on the ground – unluckily fails to have the same effect upon today's viewers.

However, in all Piero della Francesca's surviving paintings much depends upon the parts of painting that he says in the introduction to his perspective treatise that he does not propose to write about, namely rendering the fall of light and handling colour. Piero is immensely skilful at both. This is, of course, also true of the pictures of two other painters notable for 'stillness', Pieter Saenredam (1597–1665), who is known to have made detailed perspective constructions, and Johannes Vermeer (1632–1675), who used a *camera obscura*.[11] In a certain sense these two, and Piero with them, can be seen as taking a 'scientific' attitude to painting; but that is not what makes their pictures worth looking at. All the same, Piero's very considerable mathematical talent does seem to have contributed to his personality as a painter.

The structure of this study

Even historians are allowed to dream. In ideal conditions, if writing largely about some particular individual, there would be documents establishing a secure framework of dates for the subject's biography and for his or her works. Thanks to the efforts of various scholars, and particularly those of James R. Banker, historians are much less under-informed about Piero

11 M. J. Kemp, 'Construction and Cunning: the Perspective of the Edinburgh Saenredam', in *Dutch Church Painters: Saenredam's 'Great Church at Haarlem' in Context*, ed. Hugh Macandrew, Edinburgh: National Gallery of Scotland, 1984; J. P. Steadman, *Vermeer's Camera*, Oxford: Oxford University Press, 2000.

della Francesca than they were even ten years ago.[12] However, while establishing dates of contracts is helpful, it has so far mainly allowed more play for the recognition that Piero was not given to fulfilling commissions promptly. Records of final payments make it possible to establish the dates at which a few works were delivered (for example the *Sant'Antonio Altarpiece* in 1468, see Chapter 6) but, for the most part, historians have been forced to rely very heavily upon their instincts for stylistic dating. My own eye in the matter seems to be rather conventional: like a large number of other people, I am on the whole inclined to agree with the opinions of Roberto Longhi. In the circumstances, it would be a rash, not to say cranky, historian who would claim that there is a sufficient framework to give a reasonably complete and unassailably correct chronological account of Piero della Francesca's life and work. All studies of Piero tend to have something of a striving for such a chronology.

I include my own work in that generalization, but the dating of Piero's works is not my main concern and I have considered it prudent to avoid using a possible chronological sequence as an organizing principle. One aspect of my interest in Piero della Francesca is his place in science, or rather in the wider intellectual culture that, in the Renaissance – and up to the mid-seventeenth century at least – included the subjects that were later to be divided as the objects of study of different university faculties and, in due course, different types of historian. The study nevertheless begins chronologically with what Piero seems likely to have been taught, discussing his mathematical education – that is, the mathematics of the 'practical' or 'abacus' tradition. Following on from this, Chapter 2 looks at perspective, both the theoretical discussion by Leon Battista Alberti and the sometimes anarchic use of mathematical constructions by practitioners in the generation before Piero, starting from the invention of a perspective construction by Filippo Brunelleschi around the time of Piero's birth. Chapter 3 deals with Piero's family background and his training as a painter. Chapter 4, 'A Sense of Space', turns to those of Piero's paintings that show no indication of a formal linear construction in conveying a sense of depth: the London *Baptism*, the two small panels of St Jerome (Venice and Berlin) and the Montefeltro portraits (Florence). Piero's treatment of abstract solid geometry in his mathematical writings is shown to display the same strong sense of three-dimensional form, a sense that was highly unusual in the mathematics of the time and led Piero to some mathematical discoveries. Chapter 5, 'The Force of Lines', deals with Piero's treatise on perspective, *De prospectiva pingendi*, and works such as *The Flagellation of Christ* that seem to exemplify the prescriptions it provides. Chapter 6, 'Optics and Illusionism', considers works that in their various ways combine the construction techniques noted in the previous two chapters, works such as *The Story of the True Cross*, the *Resurrection of Christ*, the *Madonna di Senigallia* and the *Nativity of Christ*. The chapter concludes with an assessment of what optical or pictorial 'rules' Piero seems to have followed in his practice as a painter.

This concludes specific concern with Piero's paintings. The remaining two chapters consider his work as a whole, the emphasis being on the nature and history of the intellectual tradition it exemplifies. Chapter 7, 'But is it Art?', considers Piero as a learned craftsman, specifically comparing his style of thought with that of the learned mathematical culture of the time, that is the university art of mathematics, exemplified in the work of two of his

12 A recent and most valuable contribution is James R. Banker, *The Culture of San Sepolcro during the Youth of Piero della Francesca*, Ann Arbor: University of Michigan Press, 2003.

most famous mathematical contemporaries, the theologian Nicolaus Cusanus and the astronomer Johannes Regiomontanus. The purpose is to situate Piero's work as a whole within the intellectual life of his own time, particularly in its 'scientific' aspects. The final chapter, 'From Piero della Francesca to Galileo Galilei', turns to a wider historical context, to consider the historical significance of the tradition to which Piero belonged. It examines the part that learned craftsmen such as Piero can be seen to play in the longer-term changes in natural philosophy that led to the emergence of a recognizably modern method of investigating nature, that is the movement that historians of science call the Scientific Revolution. Piero has a secure place in the history of art. I hope that my study will not only contribute to understanding his paintings and his writings – and perhaps help to give some substance to the common rather vague assertion that he was an intellectual – but will also show him as a significant figure in Renaissance culture as a whole.

1

The Background

In the early fifteenth century, painters and sculptors were regarded as craftsmen or artisans. This fact largely dictated the nature of their education: the skills they were expected to acquire were those that were directly applicable in the practice of their craft.

The training of an artist

In about 1400 a Florentine painter, Cennino d'Andrea Cennini (*c.*1370–*c.*1440), chose to write down an account of the skills necessary to a painter. These, he says, are the skills he had himself learned in his twelve years under his master, Agnolo di Taddeo Gaddi (active 1369–96), who had learned them from his father Taddeo di Gaddo Gaddi (*c.*1307–1366), who had been a pupil of the great master Giotto di Bondone (*c.*1267–1337). Cennino called his work *Il Libro dell'Arte* and, since the wool guild was known as the 'Arte della Lana', there is clearly every reason to approve the standard English version of his title, *The Craftsman's Handbook*.[1] Moreover, although Cennino was concerned, as he says, to describe the skills a painter needs, it is clear that his book is not an instruction manual designed to teach these skills. Most of them are such as could be acquired only by practical exercise in a workshop. Since books were at the time expensive items, it is not at all likely that an apprentice painter would have had easy access to a copy of Cennino's text, even if he had the inclination to read it.

The primary readership for Cennino's work was probably among patrons who wished to know something of the actual practice of the craftsmen they employed. In this Cennino's *Libro dell'Arte* can be seen as a precursor of Leon Battista Alberti's book about painting of 1435. On the other hand, Cennino's book was written in the vernacular, and thus presents itself as belonging to the everyday practical world, whereas the fact that Alberti's was written in Latin serves to confirm that the author's intentions included associating painting with the world of high learning. The relation between these two worlds was apparently not completely clear-cut, as is suggested by the appearance of Alberti's work in the vernacular in 1436, the year following its appearance in Latin.[2] Although Alberti's concerns are different from Cennino's, Cennino's treatise must nonetheless be seen as providing an important piece of background for Alberti's essay, since it indicates that the instruction of

1 Cennino Cennini, *Il Libro dell'Arte*, ed. Fabio Frezzato, Vicenza: Neri Pozza, 2003; Cennino Cennini, *The Craftsman's Handbook*, trans. Daniel V. Thompson, Jr, New Haven: Yale University Press, 1933.

2 On Alberti's work see Chapters 2 and 7 below, and J. V. Field, 'Alberti, the Abacus, and Piero della Francesca's Proof of Perspective', *Renaissance Studies* 11/2, 1997, pp.61–88.

apprentice artists was considered a matter of wide enough interest to be the subject of discussion, at least in general terms, outside the context of a particular workshop. The vernacular version of Alberti's essay on painting may perhaps be seen as a more learned companion piece to Cennino's text.

Cennino takes the reader through all stages of producing a painting, including instructions for tinting paper, making charcoal sticks, cutting quill pens, and making ink for drawing.[3] There is also a fairly extensive set of recipes for making pigments. Much of what Cennino says about pigments can be found in earlier technical treatises such as the eighth-century *Mappae Clavicula* (edited by Adelard of Bath in the twelfth century) and the early twelfth-century *On Various Arts* (*De diversis artibus*) of Theophilus the Priest. There were quite a number of such treatises, but in the age of manuscripts it is hard to know which, if any, enjoyed more than a local distribution. The texts are valuable, however, as indications of the kind of recipes that were used in contemporary workshops.[4] Other parts of Cennino's treatise also show similarities to these earlier sources. Since Cennino's explicit claim is not to originality but to tradition, the fact that such connections can be traced is of interest merely in showing that the tradition is considerably longer than Cennino's account of it implies. Some of his recipes almost certainly date back to late antiquity.

Little in Cennino's treatise is tied to any particular style of painting, except in so far as the emphasis is upon tradition rather than innovation. In the present context, the most important element in Cennino's work is that he makes it entirely clear that he considered the training of a painter took place in another painter's workshop. Ingredients for preparing pigments might have to be obtained from an apothecary, but the actual method of preparation was learned by apprenticeship, as were other necessary skills. For instance, the apprentice learned to draw by copying drawings by his master.

Cennino's treatise is detailed and may give the impression of completeness, but it does not actually deal with all the skills a painter would have required. In particular, it omits almost all reference to mathematics. The most natural explanation of the omission is that Cennino is confining himself to what was learned in the workshop. Mathematical skills were probably acquired elsewhere, before entering the workshop as an apprentice. The painter would certainly have needed some mathematics.

Mathematics for painting

The mathematics a painter would need is mainly so elementary that it is easily taken for granted today. In western Europe, at least, everyone is taught some simple mathematics at junior school. However, the mathematical skills most obviously necessary to a painter are ones that today are not acquired in the classroom but usually learned with a certain amount of painful trial and error when an amateur attempts to hang wallpaper. Getting things right, or at least looking right, is effectively a surveying task carried out on a vertical surface.

3 An excellent and thoroughly illustrated account of the making of a panel picture in the fourteenth century is given in D. Bomford et al., *Art in the Making: Italian Painting before 1400*, London: National Gallery Publications, 1989.

4 A chronological list of twenty-seven treatises on pigments, up to the early fifteenth century, is given in Paul Hills, *The Light of Early Italian Painting*, London and New Haven: Yale University Press, 1987. It starts with Isidore of Seville (seventh century). Cennino's treatise is the seventh in this list. See also Paul Hills, *Venetian Colour: Marble, Mosaic, Painting and Glass 1250–1550*, New Haven and London: Yale University Press, 1999.

1.1 (*facing page*) General view of the interior of the Arena Chapel (Capella degli Scrovegni), Padua.

A glance at the interior of the Arena Chapel in Padua (Fig. 1.1) shows that Giotto and his workshop engaged in just this kind of task in putting in the neat horizontal and vertical borders to the rows of scenes around the walls. Painters presumably also sometimes had to use a certain amount of mathematical initiative in dealing with walls whose top and bottom edges were not exactly horizontal and whose side edges were not exactly vertical. Somewhere along the line, Giotto must also have needed to answer a question of the form 'If a chapel is 15 *braccia* wide, and 35 *braccia* long, and has walls 15 *braccia* high, how many scenes from the Life of the Virgin and the Life of Christ can be fitted in, assuming each human figure is to be made $1\frac{1}{2}$ *braccia* tall?'[5] Alternatively if, as has sometimes been supposed, the chapel was custom built to accommodate the fresco cycle, the mathematical problem would have taken the form 'If we wish to paint so many pictures, of size so many *braccia* by so many *braccia*, illustrating the Life of the Virgin, so many illustrating the Life of Christ and so many symbolic figures and so on, what size does the chapel need to be, assuming it to be rectangular in plan, pleasing in its proportions and with neither length nor width greater than so many *braccia*?' (the last condition, or something like it, being imposed by the choice of site). The two alternative calculations would be much the same in content.

In reality, the calculations would, of course, be complicated by having to make allowances for interruptions in the walls by doors and windows. When the supervisors of builders of walls (*muratori* – the translation should really be 'brickies') had to work out how many bricks would be needed to make a wall, they always omitted consideration of openings such as doors and windows. This simplification, no doubt welcome in itself, presumably also ensured that the estimate of the number of bricks required was never too low. Since bricks were fired to order in rather large batches, each batch in a kiln made specially and destroyed in the process of extracting the bricks after firing, ordering a few more bricks as an afterthought was not a practical option.[6] Painters clearly could not employ such simplifying assumptions. Giotto had to be sure that his pictures would make a neat pattern, allowing due space for doors and windows.

The geometrical-cum-arithmetical know-how required for this kind of problem is clearly common to several crafts. So too is the more narrowly arithmetical skill that would have been needed in connection with handling raw materials for painting. For instance, since natural minerals were used to make plaster, what would now be called its chemical composition was highly variable. The same would be true of pigments, whether mineral or vegetable in origin. Thus, in the painting of a fresco, different batches of pigment might give different colours if used with the same batch of plaster, and different batches of plaster might give different colours if used with the same batch of pigment. A competent professional would avoid such complications by doing some calculations about how much of what he was going to need for any particular task.

5 The dimensions of the Arena Chapel, estimated by measuring 1:50 scale drawings, are (taking 1 Florentine *braccio* = 58.36 centimetres):

length = 20.3 metres = 34.784 *braccia*, that is approximately $34\frac{3}{4}$ *braccia*
width = 8.475 metres = 14.52 *braccia*, that is approximately $14\frac{1}{2}$ *braccia*
height of plane wall = 9.0 metres = 15.422 *braccia*, that is approximately $15\frac{3}{8}$ or $15\frac{1}{2}$ *braccia*
height, to top of barrel vault = 12.7 metres = 21.76 *braccia*, that is approximately $21\frac{3}{4}$ *braccia*.

6 See R. Goldthwaite, *The Building of Renaissance Florence*, Baltimore and London: The Johns Hopkins University Press, 1980.

Gold and some pigments, most notably ultramarine – the rich blue that is made from lapis lazuli – were very expensive. Their use in certain parts of a painting is one of the matters most frequently specified in surviving contracts, and the painter presumably priced his work accordingly. The cost of the materials was certainly one element in the finished product that impressed those who viewed any work of art. Although Enrico Scrovegni (d. 1336) must surely have realized he was employing the best artist of the day to paint his new chapel in Padua, he no doubt also relied upon the good impression that would be made by Giotto's lavish use of ultramarine. That, however, is not to say that he would have let himself be overcharged for this sure-fire way of impressing the ignorant as well as the *cognoscenti*. The calculated cost of the materials was almost certainly the basis for the calculation of the final price. Indeed, in the fourteenth and fifteenth centuries, the craftsman who made the frame for an altarpiece, who used a lot of gold leaf in his work, was sometimes paid more than the painter, who used cheaper materials to fill the panels with figures. A painter who was bargaining with a patron who was a banker would clearly benefit considerably by being good at arithmetic.

It seems to have been the bankers who gave the initial impetus to the rise of widespread education in elementary mathematics. At least, while there is little hard evidence as to which is cause and which is effect, it is clear that the beginnings of international banking operations, which take shape from the late thirteenth century onwards, are suggestively contemporary with the appointment of municipal 'abacus masters' whose specific duty was the teaching of 'commercial arithmetic' and a little geometry – usually to a specified number of boys in the town or neighbourhood. Guilds also set up abacus schools and paid masters to teach in them. In the late fourteenth and early fifteenth century the best abacus school in Florence was that run by the goldsmiths' guild. No evidence has yet emerged that its pupils in fact included Lorenzo Ghiberti (1377–1455) and Filippo Brunelleschi (1377–1446), but their careers show that the education given to a goldsmith might lead to his acquiring, and applying, mathematical skills beyond what was normally required for commerce.

Piero della Francesca provides another, slightly different, example. It is not certain that he did attend an abacus school in his early years, for recent researches have shown no evidence for the existence of such a school in his native town, though there is abundant evidence for the existence of a grammar school.[7] If Piero did attend that school, the fact might account for his later apparently having some reading knowledge of Latin.[8] Piero's father, who seems to have been fairly well off, perhaps provided him with a private tutor or taught his son the rudiments himself. There is thus no real difficulty in explaining how Piero could have learned some mathematics.

However, the apparent lack of an abacus school in Borgo San Sepolcro does raise a historical problem. In later life Piero knew enough about what was taught in abacus schools to write a treatise modelled on those used in such schools. The book follows the standard

7 See James R. Banker, *The Culture of San Sepolcro during the Youth of Piero della Francesca*, Ann Arbor: University of Michigan Press, 2003. This invalidates, or at least seriously undermines, much of what is proposed in Paul Grendler, 'What Piero Learned in School: Fifteenth-Century Vernacular Education', in *Piero della Francesca and His Legacy* , ed. M. A. Lavin (Studies in the History of Art, no. 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art ,1995, pp.161–74.

8 See Giovanna Derenzini, 'Note autografe di Piero della Francesca nel codice 616 della Bibliothèque Municipale di Bordeaux. Per la storia testuale del *De prospectiva pingendi*', *Filologia Antica e Moderna* 9, 1995, pp.29–55, and Chapter 5 below.

pattern exactly. Its problems can be traced back to earlier books of the same kind and there is the familiar patchwork effect, such as the mention of different currencies, that indicates an element of compilation. All this is entirely normal in abacus books. There can be no reasonable doubt that Piero had come across the abacus tradition in its school form. Perhaps he attended a school somewhere other than Borgo San Sepolcro?[9] In any case, the lack of a school in his native town may perhaps explain why Piero wrote his own abacus treatise. The first lines of the work tell us that it was written at the request of a friend, so it was apparently not intended for use in a school, although its form, style and contents are much the same as those of surviving texts that are known to have been employed in abacus schools, and it is entirely appropriate that the title by which the work is now known should be *Trattato d'abaco*.[10] In the absence of an abacus school in Borgo San Sepolcro, Piero's friend perhaps wished to know about what such a school might provide, perhaps with a view to one being set up in the town.

The mathematics taught at abacus schools

As has already been mentioned, the institution of abacus schools appears to be a response to the demand for mathematical skills in connection with international trade and international banking: financial transactions at a distance almost inevitably involved planning and the giving of written instructions, the rendering of written accounts of series of transactions, and the conversion of different currencies one into another. However, mathematical skills were also useful for merely local and everyday trades, and the pupils at abacus schools certainly included prospective craftsmen, tradesmen and merchants as well as prospective bankers. Many of the problems found in abacus books undoubtedly reflect actual mathematical problems encountered in the marketplace.

The problem of converting from one currency to another was by no means limited to international transactions. In many places a number of different currency systems were in use simultaneously. Coins of many origins were in circulation, and coinage was worth the price of its metal – the stamp on it being a guarantee of purity – so payment was usually weighed rather than counted out. It was a matter of practical concern to be able to work out the value of a weight of coinage or other metal that contained, say, a known proportion of silver and the remainder of copper. Problems to do with more complicated alloys are also to be found in school textbooks. There are also problems concerned with barter. Underneath the 'everyday' disguises that now make such problems seem to belong to their own time there lurks the 'rule of three' – the simplest arithmetical form of proportionality and a component of elementary mathematical education to this day.

Surviving textbooks assume various degrees of prior knowledge. Many start by describing the shapes of 'Arabic' numerals, which had been introduced into western Europe in the early thirteenth century in the *Liber quadratorum* (1202) of Leonardo of Pisa (*c*.1170–*c*.1240).

9 As Banker points out, there is a difficulty as to when he might have done so; see Banker, *The Culture of San Sepolcro*, (full ref. note 7).

10 There is a modern printed edition of Piero's *Trattato d'abaco*: Piero ed. Arrighi. Since a facsimile edition of the manuscript is in preparation (as part of an edition of the complete writings of Piero della Francesca, under the general editorship of Marisa Dalai Emiliani, Cecil Grayson and Carlo Maccagni), further references will include the numbers of the manuscript pages (which are given in Arrighi's edition).

Piero della Francesca begins his *Trattato d'abaco* by explaining how to handle fractions. He then rapidly comes to the rule of three, first described in abstract terms and then with an example involving lengths of cloth:

> The rule of three says that the thing one wants to know [about] must be multiplied by that which is not similar and the result it produces must be divided by the other; and the result is of the nature of that which is not similar, and the divisor is always similar to the thing one wants to know [about].
>
> Example. 7 *bracci* of cloth are worth 9 Pounds [*Libre*], what will 5 *bracci* be worth?[11]

The pound (*libra*) was a unit of money that had once been the value of a pound weight of silver. It was divided into twenty *soldi*, each of which was divided into twelve *denari*. Readers who received their mathematical education in Britain before its similar system of pounds, shillings and pence was reformed by the introduction of decimal currency in 1971 will recognize the way an essentially simple calculation gets bogged down in this system of changing bases. Piero does not spare his readers such useful details. His solution is:

> Do thus: multiply the quantity you want to know by the quantity that the 7 *bracci* of cloth are worth, which is 9 Pounds, thus 5 by 9 makes 45, divide by 7 the result is 6 Pounds with remainder 3 Pounds; make them *soldi*, they give 60, divide by 7, the result is 8 *soldi* with remainder 4 *soldi*; make them *denari*, they give 48, divide by 7, the result is 6 *denari* and $\frac{6}{7}$. So 5 *bracci* of cloth at this price are worth 6 Pounds 8 *soldi* 6 *denari* and $\frac{6}{7}$.[12]

The impression of reality generated by considering cloth, a commodity of great importance to the Florentine economy at the time, is somewhat dissipated by the mathematician's characteristic insistence on including the final fraction of a *denaro*. All the same, the problem is clearly closely similar to those that were to be encountered in trade.

The same is true of other exercises in the 'rule of three', such as one to do with barter, in which the unit of currency is now the ducat – a fact that may indicate a different origin for the problem:

> Two men are bartering, one has cinnamon and the other saffron; the cinnamon is worth, in money, 25 ducats the *cento*[13] and he puts it up for barter at 30, and a *cento* of saffron is worth in money 100 ducats and he puts it up for barter at 115. I ask who should adjust one against the other so that neither is cheated.

Before going on to solve the problem, Piero shows how the rule of three is used to check the equity of the proposed transaction:

> Do thus. Multiply the value of the saffron in money by the [price] at which the cinnamon is put up for barter, that is 100 times 30 makes 3000; then multiply the value of the cinnamon in money by the [price] at which the saffron is put up for barter, that is 25

11 The usual form of the plural of the word *braccio* is *braccia*, but Piero seems always to prefer the form *bracci* that occurs here.

12 Piero della Francesca, *Trattato d'abaco*: BML MS, p.5 recto; Pierò ed. Arrighi, p.42.

13 *Cento* seems to be Piero's version of the Tuscan *cantaro* (or *centinaio*), or, since the money used is the ducat, of the Venetian *centinario*. The name can be literally translated as 'hundredweight'. Each was indeed equal to one hundred pounds weight, but Tuscan and Venetian pounds were not equal. The Tuscan *centinaio* was about 33.95 kilograms and the Venetian *centinario* about 47.67 kilograms. See H. J. von Alberti, *Mass und Gewicht*, Berlin: Akademie-Verlag, 1957, pp.415, 416.

times 115 makes 2875 which is the saffron. If the results of the multiplication are equal the barter levels are just, because they are in proportion; and, since [in this case] the results of the multiplication are not equal, so . . .[14]

The result is that the man with the cinnamon must give slightly short weight to the man with the saffron, whose price has been set too high. A further complication in barter is that one trader may want part of the price in money. Piero also gives an example of this:

There are two men who want to barter, one of them has cloth and the other has wool. The piece of cloth is worth 15 ducats and he puts it up for barter at 20 and also wants $\frac{1}{3}$ in money; and a *cento* of wool is worth 7 ducats in money. What price must he put it up for barter so that neither will be cheated?[15]

The answer is that the wool must be bartered at $11\frac{1}{5}$. The calculation involves division by $8\frac{1}{3}$, that is $\frac{25}{3}$, which may not be an operation that the pupil would be expected to carry out in the head, though no other method is mentioned. It seems likely that readers, and probably also market traders, had tables of division and multiplication to which they could refer.

The examples just quoted are problems that, in reality, might have to be solved reasonably quickly, in public and in competition with an opponent in bargaining. Some other problems could have been dealt with more slowly, for instance, the problem of altering the quantity of silver in an alloy:

I have 3 pounds of silver at $9\frac{1}{2}$ alloy, I want to reduce the alloy to 7. I want to know how much copper needs to be added.

It appears from the calculation that Piero's indication of the amount of silver in the alloy is as ounces of silver per pound weight of metal. The Tuscan pound contained twelve ounces, each nearly equal to the imperial ounce still in use today. So the Tuscan pound is equivalent to about three quarters of an imperial pound (that is, about 340 grams).[16] The solution to this problem is entirely straightforward, and Piero rounds it off by stating a general rule:

Take it this way. See how many ounces of fine silver [there are in] what you have, which is $28\frac{1}{2}$ ounces of fine silver; do thus: there are $9\frac{1}{2}$ ounces of fine silver, multiply 3 by $9\frac{1}{2}$ it makes $28\frac{1}{2}$, divide by the [level of alloy] that you want [to get] by reducing, which is 7, the result is $4\frac{1}{14}$. Subtract 3, there remains $1\frac{1}{14}$ of a pound, which is 1 pound and $\frac{6}{7}$ of an ounce. That much copper must be added and it turns into [an alloy] at 7. Always multiply the pounds you have by the [level of] alloy of the silver, and divide by the [level of] alloy you want to reduce it to.[17]

On the whole, abacus books are not given to stating general rules. When such rules are stated, the form of words often seems excessively cumbersome – as, for example, in Piero's version of the rule of three given above. Mathematicians were, of course, familiar with this lumbering style, which fell into disuse only in the seventeenth century, when it became stan-

14 Piero della Francesca, *Trattato d'abaco*: BML MS, p.8 recto; Piero ed. Arrighi, p.48, l.-3 to p.49.

15 Piero della Francesca, *Trattato d'abaco*: BML MS, p.8 recto; Piero ed. Arrighi, p.49. For the *cento* see note 13 above.

16 See Alberti, *Mass und Gewicht* (full ref. note 13), p. 415.

17 Piero della Francesca, *Trattato d'abaco*: BML MS, p.12 verso; Piero ed. Arrighi, p.56, l.-8.

dard to employ an algebraic notation that is more or less the same as that used today.[18] In Piero's time, even algebra was written out in words. The thin scattering of specialized symbols became gradually denser over the following century. However, the use of symbols and abbreviations often varied from author to author, so to the uninitiated fifteenth-century algebra, as regularly found in abacus books, tends to be rather easier to follow than its sixteenth-century counterpart.

The generous supply of problems on the rule of three to be found in Piero's *Trattato* and in school textbooks is no doubt a tribute to its actual importance in everyday life. Moreover, it mirrors the widespread use of simple proportionality, geometrical as well as arithmetical, that is found in contemporary natural philosophy, and the significance accorded to proportions in the visual arts as well as in music. Furthermore, the geometrical equivalent of the rule of three – that is, the use of simple proportionality between pairs of similar triangles – is the commonest method of proof employed in contemporary work in geometry. It turns out that Piero's geometrical work is entirely typical in this respect.

There is a great continuity in problems of elementary mathematics, and another of Piero's examples of the rule of three may well be familiar, *mutatis mutandis* (but only slightly), to today's readers:

> There is a fountain with two basins, one above and one below. And each has three spouts: the first spout in the upper [basin] fills the lower basin in two hours, when all the others are closed; and the second fills it in three hours, the third fills it in 4 hours. And when they are all closed, opening the first spout in the lower [basin] empties it in three hours, and opening the second empties it in four hours, opening the third empties it in five hours. Now I open, all at once, the spouts above and those below; I ask how long the lower basin will take to fill up.[19]

In the school textbooks I used in the 1950s, the fountain basins had become water tanks and the spouts taps, but the problem was otherwise the same. The answer is 'So I say that it [the lower basin] will be full in 3 hours and $\frac{1}{3}$ of an hour'.[20] And, in answer to the supplementary question that inevitably follows, if the third upper spout is closed the lower basin will fill in 20 hours.

Piero's problems have been quoted because it is his mathematical thinking that will be in question. His problems are, in fact, typical of the abacus tradition: in their mathematical form, in the way they are stated and in the way they are solved. In the event, Piero's versions had some historical importance because they were taken over by Luca Pacioli, who seems to have had access to Piero's manuscripts (probably after his death) and they duly appeared in print, though sometimes with modified methods of solution, in Pacioli's *Summa de arithmetica, geometria, proportioni e proportionalità*, published in Venice in 1494.[21]

18 The first algebraist whose work presents a substantially modern appearance, with equations set off from the verbal text as they are today, is François Viète (1540–1603).

19 Piero della Francesca, *Trattato d'abaco*: BML MS, p.15 recto–p.15 verso; Piero ed. Arrighi, p.61.

20 The use of a fraction of an hour rather than minutes is a reminder that at this time it was not usual for timekeeping devices to indicate minutes as such, though astronomers naturally made use of minutes (and seconds) in their calculations. The habitual use of sixtieths in the work of Piero and his contemporaries parallels the sexagesimal system of calculation employed by astronomers.

21 For a detailed account of Pacioli's arithmetical and algebraic borrowings from Piero, see S.A. Jayawardene, 'The *Trattato d'abaco* of Piero della Francesca', in *Cultural Aspects of the Italian Renaissance: Essays in Honour of Paul Oskar Kristeller*, ed. C. H. Clough, Manchester: Manchester University Press, 1976, pp.229–43.

Pacioli does express his gratitude to Piero ('Pietro dei franceschi') on the first page of his 'Letter to the reader', but the effect of this is mitigated by the fact that gratitude is then equally expressed to a number of other people: 'Here in Venice . . . Gentile and Giovanni Bellini who are blood brothers . . . And in Florence Alessandro Botticelli, Philippo and Domenico Ghirlandaio. And in Perugia Pietro who is called Perugino. And in Cortona Luca [Signorelli] worthy pupil of our master Piero. And in Mantua Andrea Mantegna. And in Forli Melozzo . . .'[22]

Nor is this by any means all of Pacioli's list of the painters of his day. Since the list of names is so long, the absence of a reference to Leonardo da Vinci tends to confirm that in 1494 he and Pacioli were not yet friends. It is interesting that Pacioli should acknowledge artists in this way – particularly in view of the fact that his tiny and simple section on perspective considers only natural vision, not drawing, and apparently owes no debt to anyone but Euclid for his well-known work on optics. It is understandable that such a style of widely distributed thanks might seem to give substance to Giorgio Vasari's charge that Pacioli published Piero's mathematical work under his own name. As will become clear, there is solid justification for this comment in regard to Piero's work on polyhedra, but the present instance is by no means clear. In his sections on arithmetic and algebra, and in much of the section on geometry, Piero had not invented most of the problems he sets out nor most of the solutions he gives to them. In examining school texts, including Piero's, it becomes apparent that they were not composed by individuals *de novo* but were compiled by selection from earlier ones. Sometimes problems were varied or slightly different solutions offered, but essentially the texts are visibly derived from a long and continuing tradition.

The elementary mathematical textbooks that have been studied by historians are largely from the region of Tuscany, but there is evidence that similar works were in use elsewhere.[23] The pupils apparently took notes, or copied out the complete text, while their teacher read out problems and solutions. In any case, problems and solutions are what the works largely contain. Short passages of discursive text are few and far between. That is, the books essentially supply series of worked examples. Many of the problems can be traced back to Islamic sources, and the abacus schools take their name from the *Liber abaci* of Leonardo of Pisa, who had travelled in the East and worked for most of his life in Palermo.[24] Properly speaking, the textbooks might best be described as 'commercial arithmetics', but in the present context the name, though well deserved, seems inappropriate, because it is relevant that they also deal with problems of geometry.

Historical hindsight shows that in this period it was developments in algebra that were to have the most immediate consequences for the future of technical mathematics. Historians have accordingly tended to emphasize the algebraic content of abacus books.[25] However,

22 Luca Pacioli, *Summa de arithmetica, geometria, proportioni e proportionalità*, Venice, 1494, Epistolo (first paragraph): 'Qui a vinegia . . . Gentil e Giovan bellini carnal fratelli . . . E in Fiorenza alexandro boticelli. Phylippio e Domenico grilandaio [*sic*]. E in perosia Pietro ditto elperusino. E in Cortona Luca del nostro maestro Pietro degno discipulo. E in mantua Andrea mantegna. E in Furli Melozzo . . .'

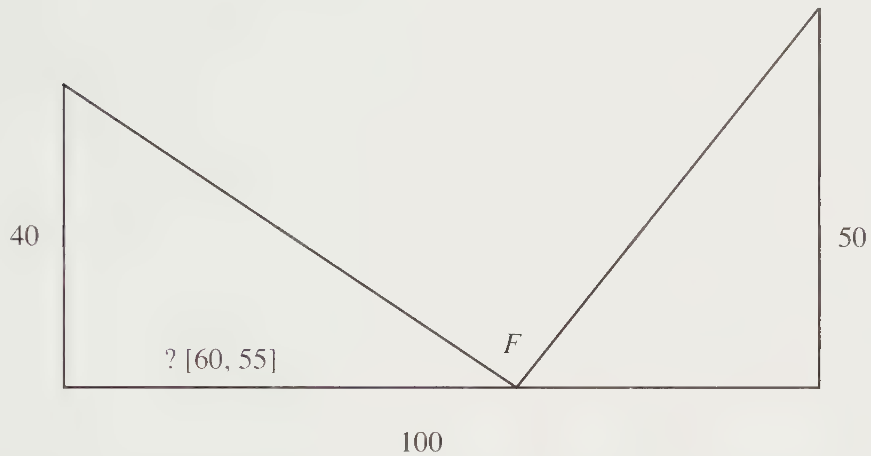
23 See R. Franci and L. Toti Rigatelli, 'Towards a History of Algebra from Leonardo of Pisa to Luca Pacioli', *Janus* 72, 1985, pp.17–82.

24 Leonardo of Pisa's treatise was not printed until the nineteenth century: Leonardo of Pisa [Fibonacci], *Liber abaci*, in *Scritti di Leonardo Pisano*, ed. B. Boncompagni, vol.1, Rome 1857–62.

25 The same is true of the useful, pioneering work on Piero's mathematics written by an art historian, M. D. Davis, *Piero della Francesca's Mathematical Treatises: The 'Trattato d'abaco' and 'Libellus de quinque corporibus regularibus'*, Ravenna: Longo Editore, 1977. This book includes a list of correspondences between problems in Piero's *Trattato* and in Pacioli's *Summa*, see also note 21 above.

in most such books the section on arithmetic is by far the longest. Moreover, arithmetical methods are sometimes proposed to solve problems that a school pupil today would probably be taught to solve by means of algebra. For instance Piero poses the problem,

> There is a plain on which there are two towers, one 40 *bracci* high, the other 50 *bracci*, and from one tower to the other it is 100 *bracci*. And on each tower there is a bird, which birds, flying at the same speed, set off at the same moment to go to drink; and they arrive at the same moment at a fountain that is between one tower and the other. I ask how far the fountain is from the tower that is 40 *bracci* [high] and how far from the one that is 50.[26]



1.2 Diagram for the problem of the fountain between two towers, solved by the method of double false position in Piero della Francesca, *Trattato d'abaco*, BML MS, p. 22 recto, Piero ed. Arrighi, p.72. Piero does not supply a diagram. Drawing by JVF.

Piero does not supply a diagram. It would look more or less as in Figure 1.2, in which the length we have been asked to find has been indicated with a '?'. To use algebra, one would begin by setting this length equal to an 'unknown'. What Piero does is to give the length a value, then calculate what that would give for the length of the flight path of the first bird (the one on the tower height 40 *braccia*), or rather for its square, using Pythagoras' theorem. He then calculates the distance of the fountain from the other tower, and finds the square of the flight path of the second bird. These two squared flight paths should be equal, since the birds arrive at the fountain at the same moment, but with Piero's first guess at a value for the unknown distance, the answers are different. He notes the difference, then makes a second guess, and again finds the difference between the squares of the two flight paths. He then uses the two guesses and the two differences to calculate the correct answer to the problem. The two guesses are usually called 'positions' (for 'positioni'), which is an

26 Piero della Francesca, *Trattato d'abaco*: BML MS, p.22 recto; Piero ed. Arrighi, p.72.

awkward but standard translation, ultimately for the Latin 'positio', which was the translation used for an Arabic word that must have meant something like 'substitution'. I have used 'substitution' in my translation of Piero's text. The technique, which is of Islamic origin, is called 'double false position'.

> Now you say: let us make the distance of the fountain from the tower of [height] 40 *bracci*, 60 *bracci*. Multiply into itself [it] makes 3600; and the tower we had made 40 *bracci*, multiplied into itself [it] makes 1600, add this to 3600 [it] makes 5200 for one side. The other [that is the distance of the fountain from the other tower] is from 60 to 100, which is 40, multiplied into itself makes 1600; and the tower is 50, multiplied into itself makes 2500, add this to 1600 [it] makes 4100. And you want 5200, subtract 4100 there remain 1100; so: for the 60 you put in you got 1100 too much. Make another substitution [*positione*] and say that from the fountain to the smaller tower it is 55 *bracci*, multiplied into itself it makes 3025, then multiply the tower which is [of height] 40 into itself [it] makes 1600, add them together it makes 4625 for one side. See what it is from 55 to 100, which is 45, multiply into itself [it] makes 2025; then multiply the taller tower which is [of height] 50 into itself [it] makes 2500, which added up makes 4525. And you want 4625, subtract 4525 there remains 100; so that: for the 55 which you put in you get 100 too much. The excesses must be taken one from another; take 100 from 1100, there remains 1000, which is the divisor; now multiply 55 by 1100 [it] makes 60500, and multiply 60 by 100 [it] makes 6000, take this from 60500 there remains 54500, which you divide by 1000 and the result is $54\frac{1}{2}$. And that is the distance from the fountain to the tower of [height] 40 *bracci*, and [it is] $45\frac{1}{2}$ from the fountain to the tower of [height] 50 *bracci*.

Piero clearly expects his readers to know Pythagoras' theorem, though no doubt only as a 'well-known' result; that is, one that would be recognized and would not be considered to stand in need of proof. A proof of Pythagoras' theorem is given in Euclid, *Elements*, Book 1, Proposition 47, but pupils at an abacus school could certainly not be presumed to be familiar with the *Elements* as such. In any case, the use of the theorem, which involves squares, is a little misleading: to put it in today's terms, the problem is in fact linear, that is it involves only the simple unknown, not squares of it. If this were not so, Piero's method of solution by 'double false position' would not work; the method is applicable only to linear problems. A proof of how (or why) it works is given in Appendix 1. All these linear problems could, of course, be solved by algebra. The advantage of the method of false position is that it allows numbers to be chosen so as to be easy to work with. The last stage may turn out to be awkward, but by choosing the initial guesses it is possible to ensure that the earlier stages are easy. Thus, at the worst, there is only a single nasty division sum. Such considerations have now been abolished by the portable electronic calculator, but in the bargaining that went on in a fifteenth-century marketplace they must have been fairly important.

Piero does, of course, recognize that the problems he has solved by the arithmetical method of double false position can also be solved by algebra. In fact, closely similar problems occur in the parts of his book concerned with the two methods. For instance, a little before the problem of the tower, there is a marketplace problem that is solved by double false position:

> There is a fish that weighs 60 pounds, the head weighs $\frac{3}{5}$ of the body and the tail weighs $\frac{1}{3}$ of the head. I ask what the body weighs.[27]

One of the problems to be solved by algebra is almost exactly the same:

> A fish weighs 50 pounds, the head weighs $\frac{1}{3}$ of the body, the tail weighs $\frac{1}{4}$ of the head. I ask what the body weighs, and what the head weighs and what the tail weighs.

The algebraic solution is essentially what would be taught today, except for the way it is expressed. Instead of starting 'Let the weight of the body be *x*' we have 'Put the weight of the body as $\overline{1}$ thing'. 'Thing' – in the original 'cosa' (sometimes spelled 'chosa') – is the normal way of referring to the unknown quantity, and it eventually gave algebra the Latin name 'cossa'. However, the addition of a bar over the 1 to indicate that we are counting in 'things' is a peculiarity of Piero's. When he has squares, he draws a small square over the number. As there was no standard notation, he explains his own at the outset of the section. For today's reader, it is not the small amount of unfamiliar notation but rather the absence of the expected quantity of familiar notation that tends to make the work hard to follow. We essentially have ordinary prose, a style that is called 'rhetorical algebra'.

> Put the weight of the body as $\overline{1}$ thing, the head weighs $\frac{\overline{1}}{3}$, the tail weighs $\frac{\overline{1}}{4}$ of $\frac{1}{3}$; and $\frac{1}{3}$ and $\frac{1}{4}$ of a third are $\frac{5}{12}$, so you have $\overline{1}\,\frac{5}{12}$ equal to 50. Make into twelfths [and] you will have $\overline{17}$ of them equal to the number 600; divide by 17 the result is $35\frac{5}{17}$, this is the value of the thing we put as the weight of the body, [so] the body weighs $35\frac{5}{17}$. The head weighs $\frac{1}{3}$ which is $11\frac{13}{17}$, the tail weighs $\frac{1}{4}$ of the head which is $\frac{1}{12}$ of the thing whose value is $2\frac{16}{17}$, so the weight of the tail is that: $2\frac{16}{17}$.[28]

Readers unaccustomed to dealing with this kind of text will probably have suffered from an almost irresistible urge to write the calculation out 'properly'. Unfamiliarity is not the only thing that makes rhetorical algebra seem clumsy. The style also obscures the simplicity of rules such as 'change side change sign', which was known to the Islamic algebraists and to their successors – and had indeed been known to Diophantus of Alexandria (active 250). The style also fails to make it obvious when powers of the unknown can simply be cancelled. For instance, Piero treats as separate cases the relationship of form 'square of thing equal to things' and that of form 'thing equal to number'.[29] It is very easy to see the equivalence of these two relationships if we use the standard notation of our own time. Piero no doubt recognized the equivalence himself, but could not assume his readers would easily do likewise.

Piero goes on to deal systematically with algebraic problems of more and more complicated forms. He lists the forms concerned before launching himself into the series of problems. Almost all of the more difficult problems are entirely abstract, that is they are merely numerical. This is an indication of the way that algebra was now developing into something that had a mathematical interest of its own, independent of its usefulness in solving

27 Piero della Francesca, *Trattato d'abaco*: BML MS, p.17 recto; Piero ed. Arrighi, p.64. Piero's solution to this problem is considered in detail in Appendix 1.

28 Piero della Francesca, *Trattato d'abaco*: BML MS, p.37 recto; Piero ed. Arrighi, pp.96–7.

29 Piero della Francesca, *Trattato d'abaco*: BML MS, p.24 recto; Piero ed. Arrighi, p.75.

problems in the marketplace.[30] Since Piero never actually taught mathematics, his own interest in the subject clearly has something of the same purely intellectual motivation in it. Piero must have enjoyed mathematics. Moreover, his algebraic work shows him to have been one of the most competent mathematicians of his day. In keeping with the abacus tradition, most of the problems are standard ones, but Piero introduces variations and improved solutions that suggest he had a thorough grasp of the matter in hand.

The greater neatness of Piero's overall treatment, compared with what is found in the books used in abacus schools, may possibly be due to the *Trattato d'abaco*'s not having been written by a working teacher. School books were probably added to over the years, as new problems occurred to the writer or new examples came to his notice. In contrast, Piero's work was presumably conceived and thought out as a whole. However, even allowing for this peculiarity in the origins of Piero's *Trattato*, the work makes it clear that Piero as a mathematician shares something of the temperament to be found in Piero as a painter. There is a notable tidy-mindedness that serves him well in both capacities.

The geometry in abacus books

Enjoying advanced algebra is certainly not an essential qualification for a competent painter. However, some interest in geometry surely is. There is not much geometry in most abacus school texts. What there is tends to have clear connections with surveying. For instance, the numerous problems of finding areas of triangles are obviously related to the practice of assessing taxes according to the area under cultivation. Since Piero's father is known to have traded in the vegetable dyestuff woad, similar to indigo, he would presumably have found it useful to have a son who was capable of checking the official estimates of the area of fields. It is therefore not surprising that Piero's *Trattato d'abaco* contains standard problems concerning areas.

The book naturally starts with triangles, as do the elementary textbooks of our time. In the fifteenth century, however, instruments for measuring angles were generally inaccurate or expensive or both, so surveyors did their best to rely upon measuring lengths. Consequently, the triangle problems frequently involve finding the height of the triangle as a preliminary to multiplying by the base and dividing by two to get the area. All the geometrical results that are used, such as the formula for the area as half base times height, are proved in Euclid's *Elements*, but are merely taken for granted in the abacus books. As with other types of exercise, the book simply gives procedures for solving problems, presented in the form of series of worked numerical examples.

After a brief preamble, Piero starts with the simplest triangle of all, one that has three equal sides:

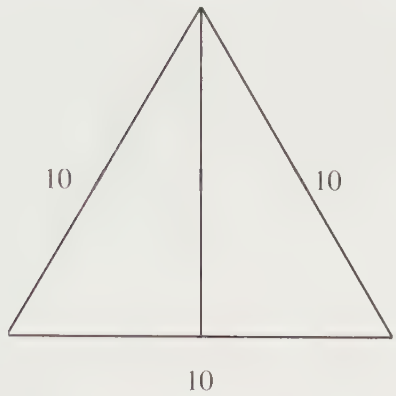
> Example. Let the triangle ABC have equal sides, and let each be 10 *bracci*. I ask what is its height.[31]

However, he immediately proposes to be systematic. His solution begins

> There are many ways [of solving this problem], I shall show two or three.

30 This development, which is of great importance for the subsequent history of mathematics, is well described by Franci and Toti Rigatelli in the paper 'Towards a History' (full ref. note 23), which also gives a detailed discussion of Piero's algebraic problems.

31 Piero della Francesca, *Trattato d'abaco*: BML MS, p.80 recto; Piero ed. Arrighi, p.169.



1.3 Equilateral triangle. Copy of figure supplied in Piero della Francesca, *Trattato d'abaco*, BML MS, p. 80 recto. Drawing by JVF.

He also supplies a diagram, which has been copied in Figure 1.3. Whereas Piero's arithmetical and algebraic sections had not been illustrated – even when, as in the last two algebra problems, the examples concerned a geometrical figure[32] – his geometry section is supplied with figures throughout. As will become clear in Chapter 4, this turns out to be important in later sections of the part of the *Trattato d'abaco* concerned with geometry, where illustrations sometimes carry information not supplied in the text. It is therefore interesting that in the manuscript in the Laurentian Library in Florence the diagrams are in the same ink as the text, and thus seem to be by the same hand. Since some of the diagrams apparently include changes of mind, it seems likely that the drawings, and the manuscript text, are by Piero himself. However, this is not a working manuscript: the occasional lapse, such as incorrect intermediate numbers in a calculation that leads to a correct answer, indicates that this is a fair copy. Perhaps the friend whom Piero addresses so respectfully in his introduction to the *Trattato d'abaco* was sufficiently important to him that he not only composed the work for him but also wrote out this copy.[33] This suggestion, which has already been made by a number of scholars, is not of crucial significance in interpreting the work: whoever did the illustrations must certainly have had access to original drawings by Piero. It therefore seems highly likely that the pattern of illustration in the work as a whole also reflects Piero's original intentions. Even the simplest geometry seems to be a cue for drawings.

Piero's first two methods of finding the height are:

> The first is if you halve one side of the triangle which you know is 10 [so the half] will be 5, multiply it into itself [which] makes 25; then multiply 10 into itself [which] makes 100, subtract 25 from it there remains 75. The root of 75 will be the height.

and

> Alternatively, multiply one side, which is 10, into itself [which] makes 100, subtract from

32 On these two problems, see Chapter 7 below.

33 The first lines of Piero's treatise are: 'Being requested that I should write some things about the abacus necessary to merchants, by a person whose requests are to me as commands, not out of presumption but in obedience I shall steel myself, with God's help, to satisfy partly that wish, that is by writing some examples relating to trade . . .' (BML MS, p.3 recto; Piero ed. Arrighi, p.39). This passage is quoted at greater length below (p.30).

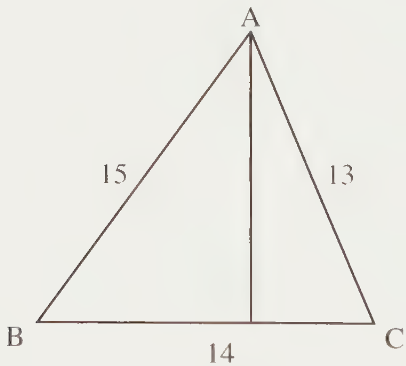
it $\frac{1}{4}$ which is 25, there remains 75; as before. This method can be used for equilateral triangles, but not for others.

Piero has again silently used Pythagoras' theorem, this time in the right-angled triangles that appear when a perpendicular is drawn to show the height. The theorem was presumably considered 'well known', but in any case Piero's book belongs to the world of useful techniques rather than rigorous proofs. No formal check on the result is suggested, as sometimes happens in such circumstances, but the multiplicity of methods may perhaps imply that the answer should be obtained in two different ways as a check? Neither here nor elsewhere does Piero demonstrate how to evaluate the square roots. Such silence is normal in abacus books, so the pupils may have been expected to have a table of square roots to hand.

After the methods for the equilateral triangle Piero provides two methods that can be used more generally, though it turns out that the particular triangle chosen, shown in Figure 1.4, has some rather simple numerical properties:

> But when they [the sides] are not equal, as happens when there is a triangle with AB 15, BC 14, AC 13 and the base is 14, multiply it into itself, [which] makes 196, multiply AB which is 15 into itself [which] makes 225, join it with 196 [which] gives 421; multiply AC which is 13 into itself [which] makes 169, take it from 421 there remains 252; divide by double the base BC which is 14 [so doubled] it will be 28, the result is 9: and 9 is [the distance] from B to the point at which the cathetus falls.[34] Multiply 9 into itself [which] makes 81, and multiply AB which is 15 into itself [which] makes 225; subtract 81 there remains 144: its root is the height [*catecto*], which is 12. And this method can be used for all triangles.[35]

The 13, 14, 15 triangle, which goes back at least to Leonardo of Pisa, has several simple numerical properties. The reason for the simple height above the base 14 can be seen by,

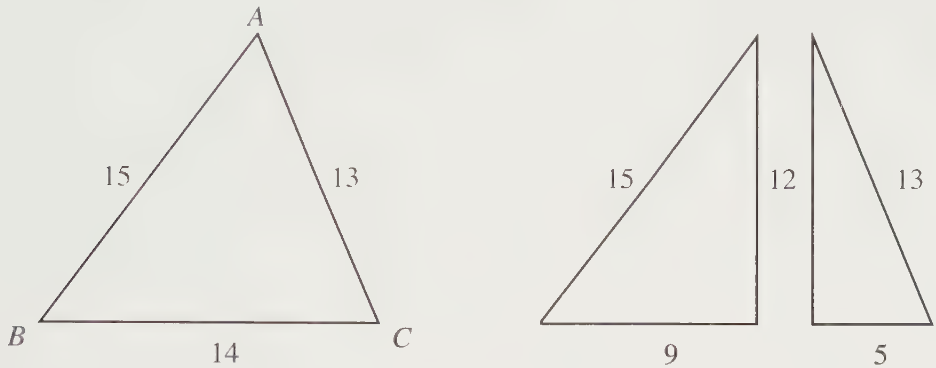


1.4 The 13, 14, 15 triangle. Copy of figure supplied in Piero della Francesca, *Trattato d'abaco*, BML MS, p. 80 recto. Drawing by JVF.

34 That is the foot of the perpendicular to the base. 'Cathetus', usually 'catecto' in Piero's Tuscan, is the standard word for a perpendicular or a perpendicular height. As can be seen in the following sentence, mathematical usage of this time does not distinguish between the thing itself (the line segment) and the value of its length.

35 Piero della Francesca, *Trattato d'abaco*: BML MS, p.80 recto; Piero ed. Arrighi, p.170.

as it were, pulling the triangle apart along the height Piero has drawn. This gives triangles with sides as shown in Figure 1.5. That is, the 13, 14, 15 triangle is made up of two well-known right-angled triangles: one 5, 12, 13 triangle and one 3, 4, 5 triangle (scaled up by a factor 3) put side by side. This triangle is used again and again in abacus books.



1.5 The 13, 14, 15 triangle dissected into two right-angled triangles, one 5, 12, 13, the other 9, 12, 15 (that is 3 × 3, 3 × 4, 3 × 5). Drawing by JVF.

After this method for finding the height in any triangle, Piero gives another:

> Again one can find the height by another method; that is add up the two sides on either side of the base, that is 15 and 13, [which] makes 28, divide by 2 [which] gives 14, and dividing by 2 again gives 7; add the divisor, which is 2, [which] makes 9; and 9 is [the length of the line from] B to D where the height falls.[36] Multiply it into itself [which] makes 81 and 15 into itself makes 225, subtract 81 from it there remains 144; its root is the height, which is 12.

He then explains why another method was necessary:

> The best [method] is the one just given; for the one given before that it is necessary that the height always falls in between [that is the foot of the perpendicular from A lies between B and C], so choose the second method, which is better.

That is, the last method is the most general of all, since it does not depend upon the angle at *B* or *C* being acute. Despite using numerical examples, Piero is thinking about triangles as such, not merely about the particular example he has drawn. Unfortunately, however, the last and most general method seems to be incorrect. As is shown in Appendix 2, it appears to be a garbled version of the previous method. Piero gets the right answer merely by numerical coincidence. A hint about his later thoughts on the matter may be found by looking at his *Libellus de quinque corporibus regularibus* (Short book on the five regular solids), which was probably written after the *Trattato d'abaco*. A great deal of the geometry to be found in Piero's *Trattato d'abaco* reappears in a neater or more developed form

36 That is, *D* is the foot of the perpendicular from *A* to *BC*.

in the *Libellus*.[37] The work on determining heights of triangles comes at the very beginning of the latter work. Here we may note that the number of methods has shrunk to two: one for the equilateral triangle and one for the 13, 14, 15 triangle. The latter is a slight variant on the first of the two general methods given in the *Trattato*. The faulty second method has vanished without trace.

Having shown how to find the height of the triangle, Piero then gives instructions for finding the area as half base times height. His example is again the 13, 14, 15 triangle. Since the methods described for finding the height were all at least moderately cumbersome it is somewhat disconcerting that Piero next gives a method of finding the area without finding the height:

> You can know it [the area] without finding the height. That is by adding all the sides of the triangle together 13, 14, 15 make 42, take half which is 21; then see the difference there is from 13 to 21, which is 8; multiply 8 by 21 [which] makes 168. Take the difference there is from 14 to 21, which is 7, multiply 7 by 168 [which] makes 1126. Then you see [the difference] from 15 to 21, which is 6, multiply 6 by 1176 [which makes] 7056. The root of 7056 will be the square of the triangle, which is 84, as above.[38]

Piero gives no source for this method, which is not commonly found in abacus books, but is known in both the artisan and the learned traditions of mathematics. It is derived from a formula for finding the area of a triangle that is now ascribed to Heron of Alexandria (active 62).[39] In today's terms, if we take the sides of the triangle as a, b, c, and let half their sum be s, then we have

$$\text{Area of triangle} = \sqrt{s(s-a)(s-b)(s-c)}.$$

This method looks simple, but in practice it might involve heavy calculation since the sides of the triangle need not be easy numbers and at the end of the procedure it may well be necessary to take the square root of a large number.

The series of problems on finding heights and then areas is followed by a number of other problems concerning triangles. Some of them are solved by using algebra. For instance:

> There is a triangle which is ABC, and BC is 1 *braccio* greater than AB and AC is 1 *braccio* greater than BC; and I find that the area of the triangle is 84 *bracci*. I want to know its sides.[40]

The triangle in question is, of course, the abacists' pet: the 13, 14, 15 triangle. Since its height is 12 above the base 14, the area is $\frac{1}{2}\times 14 \times 12$, which is 84. However, we do not simply want to know the answer. The purpose of the example is to show a method. Piero starts his solution

> Put BC equal to $\overline{1}$ thing, and AB $\overline{1}$ – 1, and AC $\overline{1}$ + 1.

The algebraic manipulation gets fairly complicated before we arrive at the expected answer

> And so you have: AB 13, BC 14 and AC 15.

37 The dating and content of the *Libellus* are discussed in Chapter 4 below.

38 Piero della Francesca, *Trattato d'abaco*: BML MS, p.80 verso; Piero ed. Arrighi, pp.170–1.

39 For the occurrence of this formula in medieval manuscript sources, and various methods proposed for its proof, see Menso Folkerts and Richard Lorch, 'Some Geometrical Theorems Attributed to Archimedes and their Appearance in the West', in *Archimede: Mito Tradizione Scienza* , ed. Corrado Dollo (Proceedings of a conference held in Syracuse, October 1989), Biblioteca di Nuncius, Studi e Testi IV, Florence: Olschki, 1992, pp.61–79.

40 Piero della Francesca, *Trattato d'abaco*: BML MS, p.81 verso; Piero ed. Arrighi, p.172.

The use of numerical examples in abacus books makes all the geometry look like mere exercises in arithmetic or algebra. Nonetheless, there are examples that are effectively numerical versions of propositions from Euclid. For instance, Book 4 of the *Elements* contains a series of propositions about constructing circles and triangles: in a given circle to inscribe a triangle equiangular with a given triangle (Proposition 2); about a given circle to circumscribe a triangle equiangular with a given triangle (Proposition 3); in a given triangle to inscribe a circle (Proposition 4); about a given triangle to circumscribe a circle (Proposition 5). These are followed by a set of analogous propositions regarding the square then the pentagon and the hexagon.

Abacus books usually mention some polygons other than the triangle, but are neither as detailed nor as systematic as Euclid. After his rather long series of problems on triangles, Piero, whose geometry section is relatively extensive, provides shorter collections of problems concerning the square and rectangles and other higher polygons. Two of his six examples on the regular pentagon involve the ratio between its side and the first diagonal. This ratio corresponds to what Euclid calls 'extreme and mean proportion', which was called 'golden section' in the nineteenth century. Pacioli calls the ratio 'divine proportion'. Piero simply works out the numerical expression for it – as usual leaving the evaluation of square roots to the reader's ingenuity – without giving it a name and without further exploration of its properties.[41] It is not until he comes to a pentagon inscribed in a circle that Piero gives any indication of recognizing that some of his problems are based on propositions in Euclid.[42] The reference here is to *Elements*, Book 13, which Piero also uses in his treatment of regular polyhedra later in the *Trattato*.

After the pentagon, Piero gives much briefer treatments of the hexagon and the octagon. He then turns to the circle. Here there are variants of Euclid's inscription problems. For instance, whereas Euclid proposes the problem of constructing a circle inside a given triangle to touch all its sides, Piero specifies first an equilateral triangle and then a triangle with sides of 13, 14, 15, and asks for the diameters of the largest circles that can be fitted within each.[43] Thus where Euclid requires a construction, abacus books, and Piero's *Trattato*, ask for numbers. Moreover, whereas Euclid goes on to consider circles in relation to higher polygons, Piero, in common with other authors of the practical tradition, proposes to fit more circles into one triangle, first two equal circles, then three equal circles. The circles are to be as big as possible, that is they are to touch each other and the sides of the triangle. Next, a square is to be fitted into a triangle, then two circles into a square, then three, four and eventually five equal circles into a given circle, and so on. As with some of the earlier geometrical exercises, many of the solutions to these more complicated problems use algebra, so that it is not certain whether they should really be described as geometrical or algebraic. In any case, the tenor of the work is far from the pure geometrical style of Euclid, since all the problems are stated in numerical terms, and all the answers are likewise numerical.

Thus far Piero's *Trattato d'abaco* may be regarded as presenting problems in arithmetic, algebra and geometry that are either genuinely practical or recognizable as having developed from the exercise of techniques that arose in the practical tradition. In particular, the complicated abstract problems in algebra and the more elaborate geometrical problems, many of

41 Piero della Francesca, *Trattato d'abaco*: BML MS, p.88 verso–89 verso; Piero ed. Arrighi, pp.188–91.

42 Piero della Francesca, *Trattato d'abaco*: BML MS, p.90 recto; Piero ed. Arrighi, p.192.

43 Piero della Francesca, *Trattato d'abaco*: BML MS, p.94 verso; Piero ed. Arrighi, pp.200–01. The diameter of the required incircle of the 13, 14, 15 triangle is 8.

them solved with the help of algebra, are clearly useless except that they provide more advanced exercises for pupils who are sufficiently talented or sufficiently interested in mathematics for its own sake. What Piero has written in his *Trattato d'abaco* seems to be a rather idealized version of a textbook. It goes much beyond what is promised in the preamble:

> Being requested that I should write some things about the abacus necessary to merchants, by a person whose requests are to me as commands, not out of presumption but in obedience I shall steel myself, with God's help, to partly satisfy that wish, that is, by writing some examples relating to trade such as bartering, [fair] prices and [dividing among] partners; beginning with the rule of three things, going on to [rules of false] position and, if it please God, some things of algebra; first saying some things about fractions . . .[44]

This sounds as though Piero had originally intended to write a conventional treatise on 'commercial arithmetic', containing no geometry to speak of. In the event, presumably in accordance with his own inclinations, he did include a substantial quantity of geometry of the kind useful to such people as land surveyors or supervisors of building works. In any case, Piero's *Trattato* tells the student considerably more about mathematics than he would need to know merely to cope with the problems encountered in commerce, surveying and so on. Nonetheless, the style of the work is exactly that of the more down-to-earth textbooks on which it is modelled.

The same is true of the style of the remainder of Piero's treatise, the last 22 of its 170 small octavo leaves in the Laurentian Library's manuscript. These pages deal with three-dimensional bodies. They are, in a sense, the geometrical equivalent of the advanced algebra to be found earlier in the *Trattato*: intellectually stimulating but not notably relevant to practical concerns. In the case of the advanced geometry, the work has some obvious connections with the learned tradition, and specifically with Book 13 of Euclid's *Elements*. This book deals with the five regular solids known to the ancients: the tetrahedron (four equilateral triangular faces), the cube (six square faces), the octahedron (eight equilateral triangular faces), the dodecahedron (twelve regular pentagonal faces) and the icosahedron (twenty equilateral triangular faces).[45] That rather unhelpful description of the solids – unhelpful in the sense that it is not a very good aid to visualizing what they look like – is probably a fair summary of what all but the most expert mathematicians of Piero's time knew about the mathematical properties of the solids in question. But they were relatively well known as a matter of general knowledge among the educated classes, because they also had, as it were, an extra-mathematical existence. They were often known as the 'Platonic' or 'cosmic' solids, a name that comes from their being discussed in Plato's dialogue *Timæus*, where each is associated with one of the five 'elements' – the cube with earth, the icosahedron with water, the octahedron with air, the tetrahedron with fire and the dodecahedron (more or less by default) with ether, the material of the heavens. There is a brief reference to this association in the otherwise markedly non-mathematical commentary on *Timæus* by Calcidius, which was known throughout the Middle Ages.[46]

44 Piero della Francesca, Incipit, *Trattato d'abaco*: BML MS, p.3 recto; Piero ed. Arrighi, p.39.

45 The word 'regular' as used here is a modern technical term to describe a polygon whose angles are all equal and whose edges are all equal (for instance a square has four right angles and four equal sides), or a polyhedron whose faces are all of the same kind and meet in the same way at each corner of the solid (for instance a cube has six square faces, which meet three by three at each corner of the solid).

46 Calcidius, whose name is sometimes spelled Chalcidius, was a Christian philosopher who lived in the fourth century. The first printed edition of his commentary, entitled *Timæi Platonis traductio*, is dated 1520.

There is very little technical discussion of the geometrical properties of the regular polyhedra in *Timæus*, and still less in Calcidius' commentary. These works might serve to introduce natural philosophers to the solids, and may well have provided an important incentive for studying them, but for the relevant geometry it was necessary to turn to Euclid, specifically to Book 13 of the *Elements*. Piero, for whatever reason, had clearly been doing just that. The final part of his *Trattato d'abaco*, and the edited and extended version of it that appears in his *Libellus de quinque corporibus regularibus*, moves away from the normal world of abacus-school learning and enters into matters that were not usually of concern to university-trained mathematicians. That is to say that Piero's work on three-dimensional geometry is original. Its being original automatically distances it from the abacus tradition that has been discussed in the present chapter, and the nature of its originality – namely the concern with three dimensions – links it with Piero's control over spatial relationships in the scenes shown in his paintings. So the work on polyhedra will be considered in Chapter 4.

The learned mathematics of the universities

Medieval university texts on mathematics are not necessarily more advanced than those used in abacus schools in the sense of dealing with more complicated problems or ones that are notably more complex in conceptual terms. They do, however, teach a different kind of mathematics. This mathematics is part of the standard university education in the seven liberal arts: the *trivium* of grammar, rhetoric and dialectic and the *quadrivium* of geometry, arithmetic, astronomy and music. The titles of the subjects, and their echoes of Boethius (*c*.480–524), make things sound much grander than they were. To judge by surviving university texts, the level was mainly elementary – and it must, of course, be remembered that students were usually in their early teens. In today's terms, the *trivium* might more aptly be described as largely spelling, accidence and syntax (in Latin) plus a little logic.

In a university the student would learn geometry chiefly by 'reading' the first three or four books of the *Elements* and would study arithmetic in a similarly abstract way, using texts based on the late antique arithmetical treatise of Nicomachus of Gerasa (probably first century). The first three books of Euclid present some properties of triangles and of circles. The fourth book considers relationships between polygons and circles. Most significantly, if properly understood, these books of Euclid also teach the reader what is meant by a rigorous mathematical proof. The works on arithmetic read at university derive from simplified versions of the arithmetical books of the *Elements* (Books 7 to 9), and are much less intellectually stimulating than this origin might suggest. They are memorable – to the historian, at least – chiefly for the proliferation of names for particular types of ratio and for some special ratios, such as 'superparticulate' for the ratio whose first term is one greater than its second and 'sesquialterate' for the ratio 3:2. It is sometimes necessary to know such words in order to understand later texts, but the classifications they describe did not have great historical importance.

The two mathematical sciences of geometry and arithmetic found their applications in the two further mathematical sciences of astronomy (for geometry) and of music (for arithmetic). These applications were not practical, except in principle, since the astronomy and music concerned were both highly theoretical. The astronomy was an account of the motion of the principal spheres of the heavens, based on the first two books of Ptolemy's *Almagest* (second century) or textbooks such as the *Sphere* of Sacrobosco (thirteenth century) that were derived from it. The music was music theory, dealing with such matters as the correspondence between certain special arithmetical ratios and the ratios between the lengths of strings that produced notes separated by particular musical intervals.

These four mathematical sciences were understood as preliminary studies, leading to higher disciplines, in which it was possible to take a degree, such as medicine and theology. The usefulness of the mathematical sciences for theology was, of course, in understanding the structure of the Universe. For those studying medicine, they were the preparation for the application of astrology in the diagnosis and treatment of disease. Not absolutely all of this latter use is as irrational as it may at first seem to a reader today: the movements of the heavens were believed to be responsible for the weather – it being obvious that the changing position of the Sun caused the alteration in the seasons – and the weather was generally believed to affect the balance of the four 'humours' in the human body.

There were, however, also believed to be direct astrological effects, which do not stand up well to closer examination. Nonetheless, it may also be noted, in defence of pre-modern astrology, that by definition the subject dealt with all effects of heavenly bodies upon the Earth. So the strongest proofs of the validity of astrology were that the Sun caused the seasons and, generally more doubtfully, that the Moon caused the tides. All other heavenly bodies were believed to have much weaker effects than these. An essentially modern explanation of the effect of the Sun as being the result of the reception of its light was given by Johannes Kepler (1571–1630) in 1602; and an explanation of the effect of the Moon on the sea as being the result of gravitational attraction was given by Isaac Newton (1642–1727) in 1687.[47] Before these crucial undercuttings of its most substantial observational foundations, astrology had a reasonable claim to be taken seriously, in principle if not always in the details of practice.

University education was acknowledged as a preparation for a career in the Church, in the State or in one of the learned professions. What was taught there was not necessarily expected to be useful except as a contribution to a general understanding of how things were. It is thus not surprising that there is a wide cultural division between the Latin mathematical tradition of the universities and the practical mathematics taught in the vernacular to prospective traders and artisans in abacus schools. Socially, the traditions might be designated as 'high' and 'low' mathematics respectively, but this should not be taken to imply any other judgement as to their intellectual value or historical destiny. In fact, the 'high' mathematics was a debased version of a learned tradition going back to ancient times, while the 'low' mathematics drew on highly sophisticated Islamic work and, in furthering the development of algebra, was to have effects of huge cultural and intellectual import – not least by its participation in the later evolution of the 'high' mathematical tradition.

Also, more relevantly to the concerns of the present study, the existence of this vigorous artisan tradition of mathematics must have gone a long way to aid the adoption of more sophisticated mathematical techniques in the craft of painting in the fifteenth century, in particular the use of 'artificial perspective' to give a naturalistic effect of depth. It is clear that Piero della Francesca's general level of mathematical skill was high. Even within the traditional range of subject matter he was undoubtedly highly competent and apparently well read. In regard to perspective, it will again be apparent that Piero provides a good but rather extreme example of what was going on. The next chapter will therefore consider the early history of the use of perspective, sketching in some of the craft background against which Piero's later contribution must be assessed.

47 Johannes Kepler, *De fundamentis astrologiae certioribus*, Prague, 1602 (English translation in the second part of J. V. Field, 'A Lutheran Astrologer: Johannes Kepler', *Archive for History of Exact Sciences* 31, 1984, 189–272); Isaac Newton, *Philosophiae naturalis principia mathematica*, London, 1687.

2

Perspective

Filippo Brunelleschi (1377–1446) is credited by his contemporaries with having invented a method for making drawings that gave an optically correct rendering of what the eye sees. His biographer Antonio di Tuccio Manetti (1423–1497) calls this method a 'regola', which implies that it was mathematical.[1] 'Regola' was the standard term used in referring to types of equation or methods of solution of mathematical problems – a usage that survives in the name 'rule of three', for which similar terms are found in several European languages. There is no reason to doubt that Manetti's account of the matter is substantially reliable, but it unfortunately does not tell us exactly what the rule was, when it was invented, or how. Historians have, over the years, deployed considerable ingenuity and imagination in providing answers to these questions.[2] However, none of the resulting accounts has won complete acceptance, with the doubtful exception of the suggestion that the discovery was probably made before 1413.

It is not difficult to see why this has been so. The hard facts are few, as is indeed usual for this period, so all attempts to construct a narrative full enough to be convincingly coherent must rely rather heavily upon what historians of science warily call 'rational reconstruction'. This is recognized as a dangerous procedure in the history of science. Results, once obtained, particularly correct mathematical results, can look obvious, and one tends to be drawn to a historical reconstruction that follows the logic of a scientific proof of a

1 Antonio di Tuccio Manetti, *Life of Brunelleschi*, ed. H. Saalman, trans. C. Engass, University Park and London: University of Pennsylvania Press, 1970, pp. 42–3. As mentioned in Saalman's introduction, the secondary literature (and indeed the primary literature) on perspective is very uneven. The former is also unmanageably voluminous. I trust readers will forgive me for not engaging with much of it. In particular, I have omitted references to recent work that seems to me merely to recap what is to be found in older sources. There is one very old source that perhaps requires special mention: Erwin Panofsky's classic *Die Perspective als 'symbolische Form'* (1924), available in English as *Perspective as Symbolic Form*, trans. Christopher S. Wood, New York: Zone Books, 1991. The semiotic aspects of the work are still of interest to historians of art. The historical investigation of perspective is very much in the style of the positivistic history of science of the time: it assumes that the 'correct' perspective theory of Brunelleschi must have had precursors that were only approximately correct. The nature of the earlier perspective schemes that Panofsky discusses has been deduced by drawing numerous lines over pictures. My misgivings about such investigative procedures have been expressed in the Introduction above. In both style and content this part of Panofsky's work seems me to be too dated to be worth detailed consideration in the present context. There may be a history of the historiography of perspective out there among the Platonic Forms waiting to be written. I do not propose to write it.

2 See, for instance, J. White, *The Birth and Rebirth of Pictorial Space*, 3rd edn, London: Faber and Faber, 1987; S. Y. Edgerton, *The Renaissance Rediscovery of Linear Perspective*, New York: Basic Books, 1975; M. J. Kemp, 'Science, Non-Science and Nonsense: the Interpretation of Brunelleschi's Perspective', *Art History* 1, 1978, pp.134–61; M. J. Kemp, *The Science of Art: Optical Themes in Western Art from Brunelleschi to Seurat*, New Haven and London: Yale University Press, 1990.

kind acceptable to the historian. Experience suggests that the historian's being ill-informed about the science of his own time does not greatly impede this process of distortion. In fact, when we do know a reasonable amount about how some scientific discovery was made, we usually find a pattern that bears little resemblance to simple logical progression from truth to consequent truth.

A classic example of this is Johannes Kepler's hard-won discovery that the orbit of the planet Mars was an ellipse with the Sun in one of its foci – a result arrived at in 1605 but not published until 1609, in a work aptly titled *New Astronomy*.[3] The orbit he obtained was not what anyone expected, which is probably why Kepler decided to write up his work as 'Commentaries on the Motions of Mars' – the part of the title by which he himself always refers to the book – describing the progress of his calculations in some detail, rather than tidying everything up into a more standard form based on the logic of exposition. We know this partly because he tells us so and partly because there are about a thousand folio pages of astronomical arithmetic that survive to confirm what he says.[4] Battles still rage, however, over the details of his calculations, and the clearest messages from the computer-assisted rational reconstructions are first that 'garbage in garbage out' is a universal rule and second that Kepler is immensely better at mathematics than any of his historians.

Luckily, Brunelleschi's discovery, though probably not expressible in such simple terms as Kepler's, cannot possibly have involved a similar weight of calculation. On the other hand, what one might be tempted to describe as 'simple geometrical optics' can actually become quite complicated. Here we shall attempt to confine ourselves to essentials.

Brunelleschi used his 'rule' to construct two demonstration panels. What we are told about them by Manetti is reassuring. The first panel showed the Baptistery of Florence as it would appear when seen from just inside the central doorway of the cathedral, and the panel was mounted so that the eye of the person looking at the picture was in a fixed position. Any correct perspective construction requires, in principle, that the resulting image be viewed in this way. However, images in correct perspective – such as photographs – continue to look convincingly three-dimensional even when seen from a position considerably removed from the ideal (mathematically correct) one. Brunelleschi's second panel, showing the Palazzo dei Signori (also known as the Palazzo della Signoria, now called the Palazzo Vecchio), seen at an angle across the open space in front of it, differed from the first in not being arranged as a peep-show. The fact that Brunelleschi set up his first panel with a fixed viewing point confirms that he had used a construction that took a fixed position for the observer's eye; and the fact that the second panel was not set up in the same way confirms that he had noticed the robustness of the illusionistic effect.

This robustness presumably came as a surprise. Since he was a good enough mathematician to invent the rule in the first place, it seems likely Brunelleschi was also good enough to realize that he had a mathematical monster on his hands. In Euclid's world there are no construction procedures that work if you violate the initial conditions that determine them.

3 Johannes Kepler, *Astronomia nova seu physica coelestis αιτιολογητος tradita commentariis de motibus planetæ Martis...*, Heidelberg, 1609, reprinted in KGW, vol.3, 1940. (English translation, *Johannes Kepler: New Astronomy*, trans. William H. Donahue, Cambridge: Cambridge University Press, 1992; this version plays down the controversies by having almost no notes. It lacks an index.)

4 Kepler, a deeply religious Protestant, apparently believed telling lies was a sin. The manuscript calculations for the *Astronomia nova* are now preserved among the Pulkova Observatory papers at the St Petersburg Academy of Sciences, Russia. Kepler's calculation, which is interesting for its use of novel methods, is discussed in Chapter 8.

In the real world, however, and very notably in engineering activities, with which Brunelleschi was much occupied, practical success was what counted. Buildings were constructed slowly, and all parts of them remained accessible, by stairs and walkways, so that the builders could check for the appearance of cracks or deformations, and take remedial action if necessary. Forward planning did not include much that could be called structural analysis.[5] As an engineer Brunelleschi could afford to wave away questions about the unexpected robustness of the illusions produced by his rule for perspective.

After Manetti's account, our best evidence for what Brunelleschi discovered must surely lie in works of art produced before Leon Battista Alberti (1404–1472) gave the first written account of a perspective construction in his treatise on painting. In fact, there are not many surviving works dating from the period from about 1413 to 1435 that show anything to help us. This apparent rarity of perspective is itself a piece of evidence, but to interpret it, and the few examples, we require a technical vocabulary that is lacking until it is supplied by Alberti. We shall accordingly examine what Alberti said before we examine what Donatello (1386–1466) and Masaccio (1401–1428) did, or seem to have done.

Alberti *On Painting*

Alberti, who had received a university education, wrote his treatise on painting in Latin. He was certainly addressing himself not to prospective practitioners of the techniques he describes, but rather to the potential patrons of such practitioners. Moreover, the style makes it clear that the work was intended as a further contribution to the humanist programme of reviving ancient learning. The original readers of *De pictura* may well have directed most attention to the very parts that a present-day reader is most inclined to disregard, such as the many references to Pliny's account of Greek painting in *Natural History*, Book 35.[6]

It is, in any case, clear that Alberti expects his readers to be sufficiently knowledgeable about ancient literature to understand, for instance, without further explanation, for instance, that different physical characteristics are to be associated with the twins Castor and Pollux, that Polyphemus must be shown as large in size, and that Io should be portrayed as a cow.[7] The reader is also expected to know what it is that is going on in a picture entitled *The Sacrifice of Iphigenia*.[8] Such references to the ancient world are extremely numerous and one usually needs to understand them in order to follow Alberti's argument.

There are no comparable references to the ancient literature on optics or mathematics. When he mentions Marcellus' capture of Syracuse, Alberti even fails to mention that the great mathematician Archimedes (*c.*287–212 B.C.) was associated with the defence of the

5 The detailed discussion in H. Saalman, *Filippo Brunelleschi and the Cupola of Santa Maria del Fiore*, London: Zwemmer, 1980, suggests the trusting of hunches rather than much careful mathematical modelling. On the practical mathematics of building the dome see T. B. Settle, 'Brunelleschi's Horizontal Arches and Related Devices', *Annali dell'Istituto e Museo di Storia della Scienza* 3.1, 1978, pp.65–80.

6 In accordance with the normal custom of the time, Alberti does not actually cite particular passages, but what he says is sufficiently often at variance with Pliny (in matters of detail only, be it said) to suggest that he may be quoting from memory. This serves to remind one that humanist authors were often thoroughly familiar with texts that most modern readers tend merely to look up occasionally in search of some particular passage.

7 Castor and Pollux are mentioned in §38, Polyphemus in §18 and Io in §44.

8 There are comments on Timanthes' painting of the sacrifice of Iphigenia in §42, but there is no account of the story it illustrates.

city.[9] Moreover, Alberti makes very heavy weather of explaining the bits of mathematics that he uses. There is no doubt that *De pictura* is written for a readership much better educated in literature than in mathematics or in natural philosophy.

All the same, Alberti plunges straight into natural philosophy at the very beginning of Book 1 of *De pictura*. As becomes clear in his second and third books, he believes that in order to paint well an artist must have a good grasp of the theory of vision. Alberti apparently ascribes this opinion to his ancient sources, in which he may be correct, though explicit statements seem to be lacking.

On the other hand, it is easy to locate authoritative sources for the theory of vision. The standard text on the subject in Alberti's day, and in Piero's, was that by John Pecham, probably written between 1277 and 1279, and now usually called his *Perspectiva communis*.[10] In fact, what Alberti says about the 'pyramid of vision', a sheaf of rays joining the eye to all points on or within the outline of an object, is so standard that one cannot ascribe it to any particular source. Within this sheaf of rays is the 'centric ray'. It is not necessarily in the geometrical centre of the sheaf (which can, of course, be of irregular shape), but it is defined as the ray that meets the outline – that is, apparently, a plane figure defined by the outline – at right angles.

Alberti's mathematical description is far from precise, but readers are offered several other descriptions to help them visualize what is going on. For instance, there are alternative figurative terms for what I have called the outline. Alberti is writing, and knows he is writing, for the kind of reader who will not fret at the inadequacy of the mathematical explanation, and is much more likely to be bored by what is perceived as too much mathematics. The work is, after all, presented as something to read, not something to work through.

As far as the end of the description of the 'pyramid of vision', the gaps in Alberti's mathematics can be filled in with some confidence. We are dealing with entirely ordinary fifteenth-century notions about the nature of light and vision. Indeed, Alberti even goes so far as to tell us that for his present purposes there is no need to decide whether sight is by the emission of visual rays (what is now called the 'extramission' theory) or by the reception of light rays (now called 'intromission'), since the geometry of the pyramid of vision will be the same in either case.[11] This is an intelligent but commonplace remark. Its importance in this context is that it shows Alberti is aware that he is dealing with *perspectiva* – that is, the complete science of sight – and not with some autonomous or quasi-autonomous subdiscipline. Moreover, although what he has to say is, in principle, merely mathematical – being concerned with what would now be called 'geometrical optics' – Alberti does sometimes refer to 'visual rays', and the physical theory underlying his mathematics would thus appear to be that of extramission. This was the theory most favoured by natural philosophers of the time, though one of the most respected of all writers on the subject, Ibn al-Haytham (*c.*965–1040, known in the West by the Latinized names Alhacen and, later,

9 Alberti, *De pictura*, §26. Archimedes had been known by name throughout the Middle Ages, and various semi-miraculous feats were ascribed to him. He became a very important symbolic figure to Renaissance mathematicians – partly because the wider dissemination of his works showed he had truly been one of the greatest mathematicians of all time. On his engineering activities, and his part in the defence of Syracuse, see D. L. Simms, 'Archimedes the Engineer', *History of Technology* 17, 1996, pp.45–111 (for Syracuse, see esp. pp.60–71).

10 See D. C. Lindberg, *John Pecham and the Science of Optics*, Madison: University of Wisconsin Press, 1970.

11 See Alberti, *De pictura*, §5.

Alhazen), had preferred intromission, and various intermediate theories were also on offer.[12] The idea that the chief contribution to vision is through the 'centric ray', which was widely accepted, was in fact derived from the work of Ibn al-Haytham. The extremely intelligent suggestion has recently been made that knowledge of Ibn al-Haytham's intromission theory may have aided the development of 'artificial perspective', since an intromission theory made geometrical relationships with the outside world, together with colours, the only elements that the eye could apprehend.[13]

What Alberti says in his description of the 'pyramid of vision' is effectively a somewhat sketchy summary of what a student of medicine, say, might have expected to learn about the subject at university. Alberti may thus have expected some of his readers to be familiar with this material. Matters stand entirely differently, however, when we come to the use Alberti makes of this theory, namely his claim that it forms the basis for the construction of an exact 'intersection' that allows the painter to reproduce the pyramid of vision and thus obtain a perfect representation, initially a perfect representation of a square-tiled pavement with one edge lying along the line in which the picture meets the ground (the 'ground line').

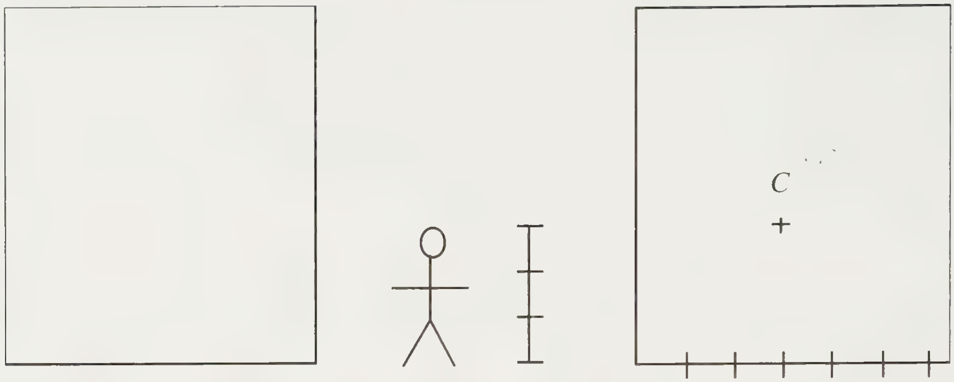
Alberti's picture seems to start at ground level, which is not realistic but does not affect the construction. The style of Alberti's description of the method of constructing the 'intersection' suggests strongly that he expects it to be new to his readers, so it is rather disconcerting to find that the account is in several respects incomplete. Since *De pictura* as a whole is certainly addressed to readers whose chief interests are not in mathematics for its own sake, a determination to be suitably brief may well be the correct explanation for the intellectual gaps.

The first gap is that there is no explicit link made between the fairly detailed description of the pyramid of vision and the first stage of the mathematical construction used for the intersection. This first stage consists of choosing a point that is 'opposite the eye' – that is, the point where the centric ray will meet the picture (the 'centric point', called C in our figures) – and then joining this point to the points along the base of the picture that divide the width into units that will each represent a *braccio* in the picture, the size of the scaled *braccio* having been decided by taking the height of a standing human figure as 3 *braccia* (Figs 2.1 and 2.2).

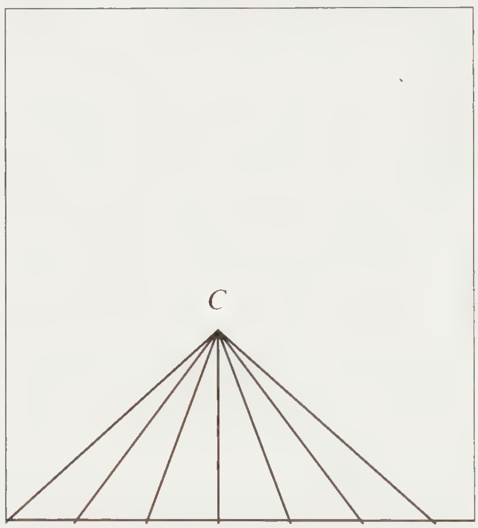
Contrary to the impression that may be given by the diagrams supplied in most editions of his work, Alberti does not specify the shape of the picture, or that the 'centric point' should be in its centre, or that the width of the base of the picture should be an exact number of the scaled *braccia*. He merely prescribes that the centric point be at a height less than or equal to that of a standing man, that is 3 scaled *braccia*. It is only later that the reader learns that the height of C has been made the eye height of a standing figure. This inconsistency is noticeable if one is attempting to draw diagrams in accordance with instructions given. It would no doubt pass unnoticed by a 'normal' reader. Alberti probably merely wished to avoid overburdening his reader with detail.

12 See D. C. Lindberg, *Theories of Vision from al-Kindi to Kepler*, Chicago: University of Chicago Press, 1976. The decision in favour of intromission was finally made in the early seventeenth century, on the basis of considerations of the functioning of the eye; see Lindberg, *Theories of Vision*, and J. V. Field, 'Two Mathematical Inventions in Kepler's *Ad Vitellionem paralipomena*', *Studies in History and Philosophy of Science* 17(4), 1986, pp.449–68.

13 See Gérard Simon, *Le regard, l'être et l'apparence dans l'optique de l'antiquité*, Paris: Éditions du Seuil, 1988; Gérard Simon, 'Optique et perspective: Ptolomée, Alhazen, Alberti', *Revue d'Histoire des Sciences* 53–4, 2001, pp.325–50; and Gérard Simon, *Archéologie de la vision*, Paris: Éditions du Seuil, 2003, esp. pp.167–81.

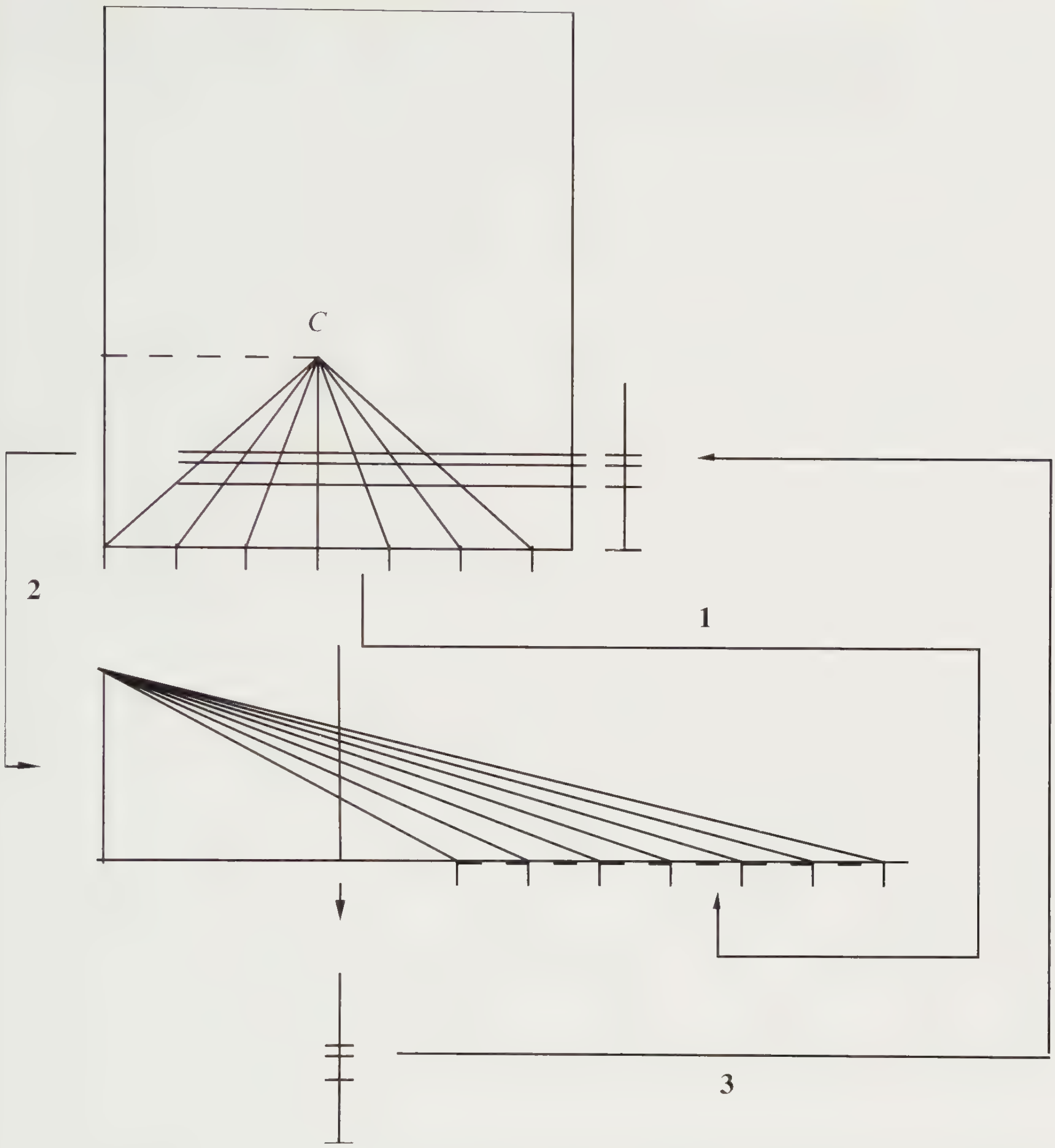


2.1 Figure for Leon Battista Alberti, *On Painting*, §19. Preliminaries for the perspective construction. Alberti first chooses the shape and size of the picture. Then he decides what size a human figure is to be and divides this size into three units (each to correspond to a *braccio*). These units are then marked off along the base of the picture. Then he chooses the position of the 'centric point' (C, the point opposite the eye of the ideal observer), making its height above the base three of these *braccio* units, that is a little greater than that of the eye of a standing man in the picture. Drawing by JVF.

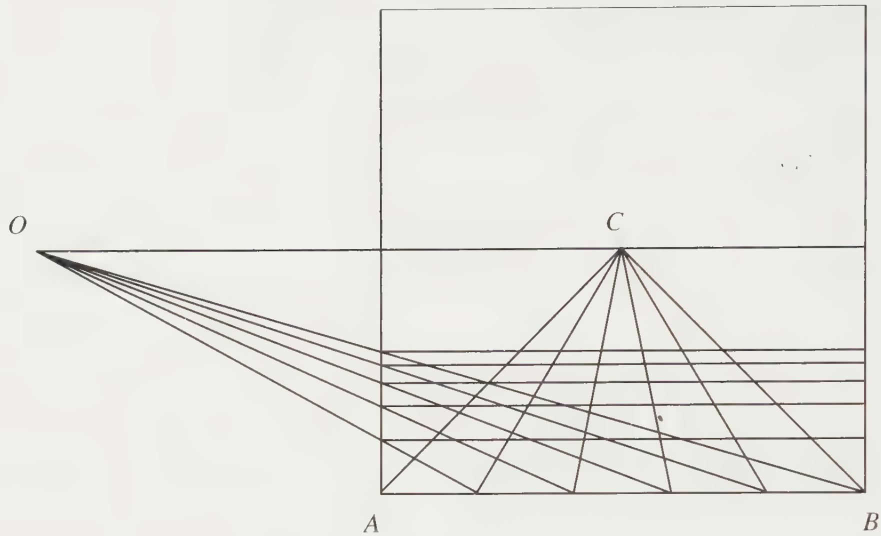


2.2 Figure for Leon Battista Alberti, *On Painting*, §19. Perspective construction. Joining divisions to the centric point. Drawing by JVF.

Many further necessary details have been omitted. For instance, although he gives dimensions for the pavement that is to be drawn in perspective, Alberti does not tell us the distance of the observer from the picture. Of course, it does not matter what distance one chooses, but a choice must be made, and others, equally free, have been made for us, with lengths given in *braccia* in the normal manner of the worked numerical examples of the abacus texts. Using mathematical initiative where necessary, the remainder of the perspective procedure described by Alberti can be summarized as shown in Figure 2.3a.



2.3a Figure for Leon Battista Alberti, *On Painting*, §20. Perspective construction. Alberti begins with the points of division along the base joined to the centric point, C (Fig. 2.2). The divisions from the base are transferred to a horizontal straight line (operation 1). A point is constructed above one end of this line at the same height as C (operation 2). Lines are drawn as shown connecting this point to the division points on the line. A vertical line is constructed at the distance of the intersection. The pattern of points of intersection produced is transferred to the edge of the picture; for clarity, our diagram shows only three divisions (operation 3). Horizontals are drawn through these points as shown. Drawing by JVF.



2.3b A simplifed version of the diagram, showing lines superimposed. This is the form in which the construction usually appears in perspective treatises of the sixteenth century and later. Drawings by JVF.

A great deal of scholarly effort has been expended on attempting to decide exactly what operations Alberti means to carry out. The absence of consensus is a measure of the incompleteness of what Alberti actually says.[14] However, the main outlines of the procedure are clear. Alberti begins with the points of division along the base joined to the centric point, *C* (see Fig. 2.2). The divisions from the base are then transferred to a horizontal straight line (this operation has been marked 1 in Fig. 2.3a). Next, a point is constructed above one end of this line at the same height as *C* (operation 2). We are not told how long to make the horizontal line, so we are not given the distance of the observer from the pavement. Lines are next drawn to connect this point with the division points on the line. A vertical line is then constructed at the distance of the intersection (not specified). The pattern of points of intersection that is produced on this vertical line is then transferred to the edge of the picture (operation 3). Horizontals are drawn through these points to cut the lines joining the divisions of the base to *C*.

In the simple form in which the construction has been interpreted in Figure 2.3a, it is clear that we are not worrying about how much space we are using for the drawing. Something more than the size of the completed picture field would be required if everything were done full scale. And if it were not, there would be the risk of errors becoming visible when the construction was scaled up again. The repeated transfer of lengths seems impractical for the same reason: it too is liable to introduce errors. All told, it is difficult to imagine that a procedure exactly resembling what is shown in Figure 2.3a, or indeed the simplified

14 A survey, and some new suggestions, are given in Pietro Roccasecca, 'Il "modo optimo" di Leon Battista Alberti', *Studi di Storia dell'Arte* 4, 1993 [1995], pp.245–62.

version of it usually presented by historians (Fig. 2.3b), was actually carried out in the construction of pictures – at least before it was securely established that in practice even large errors in the construction did not make the picture look incorrect. Tolerance of errors in construction is, of course, not quite the same thing as the tolerance of the eye in looking at pictures from the 'wrong' position. However, neither form of tolerance is compatible with a Euclidean attitude to mathematical truth, and neither is mentioned by Alberti. Indeed, part of his purpose in *De pictura* seems to be to claim scientific (that is mathematical) respectability for painting – a matter to which we shall return below.

Alberti has, however, shown us the principles of the construction he describes. Moreover, the construction itself is mathematically correct, though Alberti merely asserts this without attempting to prove it. He tells us that the lines he drew to the centric point (shown in Fig. 2.2) determine the change in size of magnitudes parallel to the ground line of the picture. He does not explain why this is so. Indeed, he does not appear to regard it as worthy of remark that the lines that in reality were perpendicular to the picture ('orthogonals') appear in the perspective picture as a set of lines converging to the centric point, that is the foot of the perpendicular from the eye to the picture.

As far as is known, this result had not been discussed in writing before Alberti used it here, so it seems unlikely that he could rely upon his readers understanding for themselves without any further help. Probably he expected them merely to accept what he said, and be glad to avoid the technicalities of a full mathematical explanation of it. The historian may, however, be permitted to wonder how Alberti came by his construction. Perhaps Alberti's silence should also be taken as confirmation that the discovery belonged to someone else, since he does specifically lay claim to having invented the next part of the construction, the method of putting in the lines parallel to the base (the 'transversals') that is shown in Figure 2.3a.

Although Alberti does not say so, this procedure has assumed that the grid to be drawn is square overall and is divided into squares. It is possible that Alberti supposed his readers would recognize the relationship between this part of his construction and a diagram showing sightings taken by surveyors. However, since Alberti does not actually mention surveying, it is not absolutely certain that he himself recognized this similarity – though later in his life, as one who was concerned with building, he certainly knew something about surveying, and he does indeed show his familiarity with its techniques in his *Ludi matematici* and *De re aedificatoria*, both written after *De pictura*.

As was mentioned in Chapter 1, taxes were levied according to the area under cultivation, so landowners of this time had good reason to learn at least a little about surveying. Brunelleschi, to whom Alberti dedicated the vernacular version *Della pittura* in 1436, had professional reasons for understanding surveying. As a landowner, Gianfrancesco Gonzaga (1395–1444), the noble dedicatee of the Latin *De pictura*, might also have been expected to recognize the elements of contemporary surveyors' practice that seem to have been silently incorporated into this part of Alberti's account of his construction of the 'intersection'.[15]

15 On the connections of Alberti's construction with surveying see W. M. Ivins, *On the Rationalisation of Sight, with an Examination of Three Renaissance Texts on Perspective*, New York: Da Capo Press, 1973, first published in *Papers of the Metropolitan Museum of Art*, no.8, 1938. On more general connections of perspective studies with surveying see Kim Veltman, 'Military Surveying and Topography: the Practical Dimension of Renaissance Linear Perspective', *Revista da Universidade de Coimbra* 27, 1979, pp.329–68.

The additional diagram constructed starting from the horizontal line may be imagined as a side view of the three-dimensional set-up, with the lines coming out from the point on the left being sight lines to the transverse divisions between the tiles of the pavement.

As we shall see, Piero della Francesca's treatise on perspective, probably written about thirty years later, proves the correctness of his own, slightly different, method of construction. The first proof of the correctness of Alberti's construction was given by Giovanni Battista Benedetti (1530–1590), a professional mathematician, in a work published in 1585.[16] In 1625, a misreading of Benedetti, taking him to have proved that it was only Alberti's construction that was correct, led to the construction being given the title 'legitimate construction' (*costruzione legittima*), by which it is sometimes still known.[17] Artists had apparently felt no urgency about providing a proof of the result.

Alberti's first book ends with the construction of the horizontal pavement. The second book does not pick up the mathematical work where the first one left it. Instead, it begins with a non-mathematical discussion that asserts the importance of studying painting and thus both justifies Alberti's having made the reader work so hard and at the same time gives him a respite from mathematical science. Alberti then returns to his discussion of the 'intersection' and provides some indications as to how the horizontal grid constructed in the first book may be used to construct images of three-dimensional structures. As with the instructions for drawing the grid, these accounts tend to be somewhat elliptical. However, for our present purpose the important thing about them is that they confirm the impression given by the choice of the square-tiled pavement as the first object for drawing, namely that we are dealing with a system in which bodies are to be located in three dimensions by reference to something very like a coordinate grid. Coordinate systems themselves were no novelty. In Alberti's day, astronomers and geographers used systems of latitude and longitude to describe positions in the heavens and on the surface of the Earth.[18]

What Alberti describes does not correspond very closely to what we find by examining the use of perspective in the works of Donatello and Masaccio. The divergence no doubt bears testimony to the habitual gulf between theory and practice as well as to the social division between Alberti's primary readership and artisan practitioners. However, the nature of the differences is also such as to suggest that the method employed by artists – with due disregard of exactness – was not precisely the same as Alberti's. Since Donatello's attitude to perspective might most politely be described as anarchic, we shall first consider what is to be found in the work of Masaccio.

Masaccio

The works by Masaccio that show clear signs of the use of some kind of perspective construction are the *Pisa Madonna* (1426) (National Gallery, London), his contributions to the frescos in the Brancacci Chapel in Santa Maria del Carmine, Florence, in 1425, and his *Trinity*

16 Giovanni Battista Benedetti, *De rationibus operationum perspectivae* in *Diversarum speculationum . . . liber*, Turin, 1585. See J. V. Field, 'Giovanni Battista Benedetti on the Mathematics of Linear Perspective', *Journal of the Warburg and Courtauld Institutes* 48, 1985, pp.71–99.

17 The name appears in Pietro Accolti, *Lo Inganno de gl'occhi*, Florence, 1625.

18 The use of celestial and terrestrial coordinate systems in fact dates back at least to Hellenistic times, to the work of Claudius Ptolemy. Geographical longitudes were very difficult to determine, but that does not affect the principle.

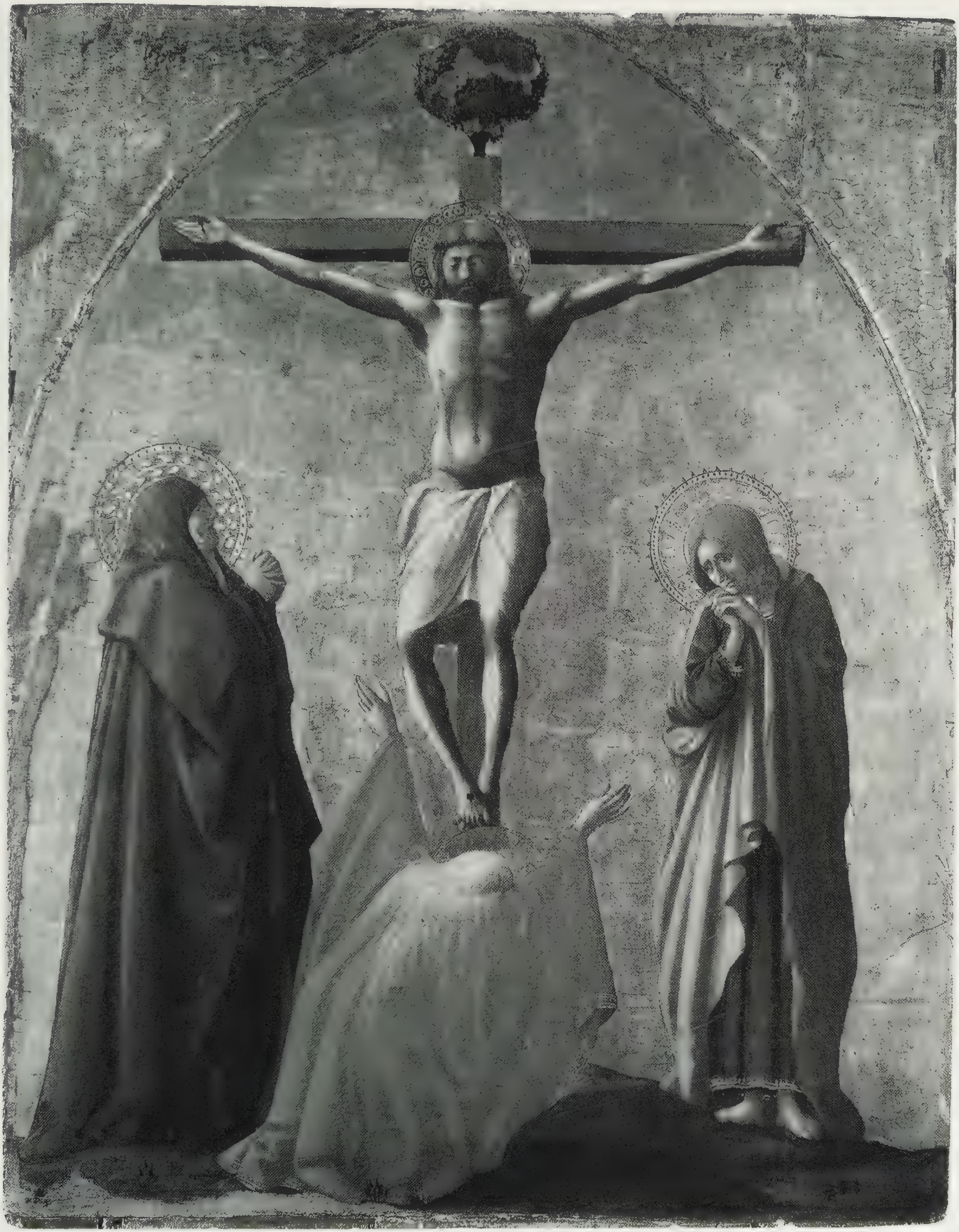
2.4 Masaccio (1401–1428), *Madonna and Child with Angels* (*The Pisa Madonna*), 1426, tempera on panel, 135.3 × 73 cm, National Gallery, London. Photograph courtesy of the Trustees of the National Gallery.

fresco (*c.*1426) in Santa Maria Novella, Florence. The predella panels from the *Pisa Altarpiece* (1426) (Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem) also show strong definition of pictorial space, but there is little one can use as a basis for mathematical analysis. This is almost as true of the panel showing the Madonna and Child (Fig. 2.4). The only elements that one can seize upon as possibly belonging to a perspective scheme are the receding edges of the classicizing throne and corresponding edges in its panelling, which are no doubt meant to be perpendicular to the plane of the picture. Unfortunately for the would-be investigator, these receding lines are short. They have to be extended for something considerably more than their own length before we can check whether they will meet.

Like all other mathematical procedures involving extrapolation, the process of extending lines is in general to be avoided, and its results treated with caution. The shorter the lines in question, the greater the caution. However, the images of orthogonal lines in the throne of the *Pisa Madonna* do, once extended, meet at a point. As there are at least six such lines, their meeting is fairly convincing, as is the fact that the point of concurrence is on the vertical centre line of the picture (in the foot of the Child), and at a height that seems reasonable for the eye of a standing viewer of the panel when it was in position as part of an altarpiece. The naturalism for the eye height seems significant because the panel showing the Crucifixion that was placed high up on the top of the altarpiece is clearly designed to be seen from below (Fig. 2.5). However, there is no way of making an eye height totally naturalistic since worshippers may be expected both to stand and to kneel in front of an altarpiece. All the same, the drawing of the throne of the *Pisa Madonna* does seem to have been done in accordance with a mathematical rule that took account of the properties of what Alberti called the 'centric point'. Moreover, the three-dimensional effect is strongly enhanced by the use of directional lighting, including cast shadows, and the bold modelling of the drapery. The gold ground, which was restored in the nineteenth century, is probably more of a distraction for the modern viewer than it would have been for Masaccio's contemporaries. It was, after all, traditional and universal at the time, and might well have been specified in the contract for the commission.

The receding lines in the *Pisa Madonna* provide a set of orthogonals that can be used as evidence for a perspective construction, but the picture does not supply any measurable transversals. The same is true of the fresco *The Tribute Money* in the Brancacci Chapel, Santa Maria del Carmine, Florence (Fig. 2.6). Here, however, the scale of the work is much larger – the human figures are about life-size – so extending the orthogonals is a more trustworthy procedure, provided it is done on the actual picture and not on a tiny photograph of it. In the 1980s, a team of restorers working on the Brancacci Chapel duly took advantage of the opportunity this afforded them for using strings to extend the lines that Masaccio had apparently 'snapped' and incised onto the damp plaster in making his drawings of the building on the right in his composition. The results they obtained were a little disconcerting. Whereas drawing lines on a photograph, at much reduced scale compared with the actual picture, apparently gives a set of lines that meet near Christ's eyes, the more accurate procedure carried out on the wall showed two points of concurrence: one on the bridge of Christ's nose, and one in the centre of His forehead.[19] As the figure is life-size, the dis-

19 O. Casazza, 'Il ciclo delle storie di San Pietro e la "Historia Salutis". Nuova lettura della Cappella Brancacci', *Critica d'Arte* year 51, 4th series, no.9, April–June 1986, pp.69–84.



2.5 Masaccio (1401–1428), *The Crucifixion* from the *Pisa Altarpiece*, *c.*1426, tempera on panel, 83 × 63 cm, Museo di Capodimonte, Naples.

2.6 Masaccio (1401–1428), *The Tribute Money*, from the cycle of scenes from the life of St Peter, 1425, fresco, 247 × 597 cm, Brancacci Chapel, Santa Maria del Carmine, Florence. The figures are about life size.

tance between these two points is not negligable. It is not large in comparison with the overall dimensions of the picture, but it is too large to be explained by some accident such as a clumsy knot altering the direction of the string attached to a nail. It seems as though in drawing his orthogonals Masaccio must, unaccountably, have used two nails and string attached to each. The points at which the nails were hammered in were in the remaining area of rough plaster, since Masaccio seems to have begun his painting at the edges and worked inwards to the centre.[20]

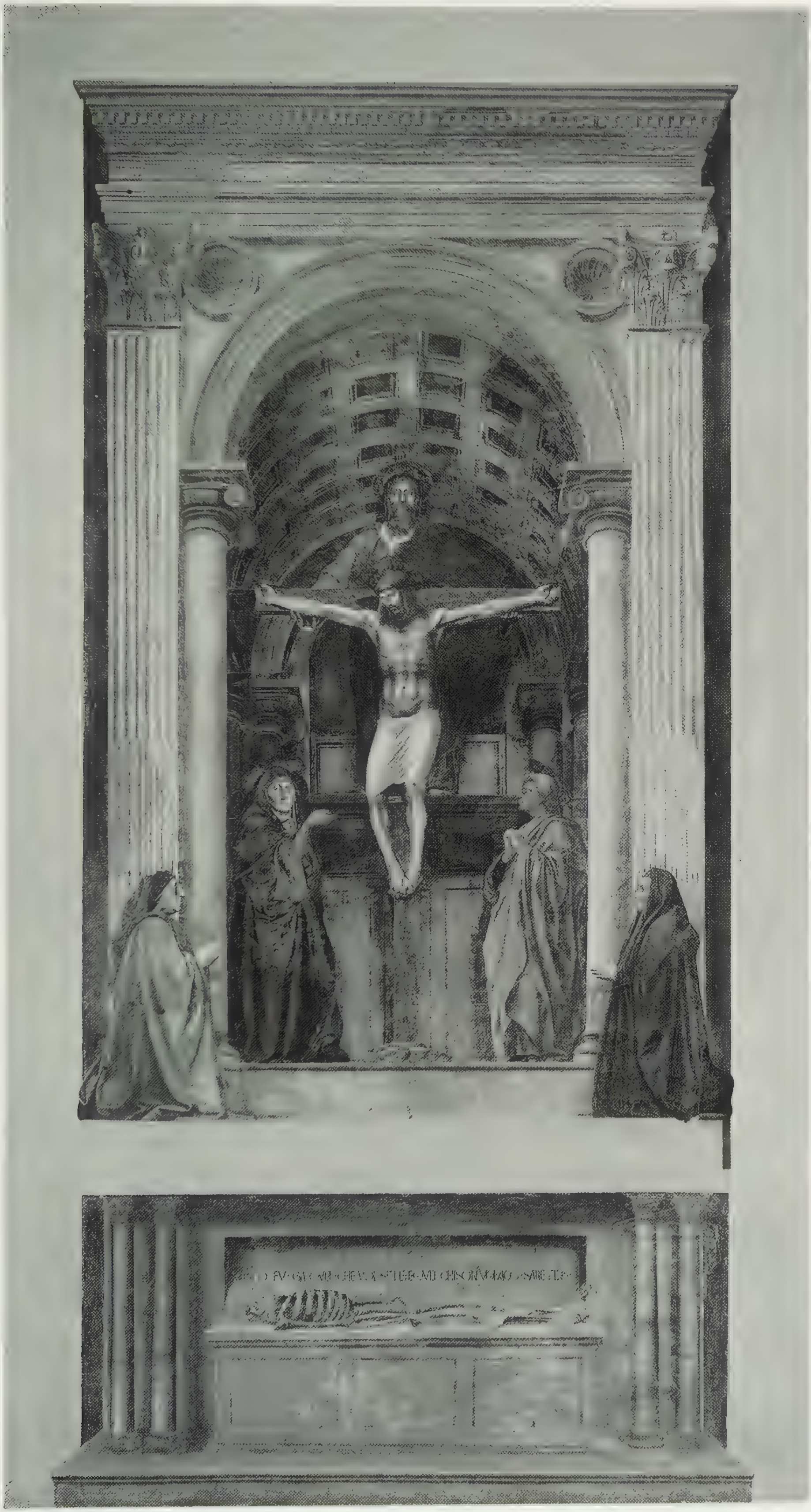
In any picture, it is architectural elements that are most likely to provide sets of orthogonals and transversals. Masaccio's *Trinity* fresco (Fig. 2.7) thus provides a good example in which to search for indications of the method by which the perspective scheme was constructed. The powerful illusion of depth in this picture was commented on by Giorgio Vasari in his *Life* of Masaccio:

> But the most beautiful thing in it, apart from the figures, is a semicircular barrel vault drawn in perspective, and divided into squares containing rosettes that diminish in size and are foreshortened so well that it seems as though the wall is pierced.[21]

This illusion strongly suggests that some mathematical construction has been used. From scaffolding, using string, it is easy to confirm that the images of the orthogonals defined by the edges of the longitudinal ribs of the barrel vault, those defined by the receding edges of

20 This can be deduced from the overlapping of the areas of plaster applied successively day by day as the work progressed (so-called *giornate*). See Casazza, 'Il ciclo delle storie di San Pietro' (full ref. note 19).

21 Giorgio Vasari, *Le Vite dei piu eccellenti architetti, pittori e scultori italiani da Cimabue a tempi nostri*, ed. P. Barocchi and R. Bettarini, Florence: Sansoni, 1971, vol.3, p.127.



2.7 Masaccio (1401–1428), *Trinity*, *c.*1426, fresco, 667 × 317 cm, Santa Maria Novella, Florence. The figures are about life size.

the abaci on the column capitals at the front and back of the painted chapel, and the orthogonals defined by the lines of the mouldings at the base of the vault, do indeed converge. They meet at a point on the central vertical axis of the picture, in the step on which the donors are kneeling. That is, the centric point is in an area that has been restored, Masaccio's original *intonaco* having been lost, either when the picture was covered by a wooden altar in the sixteenth century or when it was removed from the wall in the nineteenth. The convergence does not seem to be quite perfect, but the 'errors' are all small enough to be explained by Masaccio's having used fairly coarse string – as can be seen from the 'snapped' lines – and a correspondingly stout nail to which the string was tied.[22] We thus have a convincing set of convergent images of orthogonals. The height of the centric point, and thus of the eye of the ideal observer, is about right for someone standing in front of the painting, assuming that no significant change has been made to the floor level.

We cannot see the floor of the imaginary chapel. Transversals are provided only by the edges of the abaci that lie parallel to the picture plane. We could, in principle, construct some horizontal transversal lines of our own by joining the points at which each of the transverse ribs of the barrel vault meets the mouldings at its base. However, the artificiality of this way of checking up on Masaccio is exposed when we look closely at the relevant areas of his picture. The compass sweeps that define the edges of the ribs do not reach as far as the mouldings. The points we thought of constructing were clearly of no account to Masaccio.

The *intonaco* shows quantities of traces of the procedure whereby Masaccio laid out his lines as a preparation for painting, so the absence of emphasis upon these points, indeed the absence of the points themselves, is a clear indication that the positions of the ribs were not found by making a pseudo-pavement on the level of the base of the vault.[23] Moreover, measurements carried out on the *intonaco*, followed by a certain amount of arithmetic, have shown that the shapes given to the abaci were not correct, that is to say that the shapes Masaccio drew were not correct perspective renderings of squares. Since the Albertian construction privileges squares, the fact that the only squares in the fresco are incorrect must surely be seen as further evidence that Masaccio did not construct the perspective of his picture by a method that used the Albertian construction of transversals.[24]

It seems that adjustments were made to the shapes of the images of the abaci in order to bring their edges into a more satisfactory relationship with the pattern of ribs against which they are seen. There are also adjustments to the ribs themselves. In particular, there can be little doubt that we are meant to read the coffers of the barrel vault as square, that is they would be square if the semi-cylindrical surface of the vault were unrolled to make it flat. However, the lowest row of coffers on each side has been drawn to be considerably larger than the higher ones. The 'error' is not readily noticeable as such, partly, no doubt, because this part of the scene is not brightly lit, but measurement shows that the lowest coffers are half as wide again as the ones above them, which are all rigorously equal. By raising the lowest ribs, Masaccio has made them run between the volutes of the capitals on the front

22 A more detailed account is given in J. V. Field, R. Lunardi and T. B. Settle, 'The Perspective Scheme of Masaccio's *Trinity* Fresco', *Nuncius* 4.2, 1988, pp.31–118.

23 Such a pseudo-pavement has been suggested by a number of scholars. For references, and alternative suggestions, see Field et al., 'The Perspective Scheme of Masaccio's *Trinity*' (full ref. note 22).

24 See Field et al., 'The Perspective Scheme of Masaccio's *Trinity*' (full ref. note 22), where it is suggested that the positions of the transverse ribs may have been found simply by drawing appropriate lines of sight.

columns and the hands of Christ on the cross. Thus, together with the rib that runs into the figure of God the Father, these two lowest ribs have an important compositional function in tying the architectural setting to the figures. The setting is there to give weight to the figures. Another adjustment has also been made, this time apparently to avoid the setting becoming too distracting by being difficult to read. The first visible transverse rib should, strictly, have been completely visible, but it would then have been seen in possibly disturbing proximity to the wide pink moulding round the entrance arch. It has accordingly been pushed up on the picture, so that it is partly hidden by the moulding, and the next transverse rib has been moved up a little to avoid making the first coffer seem too deep. The remaining ribs, lower in the picture, are where theory dictates. The positions of all transverse ribs are marked with compass sweeps, all made round different centres. Edges and centres of ribs are marked on the left part of the vault, which was painted first, edges only are marked on the right.

The adjustments to the positions of the ribs would clearly have required calculations for radii and centres, so one can hardly imagine them to have been a last-minute affair. In any case, viewing the work from scaffolding does not enable one to form any sense of the composition as a whole or of the depth conveyed when the picture is seen in the normal way. What it does do, for a present-day viewer, is put one in a position of unnerving closeness to Masaccio's magnificent calligraphic brushwork. This is the ultimate reminder, if one needs it, that we are dealing with a complete work of art, not a mere perspective construction.[25] From the ground, the impressive nature of Masaccio's achievement allows no doubts. There is, as Vasari says, a powerful effect of the creation of a pictorial space that extends behind the picture surface. The figures, which Vasari also praises, are so convincingly modelled that they too contribute to the illusion.[26] The viewer has no sense of seeing anything that is incorrect in optical terms. That is, the adjustments are effectively invisible. Nevertheless, the work seems to have found few imitators. Barrel vaults are not common in perspective pictures, and it seems that most artists preferred to use architectural elements whose structure could be clearly defined by means of straight lines. We may note, also, that in the *Trinity* Masaccio has not provided an exact location for the figures, whose depth within the architectural setting is indicated only by their size. Again, this is rather unlike what we find in most later fifteenth-century works of art, in which we are usually supplied with a fairly clear set of coordinates more or less in the way Alberti describes (though that is, of course, not direct evidence for the use of his methods).

Donatello

The message from looking at Masaccio's use of perspective is that there is good evidence he did use a centric point, no evidence that he used anything that resembled the Albertian construction for transversals and some evidence that he did not. Moreover, it is clear that

25 Measurements and a detailed discussion of Masaccio's adjustments are given in Field et al., 'The Perspective Scheme of Masaccio's *Trinity*' (full ref. note 22), and are summarized in J. V. Field, *The Invention of Infinity: Mathematics and Art in the Renaissance*, Oxford: Oxford University Press, 1997, Chapter 3.

26 If one stands so far from the picture that the vault no longer looks three-dimensional (say with one's back against the outer wall of the far aisle of the church) the figures create their own space to such an extent that they seem to stand out from the wall.

2.8 Donatello (1386–1466), *St George and the Dragon*, relief from base of *St George*, *c.*1417, marble, 39 × 120 cm, Or San Michele, Florence.

Masaccio knew that mathematical correctness was not of the essence in obtaining the required illusion of the third dimension. Donatello provides such strong evidence of having taken this last point that one hesitates to express an opinion on the previous ones.

One of the characteristics that make Donatello a difficult case for the historian is his capacity, or willingness, to work in what one would prefer to regard as quite different styles within a relatively short period. Had the works been anonymous, it is unlikely that any historian would have retained a reputation for sanity if he or she had proposed to construct a 'Master of the Cavalcanti Altar (Santa Croce, Florence) and the St John the Baptist (Frari, Venice)' who was active in the 1430s. The subjects of the two works are very different from one another, and idiomatic exploitation of the possibilities of two different media is not a normal ground for ascription. Donatello's use of perspective is subject to variation in about the same degree as other elements in his style. On the whole, however, the most appropriate term for most of his apparent use of mathematical construction would seem to be 'mayhem'.[27]

Avoiding plurality of media, we may look at some examples of low relief in white marble. It has often been claimed, with some justification, that the earliest example we have of a perspective construction in the Renaissance is found in the relief of St George and the Dragon that Donatello made to go on the plinth of his statue of St George for Or San Michele (Fig. 2.8). The date of this is about 1417. We know that at this time it was the custom for sculptors to set up their workpiece at an angle, a practice that helped them to make due allowance for how the finished work would look when viewed obliquely. It appears that Donatello took advantage of this in many of his works, including *St George*.[28] It seems likely that, having designed the statue to look good when seen from below,

27 This is put much more obliquely and politely in John White, 'Developments in Renaissance Perspective II', *Journal of the Warburg and Courtauld Institutes* 14, 1951, pp.42–69.

28 See R. Munman, 'Optical Corrections in the Sculpture of Donatello', *Transactions of the American Philosophical Society* 75(2), 1985, pp.xiv, 96. For *St George* see pp.17–18.

Donatello would have taken similar precautions with the relief. However, these did not include using a naturalistic level for the eye height.

One cannot reasonably claim that the relief has a complete perspective scheme, but the treatment of the arcaded portico on the right does seem to take account of the convergence of images of orthogonals, in this case the line of the top of the building, the line at its base, and the lines between rows of tiles within the arcading. All these lines meet, fairly precisely, at a point towards the left of the body of St George and on the eye level of the princess. Since we are dealing with a relief, it is not reasonable to expect the lines to meet absolutely precisely at one point, but they come close enough to doing so for it to seem overwhelmingly likely that the convergence was intended. Moreover, the inclusion of a square-tiled floor, whose pattern provides a clear but unobtrusive set of orthogonals, suggests that Donatello had been thinking in terms of perspective effects. The tiled floor also shows a number of transversals, but as all of them are very short there does not seem to be enough evidence to attempt to find a viewing distance and check whether the drawing of the tiled floor is mathematically correct.

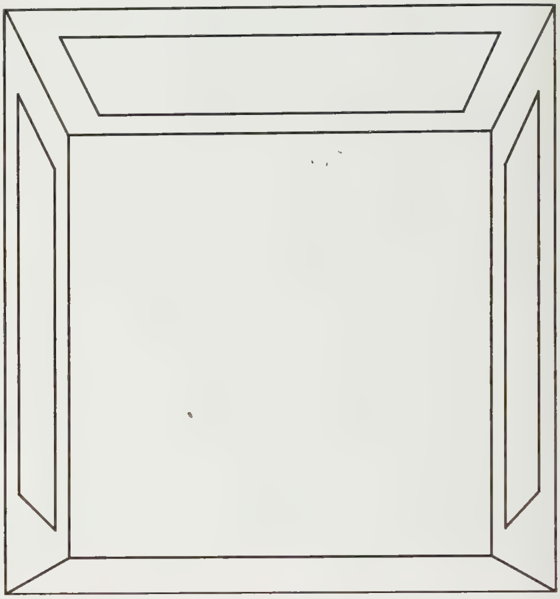
The position of the centric point is, however, of interest. Alberti's definition makes it the point where the 'centric ray' meets the picture, that is it is the point towards which the eye is directed, the point the observer is looking at. So it should be at some significant position in the composition. This is true in Masaccio's *Tribute Money*, though less so in the *Pisa Madonna*. The area in question is missing in the *Trinity*, but it is possible that it held, for instance, a coat of arms of the donors – there is one in a similar position on Donatello's Cavalcanti Altar[29] – and it might thus be a suitable point at which to direct the eye. The problem with the centric point, in both Donatello's *St George* relief and Masaccio's *Tribute Money*, is its actual height compared with that of the eye of a normal viewer of the work. In neither case is the height at all realistic. In the *Tribute Money* the point is at least 2 metres above the height of the eye of someone standing on the floor of the Brancacci Chapel, and for the *St George* relief the centric point is about half a metre above the eye height of someone standing in the street. In both cases a realistic eye height would have meant that the centric point lay outside the field of the picture.

It is clear that both Masaccio and Donatello have followed the prescription implied in the first book of Alberti's treatise on painting and made explicit in the third, namely that the centric point be put at the height of the eye of a standing figure in the picture. Perhaps it was the success of their works that inspired Alberti's prescription? In any case, since, apart from the discussion of Giotto's *Navicella*, Alberti's examples of subjects are all secular, he may perhaps have been thinking in terms of panel pictures whose viewing height could be set as convenient, since the pictures themselves were merely domestic furnishings. In some respects, Alberti's prescriptions are as out of touch with the common run of artistic commissions as Vitruvius' rules – though much invoked – inevitably were with the building of churches. However, it is common for a centric point at the eye height of a standing figure in the picture to be used in decorative schemes where such a height is far from naturalistic for the normal viewer. Its advantage in allowing one to see what is going on in the scene is unquestionable, but it nonetheless marks a notable departure from the naturalistic depiction that the use of perspective is often claimed to connote.[30]

29 If we are to judge from the height of the altar table shown by Masaccio (as reconstructed by the most recent restorers), the centric point is a little too low for a position in which a crucifix, say, might have stood on an altar table in front of the painting.

30 We shall return to this problem in Chapter 5. The departure from naturalism is, of course, much less noticeable when we are looking at pictures reproduced in a book.

2.9a Donatello (1386–1466), *Pazzi Madonna*, *c.*1422, marble, 74.5 × 69.5 × 8 cm, Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem.



2.9b Sketch of the embrasure in the *Pazzi Madonna* showing orthogonals. Not to scale. Drawing by JVF.

Another work by Donatello that, at first glance, may appear to be setting up a perspective illusion by means of orthogonals is the *Pazzi Madonna* (Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem) (Fig. 2.9a). Although this relief is not documented, the ascription to Donatello is not in dispute and the date of the work is usually accepted to be about 1422.[31] The early dating is partly in recognition that Donatello does not handle the technique of so-called flattened relief (*rilievo schiacciato* or *stiacciato*), or the perspective drawing of the Madonna's hand, with the skill he was to display in his mature works.

The overall effect of the relief is more severe than is usual for Donatello. There is no trace of gold or polychroming, so the unemphatic treatment of the panelling in the embrasure surrounding the figures is presumably what was originally intended.[32] The lines of the embrasure suggest a spatial structure, but they do not give a mathematically correct description of a plausible shape. To make their pattern clearer, a rough sketch of it has been given in Figure 2.9b. The sketch exaggerates the divergences from correct perspective but, as comparison with Figure 2.9a will show, however determinedly we may torture ourselves into accepting that the images of other orthogonal lines may converge or may be intended to converge, there is no way that such convergence could be supposed to include the lower edges of the panelling at left and right of the opening. On the other hand, these lines are not prominent. Any effect of incorrect perspective is scarcely noticeable in practice since

31 A discussion of the date of the *Pazzi Madonna* is given in H. W. Janson, *The Sculpture of Donatello*, Princeton: Princeton University Press, 1963, pp.44–5.

32 It is possible that traces of gold or paint have been removed by cleaning. The marble has a blotchy colouring that is characteristic of cleaning carried out before the development of effective techniques for dealing with marble.

2.10 Donatello (1386–1466), *The Feast of Herod*, probably 1433–5, marble, 50 × 71.5 cm, Musée des Beaux-Arts, Musée Wicar, Lille.

most attention is directed to the central figures, whose modelling is convincingly solid. All the same, although the use of the embrasure suggests an interest in perspective, the actual treatment of it suggests rather little corresponding expertise, in fact, less than is to be found in the *St George* relief. Unless we are to re-date the *Pazzi Madonna* to before 1417, the simplest explanation would seem to be that Donatello did not want to make the perspective correct. Perhaps to do so would have risked making the setting seem more three-dimensional than the figures?

The marble relief showing *The Feast of Herod* (1433–5) that is now in Lille (Musée des Beaux-Arts, Musée Wicar) (Fig. 2.10) seems to be the product of an entirely different train of thought. Here we have what almost amounts to a display of constructed perspective. The pavement has a multitude of engraved lines running through it, and images of orthogonals meet at a point whose height is that of a standing figure in the picture. The picture also conforms in other ways with what Alberti recommends in *De pictura*. This fact, together with the classicizing figures, some of which may show an interaction with the figure style of Masaccio, tends to date the work to the early 1430s.[33] We can see enough of the pave-

33 See Janson, *Sculpture of Donatello* (full ref. note 31), pp.129–31, where the relief is dated to 1433–5.

2.11 Donatello (1386–1466), *The Feast of Herod*, *c.*1425, bronze, 60 × 60 cm, Baptistery, Siena cathedral.

ment to check that its perspective is apparently correct and to find the viewing distance that was built into the construction. This distance turns out to be about two and a half times the width of the picture. At this distance, many of the details of the work, including details of the pavement, are difficult to see. In flat contradiction, close inspection shows that details have been executed with a care that demands attention. In fact, unless the lighting is good, the whole scene would probably be difficult to read from the 'ideal' distance, since the figures in the drama are small in comparison with their decorative architectural setting.

With perspective pictures, one usually finds that the 'ideal' viewing distance – that is, the distance used in the construction – operates as a minimum. If one stands closer, the picture makes one feel uncomfortable, but if one stands further away the effect is satisfactory. The tolerance of the eye makes it a little tricky to determine the ideal viewing distance without recourse to measurements, but a careful estimate made by eye is important, since the eye must be the final judge. So a picture like the Lille *Feast of Herod* that almost compels one to look at it from too close, is throwing away the perspective in doing so. One painting that is like the Lille relief in this respect is Piero della Francesca's *Flagellation of Christ* (Galleria Nazionale delle Marche, Urbino) (Fig. 5.28). Its viewing distance is also about two and a half times the picture width, and the use of perspective is similarly prominent, but the picture is finished in such detail that Piero, like Donatello, cannot possibly have expected that his viewer would wish to stand indefinitely at the 'ideal' distance. Unlike Donatello, however, Piero did arrange that his picture was at least easily readable from an appropriately large distance.[34] Donatello has made a work that presents the viewer with a carefully finished representation whose mathematical structure is to be discussed rather than seen by the eye.

Being a scene in a story that takes place largely indoors – that is, in a fully architectural setting – the Feast of Herod is a likely subject for perspective. About ten years before he made the Lille relief, Donatello had completed a very different version of the subject, in bronze.[35] This version (Fig. 2.11) also makes considerable use of perspective, but is much more harshly theatrical. It prompts the thought that with this amount of sex and violence on show no one was likely to notice details of the mathematics. They are, however, there, in an array of chased lines that mark out the tiles of the pavement in the foreground. The centric point, which is fairly well defined considering that we are dealing with a relief, is at the eye height of the seated figures. Given the position of the relief *in situ*, on the base of a font, this point is unrealistically low even for a worshipper on his knees, which in any case would not be an expected position beside a font.

No short summary of any aspect of Donatello's practice could possibly be fair – his range is simply too wide. However, that said, it does seem that his use of perspective, up to 1435, shows closer affinities with Alberti than Masaccio's does. In particular, there are square-tiled pavements that situate figures precisely in depth. However, it has been suggested that a use of pavements, probably constructed according to a definite rule, was more prevalent in Siena, which is where Donatello came to know it.[36] So the apparently Albertian element may actually be a Sienese element. In any case, it seems clear that in this period, as in later ones, Donatello did not see correct formal perspective as a necessary component of his works.[37]

The diagonal line

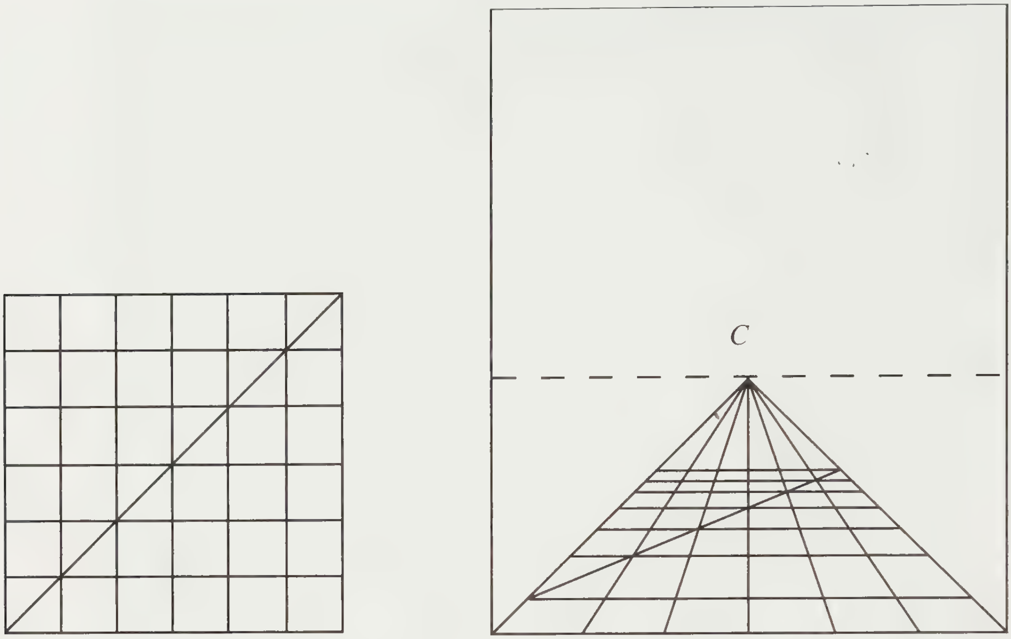
We have no direct information about constructions for drawing pavements that may have been current in Siena when Donatello was working there. However, there are correct pavements in pictures dating from the fourteenth century, most notably the one in Ambrogio Lorenzetti's *Presentation of Christ in the Temple* (Galleria degli Uffizi, Florence) dated 1342. It is tempting, though not rigorously logical, to seek some light on these earlier practices by examining the procedure that Alberti proposes as a check on the accuracy of one's drawing of the pavement in *De pictura*, §20 (Figs 2.1–2.3). He suggests one should draw a diagonal of the complete pavement, and that the drawing is certified as correct if this line is also a diagonal of the tiles that lie along it. As is habitual in *De pictura*, there is no explanation. Assuming our construction was carried out accurately, the resulting figure should look something like the one on the right in Figure 2.12. The diagram on the left in Figure 2.12 shows what the diagonal would look like if drawn on the real pavement.

34 A more detailed discussion of the *Flagellation* will be found in Chapter 5.

35 See Janson, *Sculpture of Donatello* (full ref. note 31), pp.65–75, where the relief is dated to 1423–7.

36 See Kemp, *The Science of Art* (full ref. note 2).

37 An excellent survey of Donatello's practice is given in White 'Developments in Renaissance Perspective II' (full ref. note 27).



2.12 Figure for Leon Battista Alberti, *On Painting*, §20. Using a diagonal as a check on the accuracy of the drawing of a square pavement. The diagram on the left shows the diagonal drawn on a plan of the pavement; the one on the right shows what is proposed by Alberti. Drawings by JVF.

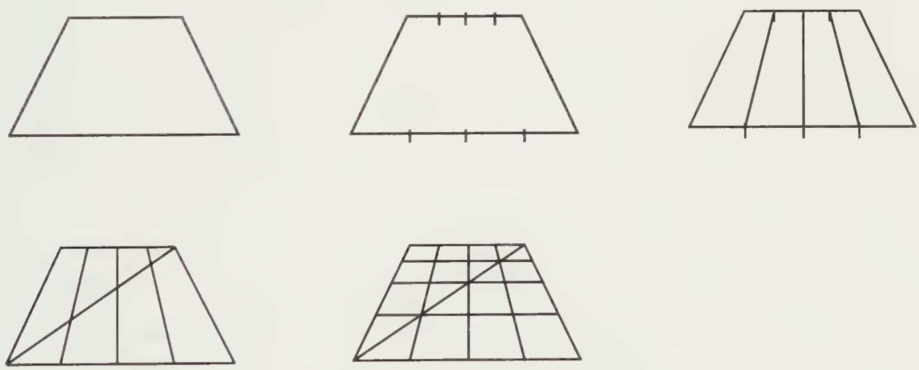
Rational reconstruction suggests that diagonals like the one in the diagram on the left might have been used as a way of checking whether a set of squares had been drawn correctly, and might have been used as a method of constructing the small squares from the larger one. As we shall see, in the mathematical preliminaries in his treatise on perspective Piero della Francesca makes use of diagonals in constructing lines across a 'perfect' rectangle, that is a rectangle in its 'true' shape.[38] The example shown on the left in Figure 2.12 is a special case of this, in which the rectangle has become a square and the lines across it divide it into equal squares.

It is, of course, a rather simple matter of observation, confirming an intuitive concept of symmetry, that the diagonal of the complete square pavement will also be the diagonal of the tiles that lie along it. Even a non-mathematician might notice such a thing when moving his bishop in a game of chess. Once we start to think about the chess board, or pavement, in perspective, it is obvious that the points where the corners of squares, or tiles, meet in the perspective version must also lie on a diagonal. To put it in formal terms: straight lines and points of intersection in the original become straight lines and points of intersection in the image. The formality is, however, liable to seem a step towards wilful obfuscation – or mathematical proof. The fact that the diagonal can be used as a check is obvious. Unfortunately, this obviousness seems to have been less apparent to some of Alberti's contemporaries, who, he tells us, drew the transversals in such a way that their renderings of tiled pavements would have failed his test. Surviving works of art tend to confirm this assertion.

38 See Chapter 5.

Either the draughtsmen in question did not care about the matter enough to make a check, or they had not thought of this way of making one. It accordingly seems possible that the origin of Alberti's check is not mere common sense or an elementary mathematical observation but a workshop rule that he happened to know about. Since Alberti presents his own construction as having a connection with the science of *perspectiva*, it is to be presumed, from his presenting it separately and almost as an afterthought, that he did not see the workshop rule, if it was one, as having any such connection with the learned science. The absence of any such connection, or rather its perceived absence, might account for Alberti's not mentioning the fact that if the diagonal can be used as a check it can surely also be used as a part of a construction procedure. Imagine we have drawn the outline of the complete square in perspective, say using Alberti's construction to obtain the images of its nearest and furthest edges. We can then divide the front edge into the required number of tiles, join the points of division to the centric point (which gives us the images of the orthogonals) and then, by drawing the diagonal, put in the images of transversals as lines parallel to the front edge through the points in which the diagonal cuts each image of an orthogonal.

This is essentially the procedure followed by Piero della Francesca when he constructs a pavement in *De prospectiva pingendi*.[39] Perhaps Piero's mathematical inclinations are a clue to why the procedure did not appeal to Alberti. What Piero does relies on mathematical results as mathematical results. What Alberti does – if we assume him to be mimicking surveying techniques – establishes the optical correctness of each transversal independently. In any case, whatever Alberti's attitude, the diagonal could be used as part of a construction procedure, even one that does not use the centric point. The repeated use of the word 'obvious' in the following description of such a procedure is intended to remind the reader that we are dealing with yet another rational reconstruction, and that certain steps in the argument could be construed as standing in need of proof. Mathematicians do not trust 'obvious'.[40] The stages of the possible construction are shown in Figure 2.13, in which, for clarity, the pavement has been made 4 × 4 instead of Alberti's 6 × 6.



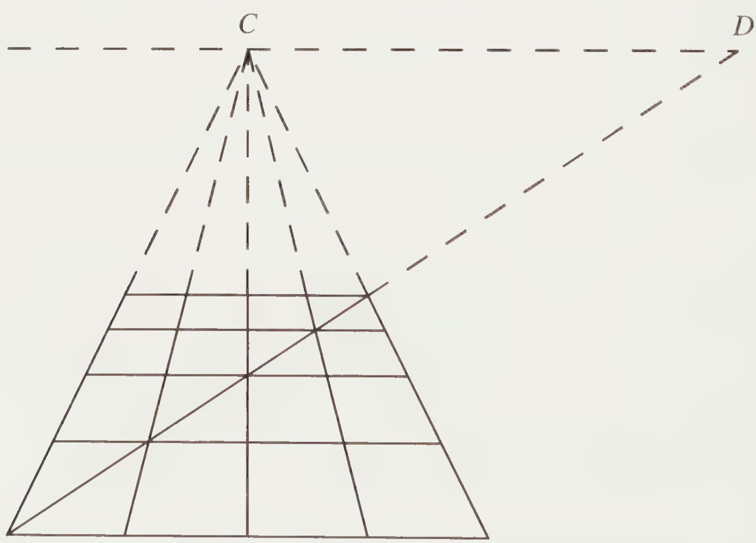
2.13 Diagrams to show a rational reconstruction of a possible method of using a diagonal of a pavement to construct transversals, without the use of a centric point. Drawings by JVF.

39 Piero della Francesca, *De prospectiva pingendi*, Book 1, sections 14 and 15; see Chapter 5.

40 A very clear and witty article about this was published in the scientific journal *Nature*: Ian Stewart, 'One Hundred Per Cent Proof', *Nature* 324, 4 December1986, pp.406–7.

The first stage is to draw the outline of the image of the square, that is we decide what shape the overall pavement is to appear to be in the picture, something that will be determined by the design of the picture as whole. In Figure 2.13, this outline has been made symmetrical, but it does not need to be so. Now, it is obvious that since the pavement is 4 × 4, the front edge of its outline in the picture must be divided into four equal parts, and the back edge likewise. Joining these points of division will give us the lines running from front to back between the tiles. Now, since it is obvious that a diagonal of the original pavement must appear in the picture as a diagonal of the image of the pavement, we can draw the diagonal as shown. In the real pavement this diagonal cuts the lines between the tiles at corners of tiles, so this will be true in the image also. Accordingly, if we draw lines through these points of intersection parallel to the front and back of the pavement, they will mark the transverse divisions between the tiles. So the drawing of the pavement is now complete. Alternatively, there may simply have been a well-known procedure, handed down from time immemorial, which gave instructions as above without any attempt at rationalization. Like any other workshop technique, this one was presumably employed only when judged appropriate, that is, in this case, when one wanted a particularly convincing pavement. The existence of such a tradition is hypothetical, but the procedure itself is mathematically correct: it could represent a degenerate or simplified form of a construction known in the ancient world.

Comparison of the third diagram in Figure 2.13 with the Albertian diagram on the right in Figure 2.12 shows that what has happened is that we have lopped off the parts of the images of the orthogonals that go to the centric point. Since the images of orthogonals in Figure 2.13 are symmetrical about the central vertical axis of the image of the pavement, it is obvious (and can be proved) that when extended they must meet somewhere on it. So the procedure summarized in Figure 2.13 has lost our centric point, which is to say that we do not know the level of the eye. Nor do we know its distance, but that can be found quite easily, though the rationale of the method of finding it is far from obvious.



2.14 Extended version of the diagram obtained at the end of the procedure summarized in Figure 2.13 which gives a centric point and a distance point. Drawing by JVF.

What we need to do is to extend the images of the orthogonals to give us the centric point, draw a line through the centric point parallel to the base of the picture (which we shall assume is parallel to the nearer edge of the pavement), and then extend the diagonal until it meets this new line, in the point *D*. The resulting diagram will look as in Figure 2.14. The distance between *D*, called a 'distance point', and the centric point, *C*, that is the length *CD*, is the distance of the eye from the picture. This diagram is what we should have obtained in constructing the image of the pavement by what is now known as the 'distance point method' (in which the position of *C* and the length *CD* are given among the initial conditions, see Appendix 3). The origin of this method is unknown. The first written mention of it is in the vernacular text of Piero della Francesca's treatise on perspective, but he does not attempt to show that the method is mathematically correct and the passage in question is not found in the Latin version of the treatise.[41]

The distance point construction was presumably known in artists' workshops when Piero wrote his treatise, but we do not know whether this method, or some simplified version of it, was known to Donatello in the 1420s. In any case, our present purpose is to employ what we know of the use of perspective constructions in the work of Masaccio and Donatello to help us to understand what Brunelleschi may have invented. It is thus more important that we do not know, either, whether the distance point construction was known to Brunelleschi.

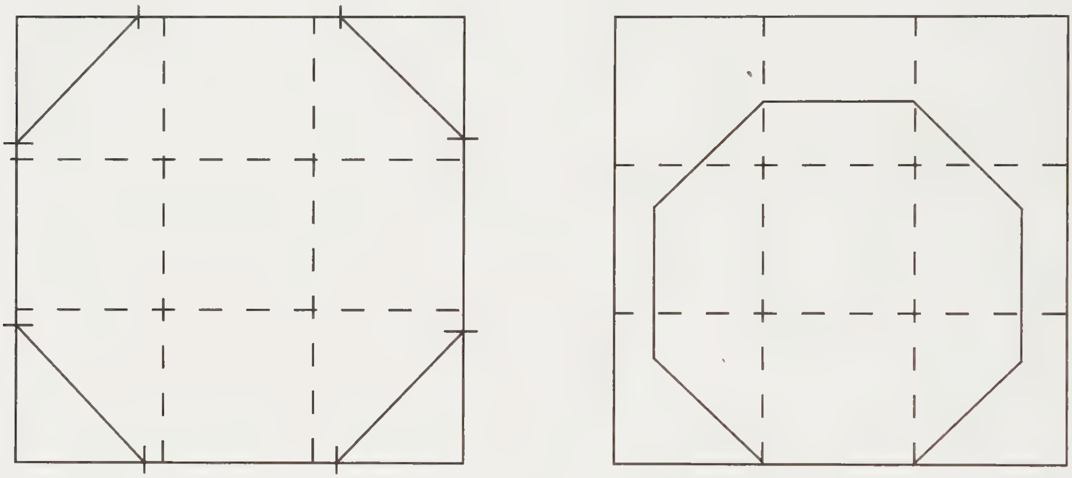
Brunelleschi's rule

One thing that the Albertian and distance point methods have in common is that both are best adapted to drawing something like a square-tiled pavement, which can then be used to set up what is effectively a system of coordinates. It is rather hard to see why the Baptistery of Florence shown in the first of Brunelleschi's demonstration panels should have formed the subject of a picture whose construction relied upon a grid of equal squares. The reason lies in the fact that the ground plan of the Baptistery is a regular octagon. No doubt it is not really exactly regular, but Brunelleschi is most unlikely to have wanted to take account of departures from perfect symmetry in a building he regarded either – in accordance with common opinion at the time – as a temple of Mars, or probably at least as exemplifying a pure Tuscan variant of the true ancient Roman style. (The building is now believed to date from between the eleventh and thirteenth centuries.)

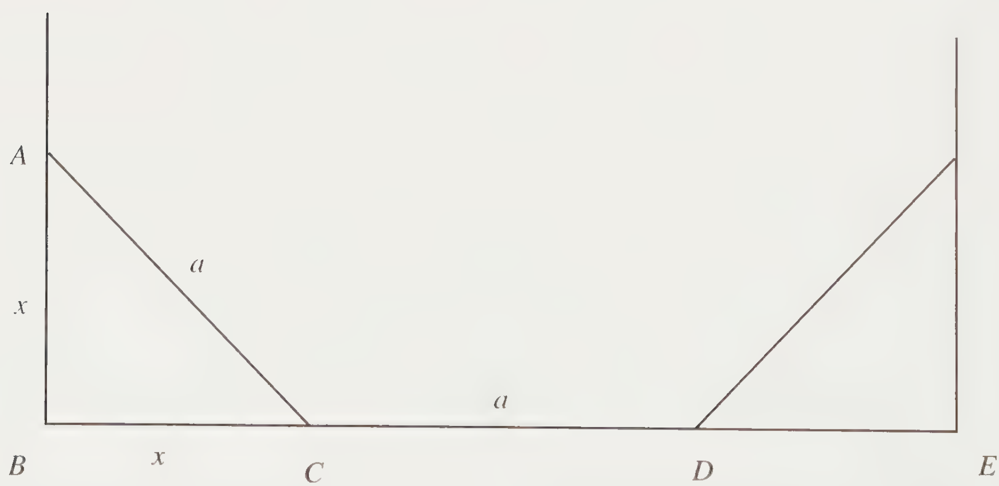
A regular octagon is a fairly easy shape to lay out because its angles are 135° (that is, 180°–45°). Renaissance texts like to start with a square and cut its corners off, symmetrically, to get an octagon. This procedure is shown in the diagram to the left in Figure 2.15 (see also Appendix 4). The construction is well adapted to being carried out on a large scale, and it may well have been used to get the octagon required at the crossing of Florence cathedral (Santa Maria del Fiore). An alternative procedure, making direct use of the side and the 45° angle is shown in the diagram on the right in Figure 2.15. The two diagrams each include a square grid. From the diagram on the left it will be seen that if we start from a square of side three units we obtain an octagon such that none of its corners lies at a grid

41 See Chapter 5 and J. V. Field, 'Piero della Francesca and the "Distance Point Method" of Perspective Construction', *Nuncius* 10.2, 1995, pp.509–30.

point, though four of the sides lie along grid lines. The diagram on the right shows an octagon constructed so that its first two corners lie at grid points. However, none of the other corners does, though two further ones do lie on grid lines. There is nothing special about the use of a 3 × 3 grid in this case: we should not get a more useful result – that is, one where more corners lie at grid points – by using a fancier number of grid squares such as 17 or 60. The proof is as follows. (Readers with a preference for the Albertian style of omitting mathematics wherever possible may skip the next few paragraphs with no loss to themselves.)



2.15 Regular octagons against a 3 × 3 grid. The diagram on the left shows the figure obtained by cutting corners off a square; the diagram on the right shows the figure obtained from sides and angles. Drawings by JVF.



2.16 Part of an octagon cut out from a square. *BE* is one side of the square. *AC*, *CD* are sides of the octagon. Drawing by JVF.

Let us redraw part of the diagram on the left of Figure 2.15 as shown in Figure 2.16, in which A, C and D are corners of the octagon, and B and E are corners of our original square. Let the side of the octagon be a, and the length of the side of the square cut off at each corner be x. That is, we have $AC = CD = a$ and $AB = BC = x$, as marked.

By Pythagoras' Theorem in triangle ABC we have

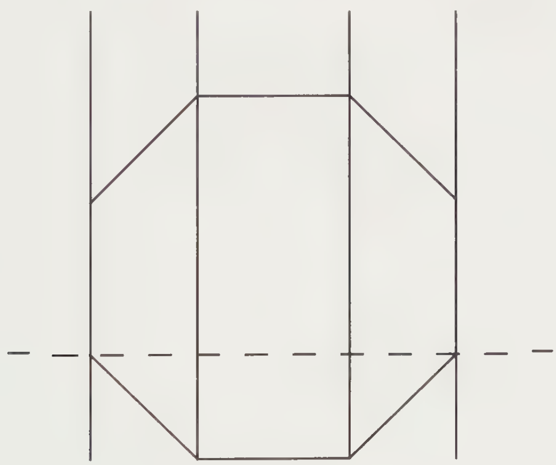
$$AC^2 = AB^2 + BC^2,$$

that is

$$a^2 = x^2 + x^2,$$
$$a^2 = 2x^2,$$
$$a = \sqrt{2}x$$

Now, as Brunelleschi undoubtedly knew, the length $\sqrt{2}$ units can be found geometrically, but the value of $\sqrt{2}$ cannot be expressed as the result of dividing one whole number by another. That is, $\sqrt{2}$ is what is called an 'irrational' quantity. So if a is a whole number, x is not, and vice versa. Thus if we put B at a grid point either C or D will not be at one, however we design the grid. And if we put C at a grid point either B or D will not be at one.

Mathematical properties being what they are, it comes as no surprise that a different method of construction will not help us. The diagram on the right in Figure 2.15 corresponds to our putting C and D from Figure 2.16 at grid points. This makes a a whole number. So x is not, and A does not lie on a grid line. By a suitable choice of grid we can get round the problem – in the sense of providing an approximate solution – but it has no exact solution.



2.17 Plan of Baptistery, showing orthogonals. Only the part of the building below the transverse broken line would be visible in Brunelleschi's picture. Drawing by JVF.

To return to the particular problem in hand: by combining elements from the two diagrams in Figure 2.15, we can see that the ground plan of the Baptistery would give us some unevenly spaced orthogonals, as shown in Figure 2.17, but, as luck would have it, they are none of them lines that will appear in the finished picture, which will show only the part of the building below the dashed horizontal line. However, whatever method he is using, Brunelleschi needs to find the images of the two corners of the octagon in which it is cut by the transverse broken line, because the verticals through these points mark the outer edges of the image of the building. Whether one uses the standard forms of the distance point method or of the Albertian method will make no difference to the problem in regard to the horizontal grid. The Albertian one seems preferable, since it can be used to construct the image of any transversal, such as the dashed line in Figure 2.17, whereas the 'distance point method' simply generates a grid.

On the other hand, the dark green and white marble patterns on the walls of the Baptistery include a number of lines that are parallel to the top and bottom edges of the wall (Fig. 2.18). Drawing these, and the lines of the horizontal edges of the side walls, would be relatively easy if Brunelleschi knew a rule for finding the images of sets of lines that in reality are parallel to one another, and lie in the same vertical surface, which makes an angle of 45° with the plane of the picture. In fact, the images of such lines will converge at a point on the horizontal line through the centric point. Without some rule to provide a short cut for drawing the patterns, the business of drawing the Baptistery would be much more laborious. One would need to find the proportions in which the horizontal lines divide the walls, and use these proportions as a means of locating the lines against the front and back edges as shown in the picture. This would be a fairly lengthy process. However, such a procedure may have been exactly to Brunelleschi's taste. He is known to have been interested in proportions, and a good case has been made for linking this concern with his invention of the perspective 'rule'.[42]

The fact that, in looking at the side walls of the Baptistery, we are considering surfaces running at 45° to the picture plane is a pointer in the direction of a short-cut construction that makes use of a diagonal of a square: the two diagonals of a square lying with one side parallel to the picture plane are each at 45° to the picture (see the diagram on the left in Fig. 2.12), that is they are parallel to the two visible receding faces of the Baptistery. It may be significant that the lines with whose images we are concerned are all in the same plane, defined by one wall of the Baptistery. In this period, to say a set of lines were parallel implied that they were all in one plane. When we are considering two lines the matter is trivial, since a plane can always be drawn through two parallel lines, but if more lines are involved, Renaissance conceptions of what is going on can become noticeably different from modern ones. However, in the present case the numerous horizontal lines in the patterning of the Baptistery wall do all lie in the same plane. And since their plane is at 45° to that of the picture, it seems Brunelleschi may be drawing on the same, possibly Sienese, source as Donatello. There are other logical possibilities as to who learned what from whom, but Donatello's being one of the people involved makes the would-be historian's life more difficult. Plausible narratives do not easily accommodate the activity of people who may decide not to use what they know.

There is, however, a serious problem with all this: if Brunelleschi believed his rule to be correct in terms of geometrical optics, as he seems to have done, then his use of the con-

42 R. Wittkower, 'Brunelleschi and "Proportion in Perspective"', *Journal of the Courtauld and Warburg Institutes* 16, 1951, pp.275–91.

2.18 The Baptistery, Florence.

vergence of images of parallel lines at 45° implies he could prove, indeed had proved, that such lines converged. From Masaccio's practice it would seem that Brunelleschi knew that images of orthogonals converged to a centric point. One of the difficulties in accounting for Brunelleschi's knowledge of the convergence of images of orthogonals is that, in mathematical terms, the behaviour of the images of orthogonals is only a special case of a more general theorem, the theorem that all sets of lines that are parallel to one another in reality will appear in the perspective picture as sets of lines converging to a point on the horizon. It so happens that it is difficult to prove the theorem in the special case without also proving it in the general one. Piero della Francesca's work demonstrates this. In *De prospectiva pingendi* he is apparently interested in the convergence of images of orthogonals, not images of any other sets of parallels, and his diagrams suggest he has proved the convergence property only in the special case. However, the tidy-minded mathematical instinct of holding on to as much generality as possible for as long as possible has led him to state the relevant propositions in the most general terms, and the proofs he provides actually add up to a proof of the convergence of images of all sets of parallel lines.[43] It is his Euclidean method of attack that leads Piero to the general theorem. If a proof were to use notions of symmetry – which Euclid notably avoids – one could probably isolate the special case of the

43 Piero may not have realized this. See J. V. Field, 'When is a Proof not a Proof? Some Reflections on Piero della Francesca and Guidobaldo del Monte', in *La Prospettiva: Fondamenti teorici ed esperienze figurative dall'Antichità al mondo moderno. Atti del Convegno Internazionale di Studi, Istituto Svizzero di Roma (Roma, 11–14 settembre 1995)*, ed. R. Sinisgalli, Florence: Edizioni Cadmo, 1998, pp.120–32, figs pp.373–375.

convergence of images of orthogonals to a point at the 'centre'.[44] However, once one notices that images of sets of parallel lines at 45° to the picture plane also converge, it surely becomes harder to avoid being edged towards the general theorem. This theorem was first proved, as such, by Guidobaldo del Monte (1545–1607) in his *Perspectivæ libri sex*, published in Pesaro in 1600. Guidobaldo's book was not an easy read, even for real mathematicians of the day, but artists seem to have realized very quickly how useful his theorem was.[45] Allowance must be made for the general rise in the level of mathematical education by 1600, but it is nonetheless difficult to suppose that the theorem had been noticed by Brunelleschi 180 years previously and had somehow not come to the attention of Piero della Francesca. We have no real evidence that Brunelleschi did know that images of parallel lines at 45° to the picture would converge. This knowledge would have provided him with a short cut for drawing the dark marble patterns, but, as already mentioned, he could have drawn the required lines by using the fact that perspective did not change proportions.

The remainder of the drawing of the Baptistery is much less of a problem than the marble patterning. The shape is a simple one, and all edges are straight: we have an octagonal prism with a pyramid on top. Thus only a few points are required to construct the image. Brunelleschi could have found them by direct use of surveying methods, with which he must certainly have been familiar. Alternatively, he might have constructed them by use of plan and section drawings of the building, using a technique like that described by Piero della Francesca in the third book of *De prospectiva pingendi* (see Chapter 5).

Piero's treatise contains four propositions directly relating to octagons and two to octagonal prisms, each following an analogous one concerning a square.[46] Since octagonal structures are not common elements in paintings – a fact that is tacitly acknowledged by the omission of the octagon propositions in most of the treatises later derived from Piero's – these propositions should perhaps be seen partly as a reminiscence of Brunelleschi's picture, or a homage to it. Piero may well have seen the panel concerned. It was probably in the possession of the Medici at the time he visited Florence in the 1430s.

To sum up the situation: the painstaking use of surveying techniques would have sufficed for Brunelleschi to make suitable drawings for both of his demonstration panels. However, Manetti says Brunelleschi had invented a 'rule' for making optically correct drawings, and Masaccio's practice suggests this included the use of what Alberti was later to call a 'centric point'. The choice of the Baptistery as his first example suggests Brunelleschi's rule may have given him a quick way of getting the images of lines that in reality were parallel to one another and lay in a vertical plane at 45° to that of the picture. In which case, he may have been familiar with – or have discovered – some version of what is now called the 'distance point construction' (which uses a centric point). However, consideration of the relevant mathematics shows that this is much less likely than it may at first seem. Donatello's practice suggests that he knew about the properties of the centric point, and that a rule

44 Field et al., 'The Perspective Scheme of Masaccio's *Trinity*' (full ref. note 22), contains, as Appendix 7, an account of a possible derivation of the convergence of images of orthogonals from the mathematics of astrolabes. This avoids naïve appeals to symmetry. It is very likely that Brunelleschi knew about astrolabes, but the argument presented – for which I rather than my collaborators take responsibility – is intended primarily as something worth thinking about: that is, as another rational reconstruction.

45 See Field, 'When is a Proof not a Proof?' (full ref. note 43) and Field, *The Invention of Infinity* (full ref. note 25).

46 Piero della Francesca, *De prospectiva pingendi*, Book 1, Sections 16, 26, 29; Book 2, Sections 2, 10; and Book 3, Section 2.

having some elements in common with the distance point method was known in the 1420s. Ambrogio Lorenzetti's pavement in the *Presentation of Christ in the Temple* suggests some such rule had been known in Siena in the 1340s.[47] One could argue that part of Brunelleschi's method derived from this postulated Sienese construction, plus, of course, his own knowledge of surveying techniques, and no doubt other bits of practical mathematics. On the other hand, everything is heavily non-proven, and there is certainly no hard evidence that Brunelleschi knew anything very like what was later to be known as the distance point method. Moreover, none of the available evidence helps greatly in telling us what Brunelleschi's rule was. However, the unsuitability of a grid for the demonstration panels and for Masaccio's *Trinity* suggests Brunelleschi's method may have included simple calculation of sight lines for key points in the picture. Brunelleschi's making pictures of actual urban scenes may suggest ordinary surveying, but the calculation of sight lines could also have been done from plans and sections (or frontal views) of buildings.

Rules and the science of sight

A connection with surveying would provide Brunelleschi's rule with a connection to *perspectiva* proper, that is the complete science of vision. As already noted, the work of Alberti makes it clear that *perspectiva* was regarded as part of natural philosophy, but it was considered a 'science' because it used mathematics. Perspective construction was known as 'perspectiva artificialis' (artificial perspective) and it is a quirk of linguistic evolution that the term 'perspective' on its own should now refer to the Renaissance invention, whereas the medieval science from which it sprang is now known as optics.

The name optics is, of course, in recognition of the Greek origins of the subject, and it came into use in the course of the sixteenth century as more Greek scientific texts were printed. However, Euclid's work on optics, which formed the chief basis for teaching in the fifteenth century, usually being known in Latin under the title *De aspectuum varietate*, is rather simple, though characteristically abstract. It deals with propositions such as that if two magnitudes are of equal size, but are seen at different distances, the one that is nearer the eye will appear larger.[48] And, again in geometrical terms, that when rectangular magnitudes are seen at a distance, their corners appear rounded.[49] This latter proof merely asserts that the triangles at the corners of the rectangle will not be seen, because they are too small. This somewhat begs the question, but is interesting because it clearly relies on unstated theories about how vision works in the eye. What would now be called geometrical optics, which is Euclid's main concern, is being allowed to shade off into what would now be called physiological optics. It is chiefly in the rigorous division it imposes between these two modes that we see the historical importance of the proof that vision was by the reception of light rays.[50] In fact, this 'intromission' theory of vision was also known in the

47 Rough measurement, carried out on photographs, suggests that if the diagonal of Ambrogio's pavement is extended to meet the horizon, it does so at a point very close to the edge of the picture field. This convenient coincidence, found fairly often in later pictures, corresponds to use of the distance point method with a viewing distance of half the picture width (see Appendix 3).

48 Euclid, *Optics*, Prop. 5; Euclid, *L'Optique et la Catoptrique*, ed. and trans. Paul Ver Eecke, Paris, 1959, p.4.

49 Euclid, *Optics*, Prop. 9; Euclid, *L'Optique et la Catoptrique* (full ref. note 48), p.8

50 See note 12.

fifteenth century, through several thirteenth-century works, most notably those by Roger Bacon (*c.*1220–*c.*1292) and by Witelo (died after 1281), which drew heavily on the optical treatise written by Ibn al-Haytham. Gradually, treatises based on Euclid – the most widely read seems to have been that by John Pecham, to which we referred above – were replaced by works that made greater use of the Islamic tradition. Advances in optics from the late fifteenth century onwards were strongly mediated not by the recovery of Greek work but by better understanding of Arabic sources.

In *De pictura/Della pittura*, Alberti mainly uses what would now be called geometrical optics, and there is nothing that goes beyond what can be found in Euclid. In fact, as we shall see in examining Piero della Francesca's treatise on perspective, everything Alberti needs can be proved by the medieval versions of ancient Greek methods. To put it simply: there was no need of a Renaissance rediscovery of anything 'lost' in order to invent perspective. The use of the technique in works of art marks a break with what went before, but the mathematics does not. As a sort of corollary to this, we may note that Euclid, or his predecessors, had the same mathematical results at their disposal, so the ancients too might have used techniques like those we find in Renaissance treatises on perspective.[51]

However, the presence of 'physiological' elements in Euclid's work, and in Alberti's, is a reminder that, although the mathematics of geometrical optics looks, to modern eyes, just like any other mathematics, to Alberti it was a part of a larger structure that belonged to natural philosophy. Thus the work carried out by artisans is presented as having connections with serious subjects that are taught in universities. Alberti's claim on behalf of painting seems to be founded in his reading of ancient sources such as Pliny, in which, in an idealistic programme that is similar to Vitruvius' claims for the architect, the competent painter is required to be master of the parts of natural philosophy relating to vision.

Alberti presents the use of 'artificial perspective' as an outgrowth of such understanding. He does not explain how the construction he describes relates to the 'cone' or 'pyramid' of vision that he has just discussed, but there is a strong implication that the mathematics of the construction is derived from that of the 'pyramid'. That is, to Alberti, the construction, 'artificial perspective', is an offshoot of the science of *perspectiva* and partakes of its intellectual respectability. With the considerations of intellectual respectability there of course go considerations of social respectability. History shows that in the latter Alberti proved to be on the winning side: the status of artists rose in the following centuries.

In regard to intellectual respectability, we run into a number of interesting problems. The first is that of hypocrisy. As we have seen, some of the earliest uses of mathematical construction techniques seem to be anything but scientific in spirit. Masaccio's practice seems rather to merit the adjective 'theatrical', or perhaps even 'Donatellesque'. All the same, the strongly modelled figures are clearly meant to be considered naturalistic, and the pictorial space is meant to be read as naturalistic also. Masaccio is knowingly deceiving us, not only into seeing the wall open to admit his vault, but also into believing that what we see is optically correct. We have already mentioned some important adjustments that seem to have been motivated by the painter's concern with the effect of his composition as a whole. There are also other more minor 'errors'. For instance, the thicknesses of the abaci on the columns

51 This is a general remark. I am not sympathetic to the specific claim made in Wilbur R. Knorr, 'On the [*sic*] Principle of Linear Perspective in Euclid's *Optics*', *Centaurus* 34, 1991, pp.193–210.

at the front (near the top of the picture) and those at the back (lower down) are all but exactly the same, the back ones being a fraction larger: 3.55 cm for the front abaci, 3.6 cm for the back ones.[52] This is contrary to the proposition from Euclid to which we referred, but it fits rather well with the 'size constancy' effect whereby, as perceptual psychologists tell us, we perceive things as the same size if we know them to be so in reality. Euclid was dealing with mere geometrical optics, an ancient and unassailable mathematical discipline, which states that at a greater distance the object will subtend a smaller angle at the eye and therefore looks smaller. My appeal to a more modern discipline is anachronistic – and helps me to feel less foolish not to have noticed this near equality until I held a piece of squared paper up to the *intonaco*. In the terms of his own day, Masaccio was certainly not engaged in getting things as scientifically correct as possible. It seems highly likely that he knew that the construction rules he was using were an offshoot of the science of *perspectiva*, but he treats them merely as artisan rules. No doubt Masaccio had been taught the usual artisan rules, such as we find in Cennino Cennini. And to judge by his attitude to the prescription that the plaster for painting should be made as flat as possible, he felt free to treat them with some disrespect. Masaccio seems to have preferred to work on plaster that was not flat.[53] It is not really surprising that he felt equally free in his use of the rules of perspective, though we may guess that he had made some experiments or sketches to convince himself that he could get away with his departures from mathematical correctness.

Masaccio provides an interesting example because we are sure that he could have asked advice from Brunelleschi about the relevant mathematics. Other artists were probably mostly in a much weaker position. The abacus-school style of mathematics in which they had learned whatever they knew of geometry would certainly have inclined them to applying limited rules rather than thinking through each situation as it arose. Donatello, of course, appears here in his natural role as counterexample, but his solutions are not by any means always accomplished in terms of correct mathematics. In any case, there seems to have been general agreement that the rules of perspective as used in painting were intellectually respectable because they were an extension of the accepted partly mathematical science of *perspectiva*. The fact that the rules were applied in a less than rigorous way does not necessarily invalidate this position.

Much the same sort of thing was happening in music. The subject was accepted as a mathematical science and taught as such in the *quadrivium*, but practice regularly departed from theory. Available theory did not, in any case, go very far in explaining compositional practice, but the practice continued to fulfil its functions – which included exciting admiration and pleasure – and it did so with no apparent detriment to its status as a practical aspect of a true mathematical science. In the course of the sixteenth and early seventeenth

52 See Field et al., 'The Perspective Scheme of Masaccio's *Trinity*' (full ref. note 22).

53 The chief restorer from the Brancacci Chapel, who came up on the scaffolding in Santa Maria Novella in 1987, waved his hand at the wall and said something like 'Ah, quest' intonaco, riconosco Masaccio. Gli altri fanno liscio. Lui non cura. Tutti, tutti fanno liscio'. ('Ah that plaster! I recognize Masaccio. The others make it flat. He does not care. Everyone, everyone makes it flat.') He then recited a long list of painters that ended with Filippino Lippi. There, I suspect, speaks a voice from the past that earned Masaccio his apparently derogatory nickname. More importantly, however, it also suggests a method of ascription. This method passed my own test: the *intonaco* of Christ's face in *The Tribute Money* is absolutely flat. Conclusion: Masolino da Panicale did it. Masaccio's preference for irregularly curving *intonaco* may be connected with the avoidance of effects that make the position of the surface too easily readable. This suggestion is, of course, mere speculation.

centuries, the agreement between musical theory and musicians' practice became worse and worse, and the subject nevertheless continued to be considered mathematical.[54] In the fifteenth century, the cases of music and painting seem to have enough features in common to explain why a divergence from the ideals of theory did not crucially weaken the claims of painting to be associated with a learned tradition. As we shall see in Chapter 5, Piero della Francesca is sure that perspective is 'a true science'. His means of defending that opinion were, of course, mathematical.

Perspective in practice

Brunelleschi had invented a method for illusionism. His not being himself a painter fits in well with Manetti's account, in which the invention comes first and the panels afterwards. Moreover, both panels were in some respects unlike a normal picture: the first was set up as a peep-show and the second had its sky cut away so that one could look at it against the real sky, or perhaps check the outline of the group of buildings against reality. It is thus no great surprise that the technique Brunelleschi had invented – whatever it was – does not seem to have found an immediate application in the making of pictures. It was only from the 1430s onwards that changing fashion in art made perspective useful, as imparting an air of greater naturalism to pictures.

However, it is difficult to tell what degree of naturalism was considered to have been achieved. For instance, in the frescos that Masolino da Panicale (*c.*1383–?1447) painted in San Clemente, Rome, in the late 1420s, there are any number of sets of converging images of orthogonals in the series of scenes running round the walls of the chapel. However, none of the scenes seems designed to convey a strong sense of the third dimension. The 'perspective' is visible, but depth is much less so. Perhaps, in such a number of scenes, repeated effects of spaces behind the wall would have appeared confusing. No painter of a fresco could forget his picture was to become part of a wall. What worked for the isolated *Trinity* fresco, standing alone above an altar, would not necessarily work for a whole cycle of small scenes in a decorative scheme encompassing a complete chapel. The scheme of the Brancacci Chapel, where Masolino collaborated with Masaccio, has relatively few separate pictures. In particular, Masaccio combines three episodes of the story into one picture in *The Tribute Money*. In San Clemente, it may have been the client who took the relevant decisions, but in any case the result is a use of perspective that, to today's eye, is much less naturalistic than in, say, *The Tribute Money*.

The modern eye may well be a reasonable judge that there has been a decrease in spatial naturalism, but we cannot be at all sure how convincing the scenes in San Clemente really looked at the time. We are now thoroughly accustomed to reading perspective pictures but they were new to fifteenth-century viewers. Perhaps Masolino's contemporaries found the scenes as difficult as the truly three-dimensional images of holograms are for us nearly six centuries later. They are real and not real at the same time, simultaneously imperfect and convincing. The perspective effects in the San Clemente frescos may have made them fas-

54 That is, a belief in the essentially mathematical nature of music even survived the huge upheavals of what Claudio Monteverdi (1567–1643) called 'the second practice'. With hindsight we can see that an intellectually respectable concord between theory and practice was not in fact achieved until the early eighteenth century, in the musical treatises of Jean-Philippe Rameau (1683–1764).

cinating just for being what they were. In any case, we have quite a number of later examples in which the use of perspective seems to be in somewhat the same style. For instance, many paintings by Paolo Uccello (1397–1475) appear to be stiff with constructed perspective rather than provided with a convincing pictorial space. Perhaps they seemed startlingly spatial to his contemporaries, but they may have been equally satisfying just for being visibly 'done in perspective'.

Centric points are often used in a way that does seem designed primarily to produce a naturalistic pictorial space, but attempts at showing scenes as if they were truly seen from where the viewer is standing – that is, usually from below looking upwards – are exceedingly rare. Everyone must have been aware that the extra naturalism of showing the scene as if it were viewed from below would be paid for by making the picture hard to read. The difficulty in employing the low viewpoint is probably not merely that, as at a pageant, figures in front block the view of those behind, but also that attempting to get round this problem led to notable departures from the forms of composition hallowed by tradition. In any case, naturalism much more usually took the attenuated form of showing the scene as Alberti prescribes, as if it were seen by someone whose eye is level with that of a standing figure in the scene in question.

The perspective schemes of the panels in Ghiberti's second pair of doors for the Baptistery (completed 1452) follow this rule, presumably because a door had to retain a sense of solidity in the same way as a wall. A few years earlier, Donatello's doors for the Old Sacristy of San Lorenzo (1437–43) had emphasized their plane in a much more direct and conventional manner: the figures are mainly simply shown as standing out from a blank surface. The figures themselves are naturalistic and lively – which may go some way to explaining why, in the early 1460s Antonio Averino (called Filarete, *c.*1400–*c.*1469), a fellow sculptor who also practised as an architect and wrote about art, was to complain that Donatello had shown Apostles who looked like fencers.[55] The blank surface against which the figures appear, sometimes with small pieces of stage property such as a desk, shown in perspective, was apparently regarded by Donatello as presenting no impediment to a naturalistic treatment of the figures. It becomes as invisible as the traditional gold ground behind the figures in Masaccio's *Pisa Altarpiece*, and later those in Piero della Francesca's altarpiece for the confraternity of the Misericordia (probably 1460–62, see Chapter 3).

It is, of course, not in any way a new observation that, in the first few decades after Brunelleschi invented his 'rule', perspective was used in a wide variety of ways, not all of which were apparently designed mainly to enhance the naturalism of the pictorial space of the work concerned. However, since Piero della Francesca had a mathematical turn of mind, it may be as well to recollect that when he came to Florence in the 1430s the new works with which he was confronted displayed mathematical rules and their usefulness in a thoroughly ambiguous manner. We may harbour doubts about whether Piero was more sharp-sighted than Vasari in noticing the departures from optical correctness in Masaccio's *Trinity*, but he may surely be supposed capable of noticing that, in the cathedral, Uccello's spectacular *terra verde* fresco of Sir John Hawkwood (1436) showed the plinth as if it were seen far more steeply from below than the horse and its rider appeared to be. As we shall

55 Filarete [Antonio Averlino], *Trattato dell'architettura*, Book 23.

see, the ambiguity of the use of perspective in its generating as well as solving problems of naturalism (and some other more narrowly pictorial problems) seems not to have been lost upon Piero. It is unlikely that his artistic education in his native Borgo San Sepolcro would have included much that served as a useful introduction to the new style. However, his mathematical education must certainly have equipped him to look at it in a critical way. Just as Alberti, in *De pictura*, immediately called on his understanding of natural philosophy to help him explain the new art he found in Florence, so we may imagine that when Piero looked at the same works it was not only as a trained craftsman but also as a competent mathematician.

3

Piero's Early Life and Continuing Ties

Giorgio Vasari (1511–1574) was born in Arezzo and went back there to live in his old age. There is thus probably an element of local loyalty in his providing a relatively long biography of Piero della Francesca, who was born in Borgo San Sepolcro (now called Sansepolcro), which is the nearest town to Arezzo and can be seen from the top of the hill on which Arezzo is built. A telescoped and lightly fictionalized version of this view towards Borgo, adding elements of the view of the town from Anghiari, on a nearby hill, appears in the background of Piero's *Baptism of Christ* (National Gallery, London) (Fig. 4.2). Whatever local opinion may have been, Vasari's *Life* of Piero is nevertheless not as substantial as that of, say, Donatello, and the stress he lays on Piero's learned activity in writing about mathematics and perspective is almost certainly an indication that Piero's reputation as a painter was not very high among cognoscenti in the mid-sixteenth century.

On the other hand, Vasari does remark on Piero's skill in such matters as the naturalistic rendering of reflections on armour in one of the battle scenes in the fresco cycle of *The Story of the True Cross* (San Francesco, Arezzo). Vasari must surely have known these pictures since childhood, and he records in his *Life* of himself that he drew them as part of his training as an artist. So there may be an element of something more personal than mere local loyalty in him writing about Piero as he does. One should perhaps bear in mind that Piero's frescos were probably the first Renaissance pictures that Vasari saw, that is they marked his first encounter with what he was later to call the 'good new manner'. Moreover, the use of perspective and the sculptural solidity of Piero's figures gives his work an affinity with that of Masaccio, a painter admired by Vasari's hero Michelangelo Buonarroti (1475–1564). So the older Vasari may not have felt compelled to reject the younger Vasari's affection for Piero's work, or even to suppress it completely. More prosaically, Vasari may have had access to local information and family papers that have since been lost. In particular, Vasari claims that Luca Signorelli (*c.*1441–1523) of Cortona, one of Piero's most successful pupils, was one of his own relations.[1] This may also have helped to increase Vasari's sense of Piero's importance. In any case, what Vasari tells us is supplemented by what is now, thanks to recent work in local archives, becoming a decidedly helpful quantity of contemporary documents.[2]

1 Vasari's great-grandfather had moved from Cortona to Arezzo in the mid-fifteenth century. On Vasari's early life see P. L. Rubin, *Giorgio Vasari: Art and History*, New Haven and London: Yale University Press, 1995, Chapter 2, pp.61ff. For the possible family connection with Signorelli, see especially p.61.

2 See especially James R. Banker, *The Culture of San Sepolcro during the Youth of Piero della Francesca*, Ann Arbor: University of Michigan Press, 2003.

Piero's family

Piero was the eldest son of Benedetto della Francesca, of Borgo San Sepolcro, and his wife Romana, who came from the nearby village of Monterchi. Piero was presumably born in about 1412.[3] He had three younger brothers who survived childhood: Francesco (1413/14–1448), Marco (*c*.1415–1487) and Antonio (1415–1502) Piero's baptismal name is that of his grandfather, who had died in 1395. In his *Life* of Piero, Vasari claims that Piero had the surname della Francesca because his father died, leaving him in the care of his mother, who as a widow would have been known, from the family surname Franceschi, as 'la Francesca'. This is now known to be untrue: Benedetto, Piero's father, did not die until 1464, and his wife Romana had died before him, in 1459. Documents discovered in the 1980s show that the surname della Francesca was used by Piero's grandfather, also called Piero (and also with a father called Benedetto), so Vasari's explanation may apply to his case.[5] In contemporary documents from his native city and its environs Piero tends to be specified as the painter and the name is usually Pietro. On his perspective treatise his name is given as 'Petrus pictor burgensis', and similar forms are found on paintings. For instance, the *cartellino* in the small *St Jerome as a Penitent* (Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem) has the words 'Petri de Burgo opus MCCCCL'.

Borgo San Sepolcro is in the upper part of the valley of the Tiber. At the time of Piero's birth it was ruled by the Malatesta, lords of Rimini, but in 1431 it passed to the pope, and in 1441 to the Florentines, as it turned out definitively, since the city is now in the region of Tuscany. During Piero's lifetime, Borgo was strategically important, which, coming down to practical matters, meant that the city had substantial fortified walls and battles were fought close to them. The nasty look of the jam-packed weapon-wielding throng in Piero's fresco of *The Battle of Heraclius* (part of the cycle *The Story of the True Cross* in San Francesco, Arezzo) may well owe something to personal observation of warfare, either directly or through reports from friends. The city walls would have provided a vantage point for viewing the Battle of Anghiari (1440), which, though named for a small town in the nearby hills, was actually fought on the plain close to Borgo.

Both Piero's parents came from well-to-do families. On his mother's side, this is uncertain, because it relies heavily on deductions and assessments of probabilities, but on his father's it is thoroughly documented. Various members of the family served as councillors of Borgo. In his testament, dated 6 December 1390, Piero di Benedetto, the painter's grandfather, made a bequest to the local confraternity of the Misericordia. The family continued to have connections with this religious society, which may help account for its commissioning Piero (the grandson) to paint an altarpiece for it in 1445.[6] Another family associated with the confraternity was that of the Pichi, whose coat of arms appears on the first

3 James R. Banker, 'Piero della Francesca: gli anni giovanili e l'inizio della sua carriera', in *Città e Corte nell' Italia di Piero della Francesca. Atti del Convegno Internazionale di Studi Urbino, 4–7 ottobre 1992*, ed. Claudia Cieri Via, Venice: Marsilio, 1996, pp.85–95. See also Banker, *The Culture of San Sepolcro* (full ref. note 2).

4 There is a family tree in Banker, *The Culture of San Sepolcro* (full ref. note 2), p.99.

5 James R. Banker, 'The Altarpiece of the Confraternity of the Misericordia in Borgo Sansepolcro', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no. 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art, 1995, pp.21–35, esp. p.21 and note 4 (p.32).

6 See Banker, 'The Altarpiece of the Confraternity' (full ref. note 5).

page of the manuscript of Piero's *Trattato d'abaco* now in the Laurentian Library in Florence. It seems highly likely that the unidentified person at whose request the work was written was a senior member of this family.[7] On the other side of his own family, it seems likely that Piero's mother's connection with Monterchi played some part in his being commissioned to paint the fresco of the *Madonna del Parto* in what is now the mortuary chapel of its cemetery.

Piero's education

Piero's father worked hard to increase both his family's wealth and its social status. This is clearly apparent from the documents surviving in local archives. The family had been one of artisans, workers in the manufacture of leather goods. Benedetto moved into retail trade in leather goods, and then into wholesale trading in skins and other commodities. He bought town houses and, eventually, a plot of agricultural land.[8] Benedetto's financial interests included woad, a very important dyestuff at the time. Thus Piero was born into an upwardly mobile artisan family and, given his father's aspirations, it would have been entirely appropriate for all his sons to be given some grounding in 'commercial arithmetic' – that is, the kind of mathematics taught in abacus schools.

However, as we have seen, the evidence is that there was no such school in Borgo San Sepolcro at the time. It is conceivable that Piero was sent away to school – perhaps to the nearby city of Arezzo[9] – but there is no direct evidence for Piero leaving his home town at an early age, and it seems more likely that he was taught some mathematics privately, perhaps by his father or some other family member or family friend. As mathematics is a subject in which a child's ability often becomes apparent at an early age, it is likely that Piero's mathematical education, once begun, would rapidly have seemed a profitable enterprise. Thus, though details are lacking, at least at present, there is no great mystery about why Piero should have acquired an education in practical mathematics. Borgo never became famous as a centre of learning, but in the fifteenth century it was not entirely a backwater and did have an intellectual life of its own. In the following century its citizens included the distinguished algebraist Rafaelle Bombelli (1526–1572), whose presence was occasioned by Borgo's proximity to the valley of the Chiana, a river that meets the Tiber near the city of Arezzo. The Chiana was liable to unpredictable flooding and Bombelli was one of many engineers employed in attempting to bring the flooding under control. His predecessors included Leonardo da Vinci, and his successors Galileo's pupil Vincenzo Viviani (1622–1703).

This continuing connection with engineering, one of the crafts closely associated with practical mathematics, suggests that mathematical education would eventually have been established in Borgo San Sepolcro, as it was in other towns in the vicinity. Bombelli and his predecessors really did need to know the answers to those rule of three problems in which one is asked how long it will take a certain number of men to dig a ditch. Curiously, there is no problem on ditch digging in the appropriate part of Piero's *Trattato d'abaco*, which,

7 See Chapter 1, p.30 for Piero's account of why the treatise was written. There is a photograph of the page concerned in Piero ed. Arrighi, p.38.

8 For details of Benedetto's transactions and the documentation, see Banker, *The Culture of San Sepolcro* (full ref. note 2).

9 On Arezzo see Robert Black, 'Humanism and Education in Renaissance Arezzo', *I Tatti Studies (Essays in the Renaissance)* 2, 1987, pp.171–237.

as promised in the author's introductory remarks, specifically addresses problems of concern to merchants.

It is unfortunate that we do not have precise information about Piero's formal education. However, as we have seen in Chapter 1, his writings do provide fairly extensive indications concerning the more significant matter of what he knew in later life. It is thus to be presumed that it was in his childhood that he learned to read and write, at least in the vernacular, perhaps by attending the most elementary classes at the grammar school in Borgo San Sepolcro, and that at the same time he also studied 'the abacus'.[10] This would have been a normal enough education for a boy with Piero's family background.

On the other hand, if Benedetto intended his son to join him in the family business activities, which seems likely since the boy's skill at mathematics must have made him seem well suited to such an occupation, there was no necessity for Piero to have been taught Latin. Piero's brother Francesco, who was to become a monk in the Camaldolese abbey in Borgo San Sepolcro from 1428 to 1448, probably attended the grammar school in the town, and it thus seems probable that Piero did so too.[11] In the first year at such a school pupils were taught to read and write in the vernacular as a preliminary to learning Latin in subsequent years. This aspect to Piero's education is of interest because the traditions of learning in Latin and the vernacular were in many ways quite separate, and some of Piero's mathematical work appears to be crossing boundaries.

Vasari's biography of Piero is not notable for presenting things in chronological order, so it is not clear whether the assertion that Piero acquired an intimate knowledge of the work of Euclid on regular bodies, which appears towards the end of the *Life*, should be taken to refer to Piero's formal education or to later independent study.[12] There is strong evidence from his own writings that Piero had indeed studied Euclid with care.[13] However, while the study of Euclid's texts would have required some knowledge of Latin, it does not require great linguistic proficiency. First, the wording in the *Elements* is moderately simple, and the vocabulary necessarily limited, with all new terms properly defined at the beginning of each book. Second, the development of arguments in proofs is rigorously logical, which allows wide scope for intelligent guesswork on the part of someone truly capable of understanding the mathematical content. For someone like Piero, with an interest in mathematics, reading Euclid's *Elements* would in fact be quite a good way of improving a limited skill in reading Latin. In any case, the humanist style of Piero's handwriting suggests an interest in Latin writings.[14]

Further evidence pointing in the same direction is provided by Piero's introduction to the third book of *De prospectiva pingendi*. This introduction consists largely of a defence of artificial perspective, first on the grounds of its scientific truth and then on the grounds that this was known to the ancients, who themselves used perspective in making their pictures.

10 As suggested in Banker, *The Culture of San Seplocro* (full ref. note 2), pp.89–92.

11 On Francesco, see James R. Banker, 'Piero della Francesca, il fratello Don Francesco di Benedetto e Francesco dal Borgo', *Prospettiva* 68, October 1992, pp.54–6.

12 The texts of the 1550 and 1568 editions of Vasari's *Life* of Piero, and passages that mention him in other *Lives*, are reproduced in Battisti, vol.2, pp.640–2, 642–3. An English translation of the *Life* of Piero is included in Giorgio Vasari, *The Lives of the Artists*, 2 vols., trans. George Bull, London: Penguin Books, 1985.

13 See Chapters 4 and 5.

14 This has been pointed out in Paul Grendler, 'What Piero Learned in School: Fifteenth-Century Vernacular Education', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no. 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art, 1995, pp.161–74.

That is, we first have a technical defence and then a characteristically humanist one. The technical defence will be discussed later, when we have examined the actual mathematical content of the treatise.[15] The humanist historical–rhetorical defence includes a list of great painters of antiquity:

> By following this practice [of perspective] many ancient painters acquired immortal fame. Such as Aristomenes, Thasius, Polides, Apello, Andramides, Nitheo, Zeusis, and many others.[16]

Except for the name of Apelles, this list appears to be taken from Vitruvius' introduction to *De architectura*, Book 3, though possibly not directly. In fact, if we omit Apelles, Piero's list agrees closely, in its non-classical orthography, with the one given in the first printed edition of *De architectura*, published in Rome in 1486, and therefore presumably to be found in the manuscript tradition.[17] However, a similar list is also given in possible indirect sources, such as Ghiberti's *Commentaries*. Apelles, who is mentioned repeatedly by Pliny and in Filippo Villani's *De origine civitatis Florentiæet eiusdem famosis civibus* (1381–2) is discussed at length by Leon Battista Alberti in *De pictura/Della pittura*.[18] There is thus nothing of exceptional erudition in Piero's list. His humanist defence of perspective may well be a product of his later self-education rather than a direct reflection of his formal education in childhood. Nonetheless, it suggests a reading knowledge of Latin that is most likely to have had some basis in Piero's early years.

This humanist passage defending perspective is the only one in which Piero's tone comes close to echoing Alberti's. However, in this context, it is perhaps worthy of note that all the manuscripts of *De prospectiva pingendi*, vernacular as well as Latin, have the Latin title, which may thus be Piero's own choice.[19] The Latin title should be seen as implying a link with the learned tradition in much the same way that Luca Pacioli's use of the Latin title *De divina proportione* for his vernacular mathematical work, published in Venice in 1509, is clearly intended to remind one of its relation to Euclid.[20] Pacioli, who was Piero's countryman, tells us that Piero originally wrote his treatise on perspective in the vernacular and that it was then translated into Latin:

> which [that is, write on perspective] he did in the vernacular; and then the famous orator and poet and rhetorician, in Greek and Latin (his constant associate and also [his] countryman) Maestro Matteo turned it into Latin most elegantly, word for word . . .[21]

15 See Chapter 5.

16 Piero della Francesca, *De prospectiva pingendi*, Book 3, Introduction: Parma MS, p.32 recto; BL MS, fol.37 recto (where Nitheo has been emended to Nitro); Piero ed. Nicco Fasola, p.129.

17 Some later Renaissance editors give slightly different readings of the names. Modern editions have 'Aristomenes, Thasius, Polycles, et Androcydes ⟨Cyzenice⟩ni, Theo Magnes ceterique'.

18 See M. Baxandall, *Giotto and the Orators*, Oxford: Clarendon Press, 1986, p.146 for Villani's Latin text.

19 The relation of Piero's perspective treatise to the learned tradition is discussed in Chapters 5 and 7 below.

20 In the part that deals directly with 'divine proportion', that is, the first section of the book, all Pacioli's propositions are lifted from the *Elements* and readers are referred to Euclid for the proofs. Moreover, Pacioli preserves Euclid's order, so the progression is far from logical, since Euclid was dealing with these matters purely incidentally. Pacioli's claims to originality in this work are tenuous. He essentially provided no more than a vernacular extract from Euclid, on a particular topic and without proofs.

21 Luca Pacioli, *Summa de arithmetica, geometria, proportioni e proportionalità*, Venice, 1494, Distinctio VI, trattato I, articolo II, fol. 68 verso, lines 28ff: 'el qual lui fece vulgare: e poi el famoso oratore: poeta: e retorico: greco e latino (suo assiduo consotio: e similmente conterraneo) maestro Matteo lo recco alengua latina ornatissimamente de verbo ad verbum.'

The existence of fifteenth-century manuscripts of the Latin text (for instance, that in the British Library, London, Add. MS 10366) confirms Pacioli's assertion that the translation was made in Piero's lifetime. Indeed, careful work in local archives has now permitted a precise identification of the translator, who was a lawyer.[22] All indications provided by verbal divergences seem to confirm that the Latin is indeed a translation, but there are also divergences of content that indicate that Piero himself took some part in the preparation of the new text.[23] At the least, Maestro Matteo must have been willing to discuss translation problems with Piero, so it would seem that Piero had access to a competent Latinist if he happened to need help in reading an awkward passage of text.

There is another piece of evidence that Piero may have read Vitruvius: the first two sentences of the introduction to the *Libellus de quinque corporibus regularibus* are a condensed and very slightly adapted version of the second paragraph of Vitruvius' introduction to *De architectura*, Book 3. (The import of both passages is, roughly, that working for the great helps artists to be remembered.) In the absence of other evidence, it is reasonable to suppose that Piero's *Libellus* was written in the vernacular and that Maestro Matteo made the translation found in the single surviving manuscript text of the work, which is in the Vatican Library and is in Latin.[24] However, it seems unlikely that this passage paraphrasing Vitruvius is entirely the work of the translator, although he may be responsible for the fact that some phrases seem to have been directly quoted from Vitruvius' text. The lists of names of successful and unsuccessful artists are very close to those given in *De architectura* – the only two additions being the names of Apelles and Praxiteles – so the most natural explanation would be that, at the time this manuscript was written, Piero was acquainted with Vitruvius' text.

Training as a painter

The dating of almost all of Piero's works – written and painted – is doubtful, but there is general agreement in learned publications over the last twenty years or so that we have no surviving paintings that can be dated earlier than 1445. The possible exceptions are the undocumented *Madonna and Child* of the Contini-Bonacossi Collection (Florence), whose ascription to Piero has been doubted, and the *Baptism of Christ*. We are thus thrown back on written sources. So far, the earliest documentary evidence for Piero's activity as a painter connects him with Antonio di Giovanni d'Anghiari in 1432, when they worked together on an altarpiece for the church of San Francesco in Borgo San Sepolcro.[25] By this time, Piero, aged about twenty, was no longer an apprentice. We may note, also, that this association with Antonio d'Anghiari does not actually prove that Piero had been his apprentice, though

22 James R. Banker, 'Piero della Francesca's Friend and Translator: Maestro Matteo di Ser Paolo d'Anghiari', *Rivista d'Arte*, 4th series, vol.8, 1992, pp.331–40.

23 See J. V. Field, 'Piero della Francesca and the "Distance Point Method" of Perspective Construction', *Nuncius* 10.2, 1995, pp.509–30, and Chapter 5 below. There are Latin annotations in Piero's hand in some Latin manuscripts of his perspective treatise; see Giovanna Derenzini, 'Note autografe di Piero della Francesca nel codice 616 della Bibliothèque Municipale di Bordeaux. Per la storia testuale del *De prospectiva pingendi*', *Filologia Antica e Moderna* 9, 1995, pp.29–55.

24 Vatican codex urbinas 632. Printed in Piero ed. Mancini. The passage in question here is on p.488. An English translation is given in Appendix 7 below.

25 James R. Banker, 'Un documento inedito del 1432 sull'attività di Piero della Francesca per la chiesa di San Francesco in Borgo S. Sepolcro', *Rivista d'Arte*, 4th series, vol.6, 1990, pp.245–7; Frank Dabell, 'Antonio d'Anghiari e gli inizi di Piero della Francesca', *Paragone* 35.2, 1984, pp.71–94.

.1 Pietro Lorenzetti (active 1320–48), *Polyptych*, 1320, tempera on panel, 298 × 309 cm, main altar of Pieve di Santa Maria, Arezzo.

this seems the likeliest explanation of it. Such an explanation is consistent with what we are told by Vasari, namely that Piero had been apprenticed to a painter (unnamed) at the age of fifteen, but this did not interrupt his pursuit of mathematical studies.

No works by Antonio d'Anghiari are known to survive. Documentary evidence suggests his reputation was purely local and that much of his activity was concerned with painting banners or heraldic devices for architecture, such as the papal arms to be placed above the city gates. This kind of work normally involved using templates to ensure exact repetition of recurring elements, and it is possible that Piero's later habit of transferring designs by means of cartoons has its roots in this early phase of his career. Work on heraldic devices might also have fostered the concern with compositional balance that we find in Piero's later work, and in particular his apparent ease with designs that, especially if one concentrates on the drawing rather than the painting, show a considerable degree of simple symmetry.[26]

Piero's father would no doubt have seen to it that his son was apprenticed to someone known to be competent at his craft, which probably indicates a painter working in a style that was generally acceptable in the locality. Thus our best guide to Antonio's style, and hence to Piero's training, is that what is known of the visual arts in Borgo, and nearby Arezzo, in the years from 1412 to 1432 shows that the predominant style was conservative and heavily indebted to that associated with Siena. The recent art that Piero saw in his early youth aspired to the qualities that are to be found in the altarpiece that Pietro Lorenzetti (active 1320–48) painted for the Pieve di Santa Maria in Arezzo (Fig. 3.1).

For all its obvious difference in style from Piero's own works, this altarpiece has certain enduring painterly qualities in common with them: subtle colours, strong control of the surface composition, and an overall calm elegance that links it with the long tradition of subordinating undue naturalism to theological truth. There were numerous more recent works in a similarly Sienese style to be seen in Arezzo in Piero's youth. It seems likely that young Piero della Francesca, like young Giorgio Vasari after him, enjoyed looking at pictures whose styles differed markedly from that which was to claim his allegiance as an adult. Piero's first experience of what Vasari was to call the 'good new manner' must have occurred some time in or before 1439, when Piero is mentioned as the companion of Domenico Veneziano (active 1438, died 1461) in painting frescos in the church of Sant' Egidio in Florence.

Unfortunately, although Domenico Veneziano is a much less unknown quantity than Antonio d'Anghiari, we do not know a great deal about his life in the years just before 1439. He is known to have executed commissions in Perugia in the mid-1430s, and it seems possible that it was there that he and Piero first met.[27] The two painters seem to have worked together again on several occasions in the following decade, so it appears they got on well with one another.[28] Since Domenico's work shows him to have been thoroughly sympathetic to the new style, we must assume Piero felt more or less the same way. In default of other candidates, it seems possible that it was Domenico, or one of his associates, who first introduced Piero to the mathematical procedures of artificial perspective. Piero was no doubt an encouragingly apt pupil.

What is 'early'? Dating some of Piero's works

Visiting Florence must have been as powerful an experience for Piero as it had been for Alberti a few years earlier. Nor did the impact fade. Detailed reminiscences of Florentine

26 These suggestions are made by James R. Banker, who has also found out much about the activities of Antonio d'Anghiari, see Banker, *The Culture of San Sepolcro* (full ref. note 2).

27 See Banker, *The Culture of San Sepolcro* (full ref. note 2), pp.213ff.

28 Piero's association with Domenico Veneziano is discussed in detail in R. Lightbown, *Piero della Francesca*, London: Abbeville Press, 1992, esp. pp.14–17.

architecture can be found in works painted late in his career,[29] and the *Montefeltro Altarpiece*, almost certainly painted after 1469, contains a high-visibility perspective rendering of a coffered barrel vault with a rosette in each coffer, which is the most explicit of Piero's homages to Masaccio's *Trinity* fresco. Even if Piero had been liable to register 'influences' instantly and with noticeably decreasing effect over time, as would seem to be the historian's ideal, one might find it difficult to disentangle the effects of working with Domenico Veneziano from those of seeing new art in Florence in the late 1430s. As it is, the task at least has the merit of being obviously not worth attempting. All the same, Piero's art can hardly be understood unless we recognize it as subject to development over time, so we do need to put works into a chronological sequence. That unfortunately lands us in a potential circularity: stylistic dating of Piero's works is to some extent unavoidable, since documentation is scanty.

We have only four pictures with signatures: *St Jerome as a Penitent* (Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem) (Fig. 3.2), *St Jerome with Girolamo Amadi* (Galleria dell'Accademia, Venice) (Fig. 3.3), *The Flagellation of Christ* (Museo Nazionale delle Marche, Urbino) (Fig. 5.28) and the fresco of *Sigismondo Malatesta before St Sigismund* (Tempio Malatestiano, Rimini) (Fig. 6.1). Two of these also have dates: 1450 on the Berlin *St Jerome* – though we may note that the *cartellino* curls round coyly in such a way that it would hide the next letter after the L of MCCCCL – and 1451 on the Rimini fresco. These two pictures look very different because of differences in scale and medium and so on, but there is no problem in seeing them as characteristic of Piero's work as a whole. It is probably this apparently innocuous fact that accounts for the much less innocuous fact that an uncomfortably large proportion of Piero's extant pictures have, at one time or another, been dated to about 1450. The two secure comparison pieces have made their dates magnetic.

The perils of stylistic dating are also to be seen in what we now know of the progress of Piero's work on the altarpiece for the Confraternity of the Misericordia. The commission for the work is dated 1445 and stipulates that Piero is to be paid the substantial sum of 150 florins (these are gold florins and each is stipulated as having the value of 5 *lira* and 5 *soldi*). The size of the payment indicates that his countrymen recognized Piero as an established painter. However, thanks to the hazards of survival we, unlike the members of the confraternity of the Misericordia, are compelled to regard the altarpiece as an 'early' commission. Whether it is an 'early' work is now open to serious doubt.[30] The archives have yielded evidence of what must be almost all the payments that were made to Piero in connection with this commission. This is essentially a simple matter of arithmetic: the payments add up to nearly a total sum of 150 florins (that is, 787 *lire* and 10 *soldi*). From the sizes and dates of the payments, it has been argued convincingly that Piero did not actually start painting until about 1460 and had finished in 1462. As far as any historical argument based on such documents can be watertight, this one is. If the paintings in question were not extant, murmurs of dissent would be non-existent, so it seems only reasonable to ignore the much shakier stylistic arguments for an earlier date and see instead what we may learn by accepting the dating for which we now have good documentary evidence. The

29 Christine Smith, 'Piero's Painted Architecture: Analysis of His Vocabulary', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no. 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art, 1995, pp.223–53.

30 What follows is based on Banker, 'The Altarpiece of the Confraternity' (full ref. note 5).

3.2 Piero della Francesca (*c.*1412–1492), *St Jerome as a Pentitent*, signed and dated 1450 on *cartellino* bottom right, tempera on panel, 51.5 × 38 cm, Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem.

3 Piero della Francesca (*c*.1412–1492), *St Jerome with Girolamo Amadi*, tempera on panel, 49 × 42 cm, signed in letters
own as carved on the tree trunk, lower left, Galleria dell'Accademia, Venice.

Misericordia Altarpiece is now seen to have been painted after the Berlin *St Jerome* and the Rimini fresco, after Piero's visit to Rome (1458–9), and almost certainly after the completion of the fresco cycle *The Story of the True Cross* in San Francesco, Arezzo. Partly thanks to the magnetic effect of the early 1450s, a date of 1460 would also put the *Misericordia Altarpiece* after the dates a variety of scholars have suggested for several other pictures.

There are, of course, some works for which the magnetic effect of 1450 seems relatively reasonable. One of them is the Venice *St Jerome* (Fig. 3.3), which was probably painted in the city where it is still to be found.[31] In this picture St Jerome is shown reading – indeed the presence of the worshipper seems to be an interruption – whereas in the Berlin picture the saint is praying, looking towards a cross at the extreme right whose outline is barely visible on the original and cannot be made out at all on a photograph. The closeness of the subjects and the repetition of several stage properties suggests that someone who had seen one of them, or had been given a detailed description of it, might have commissioned the other. Another thing the pictures have in common is their sense of drama. In the Berlin panel the story is clear: the passionate intensity of the saint's glance is directed to the cross. Since the composition of the picture is markedly asymmetrical, it is of interest that the frame, which is clearly original, gives absolutely no indication that the panel was ever part of a diptych. The asymmetry – which may have been accentuated by the loss of colour caused by damage to the lower part of the panel – increases the overall dramatic effect. In the Venice picture it is much more difficult to interpret what we see. The saint is caught part way through the action of turning a page of his book, which would seem to indicate he has looked up suddenly, startled, as the Virgin Annunciate sometimes is by the arrival of the angel. But the kneeling donor cannot have appeared as suddenly as an angel. Perhaps there is no story and Piero has merely sought to make his saint look lively.

Since we have only four signed pictures by Piero, it may be reasonable to see some significance in the fact that two of them show St Jerome. St Jerome was a favourite subject for pictures in studies, since his activity in translating the Bible into Latin from Hebrew and Greek was emblematic of the pious use of scholarship. The Berlin panel does not show St Jerome at his learned work, but there are allusions to it in the books shown behind him. Several scholars have suggested that Piero's reason for signing the two *St Jerome* panels may have been that he wished to associate himself with the humanist learning that the saint represented. In the case of the panel now in Venice, the name of the donor actually appears in the painting, so Piero's inclusion of his own name as well may reasonably be read as associating himself with the donor's veneration of the saint. The signature on this panel is rather discreet, being shown in letters apparently carved into the tree stump that supports the cross on the left of the picture. The signature on the Berlin panel is displayed more flamboyantly, using the characteristically Venetian device of a *cartellino* and, like the signature on the Venice panel, it is also associated with the support of the cross, which in this case plays an important part in the composition, being essentially the focus of the drama. His rather emphatic treatment of the signature may be read as indicating Piero's devotion to St Jerome, but it might also be read as an act of self-advertisement associated with a painting done far

31 This is argued in Lightbown (full ref. note 28), pp.79–81, who notes that the suggestion was also made by Longhi. Some of the architecture shown in Piero's paintings also suggests he knew Venice; see Smith, 'Piero's Painted Architecture' (full ref. note 29).

from Piero's native place, or, because it was portable, liable to be seen by people who did not otherwise know the painter's work.

The two paintings of St Jerome clearly bear a superficial similarity to the *Baptism of Christ* now in London. In the next chapter we shall discuss their possible relationship in more detail. For our present purposes it is enough that one of the panels of St Jerome, and possibly also the other one and the *Baptism*, were painted before the panels of the *Misericordia Altarpiece*. However, the altarpiece can reasonably be described as old-fashioned in certain respects in comparison with the Berlin panel, which shows the saint in a naturalistic landscape. On examination, all these old-fashioned features of the *Misericordia Altarpiece* can be ascribed to the patrons – but we must also bear in mind that Piero did accept, and eventually fulfil, the commission. The overall design of the polyptych was clearly fixed in advance. Surviving documents even tell us something of the story of the carpentry.[32] Unfortunately, the altarpiece was dismembered in the seventeenth century, and the original frame has now been lost.

The gold ground was also certainly given, but may well have seemed less important to Piero than it does to us; Masaccio's *Pisa Altarpiece* has convincing three-dimensional figures against gold (Fig. 3.4). Since Piero did likewise, to the extent of producing the most Masaccesque piece of painting we know by him (see, for example, Fig. 3.5), it seems as well to point out that, although we know for certain only that Piero was working in Florence in 1439, it is not at all difficult to get to Pisa from Florence. The journey could be made overland or, probably in more comfort, by water, along the Arno.

3.4 Masaccio (1401–1428), *Carmelite Saint* from the *Pisa Altarpiece*, *c.*1426, tempera on panel, 38 × 12.5 cm, Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem.

32 See Banker, 'The Altarpiece of the Confraternity'(full ref. note 5), and Banker, *The Culture of San Sepolcro* (full ref. note 2), chapter 5, esp. pp.175ff.

3.5 (*above*) Piero della Francesca (*c.*1412–1492), *St Benedict* from the *Misericordia Altarpiece*, *c.*1460–62, tempera on panel, 54 × 21 cm, Museo Civico, Sansepolcro.

3.6 (*facing page*) Andrea Mantegna (1431–1506), *St Scholastica* (?) from the *St Luke Polyptych*, 1453–4, tempera on panel, 97 × 37 cm, Pinacoteca di Brera, Milan.

It is surely not unreasonable to suppose that, having seen some frescos by Masaccio, Piero might be prepared to take a little trouble to see an altarpiece by him. The *Carnesecchi Altarpiece* in Santa Maria Maggiore, Florence, on which Masaccio had collaborated with Masolino in about 1423, seems, on the evidence of known fragments, to have had its three main panels painted by Masolino and only the predella panels by Masaccio.[33] In any case, Masaccio's strongly sculptural treatment of figures, in fresco as well as in tempera, has much in common with what we find in Donatello. In addition, if Piero was indeed in Venice in the late 1440s or early 1450s he may well have seen the figures that Andrea del Castagno (*c.*1421–1457) painted there, in the vaulting of the Cappella d'Oro of San Zaccharia. Castagno's figures stand out with his habitual metallic clarity against their blue background.[34] We find similarly strong-minded attitudes in roughly contemporary works that bear no direct relation to Piero's. For example, Andrea Mantegna (1431–1506), in his *St Luke Polyptych* of 1453–4 (Galleria di Brera, Milan), painted for the church of Santa Giustina in Padua, makes short work of any 'problem' with a gold ground. The pinks and greens mix well with the gold, the linear outlines of the standing figures have a nervous elegance equal to anything in the fourteenth century – with hindsight, pointing forward to Mantegna's

33 See Paul Joannides, *Masaccio and Masolino: A Complete Catalogue*, London: Phaidon, 1993, pp.64–9, and cat. no.13, pp.350–5.

34 If Piero returned to Florence after 1444 or so, he might also have seen Castagno's cycle of frescos of the Passion of Christ in Sant'Apollonia, in which the figures are again extremely strongly modelled. If he returned after 1450, he may also have seen Castagno's *Assumption of the Blessed Virgin* (Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem), painted in 1449 or 1450 for the church of San Miniato fra le Torre. Like Piero's *Misericordia Altarpiece*, this latter picture has strongly modelled figures against a gold ground.

3.7 Piero della Francesca (*c.*1412–1492), *Misericordia Altarpiece*, *c.*1460–62, tempera on panel, central panel 139 × 91 cm, Museo Civico, Sansepolcro.

engravings – and the tiny receding orthogonals of the marble slab bases are more decorative than real, but once one is close enough to see the modelling the figures are entirely solid (Fig. 3.6).[35] Piero himself was to show similar confidence in the altarpiece he painted for the convent of Sant'Antonio in Perugia, whose design was also highly conventional and indeed provincial as well (see Chapter 6).

More awkward than the gold or the old-fashioned framing of the *Misericordia Altarpiece* was the subject of the central panel. The iconography of the Madonna of Mercy ordained that the Virgin should be shown considerably larger than the people sheltered by her outspread cloak. Piero seems simply to have taken straightforward naturalism as far as it would go and made the central figure a giantess (Fig. 3.7). He must have decided there was no way he could make her seem natural, so he concentrated instead on giving her a dignity that would fit with her size, effectively following the well-established older convention in which size was an expression of the narrative or theological importance of the figure concerned. The result is a figure that, even by Piero's standards of sternness and austerity, is

35 It is possible that Piero visited Padua before painting the *Misericordia Altarpiece*, but Mantegna was probably not famous enough at the time for Piero to take the trouble to seek out this early work.

3.8 Piero della Francesca (*c.*1412–1492), *The Crucifixion* from the *Misericordia Altarpiece*, *c.*1460–62, tempera on panel, 81 × 52.5 cm, Museo Civico, Sansepolcro.

unusually stern and austere. His not having quite got away with this, in the naturalistic terms that apply to most of his pictures, is probably one of the reasons that scholars have, on the whole, been willing to accept a rather early date for the altarpiece.

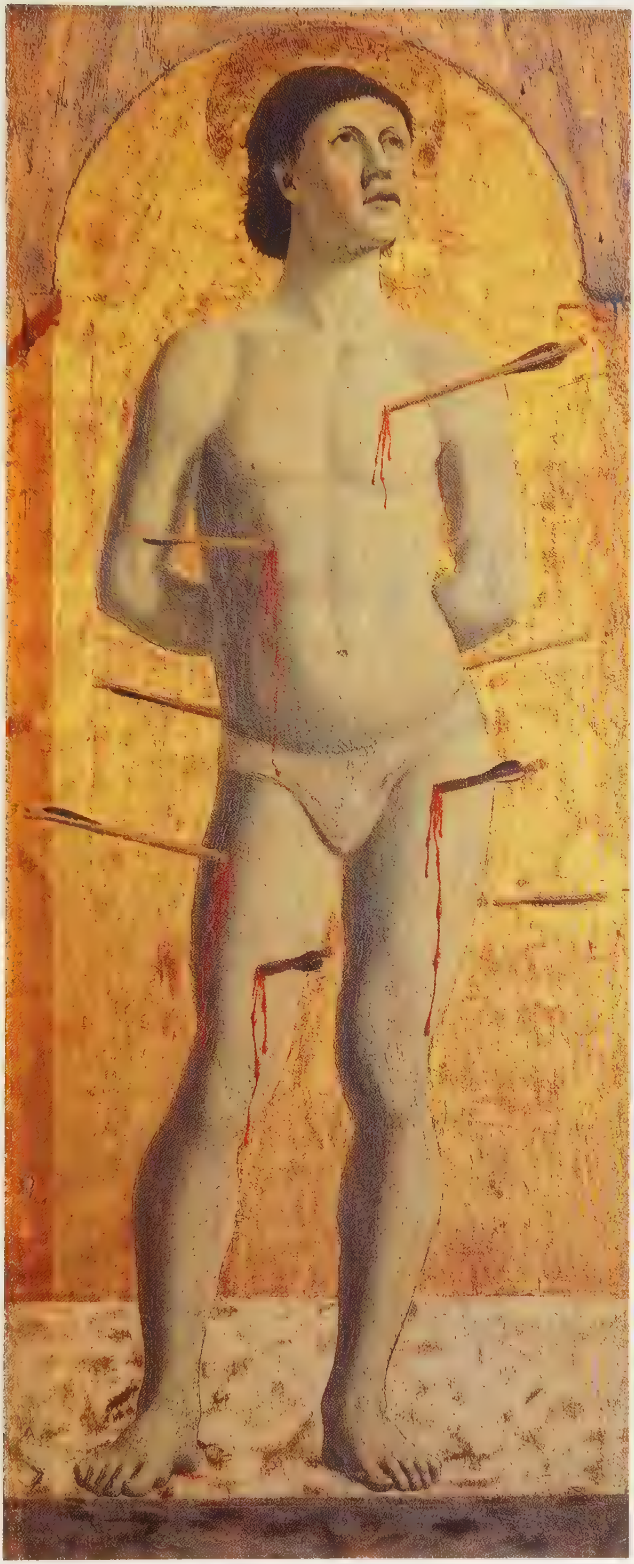
If we remove the central panel, we are left only with saints whose debt to Florentine painting is instantly obvious. However, their forceful modelling may be not only an acceptance of Florentine style but also a reaction to the gold ground. Piero is doing what Masaccio and Mantegna did on altarpieces and Donatello did on the doors in the sacristy of San Lorenzo, and he is doing it to the same effect. One may treat this as a case of 'influence' or one may prefer to regard it as a piece of problem-solving in which artists with similar stylistic allegiances came up with rather similar solutions.

What seems less accountable in Piero's treatment of the various panels, is that the crucifixion scene of the *Misericordia Altarpiece* should be so dramatic. Here the sense of drama may be not only a link with Masaccio and Donatello but also a link with the *St Jerome* panels. It might also have been part of the programme supplied by the Confraternity: penitents and flagellants presumably tended to see Christ's suffering on the Cross in dramatic terms rather than contemplative ones. Piero's crucifixion scene (Fig. 3.8) in some ways closely resembles Masaccio's (Fig. 2.5), though Piero has prudently designed his scene to be looked up at rather less steeply than Masaccio's. Moreover, the scenes are not so similar as to imply that Piero must necessarily have seen Masaccio's version before he painted his own – though it seems perfectly possible that he had.

There is, however, a serious problem with dating the *Misericordia Altarpiece* to the early 1460s. The problem relates to the St Sebastian, shown in Figure 3.9. St Sebastian is, as usual, portrayed in the course of surviving the attempt to shoot him dead with arrows. Thus a certain degree of tranquillity may be understandable, and the upward glance may be supposed to be directed towards the crucified Christ at the top of the altarpiece. In many pictures, Piero shows himself to be accomplished at conveying tranquillity. In this St Sebastian the effect is not completely happy. The strength of the lighting gives the figure a rather wooden appearance that is at variance with the relaxed stance that puts most of the weight onto the saint's left leg. The use of heavy cast shadows and what seems to be a too literal use of preliminary studies from a live model have resulted in a figure that is lacking in the assurance that one finds in most of Piero's other work. Indeed, looking at St Sebastian one might be tempted to think that it is just as well that on the whole Piero's figures tend to keep their clothes on.

Not all of them do so. In particular, there are accomplished nude and almost nude figures in the scene of *The Death of Adam* in *The Story of the True Cross* (San Francesco, Arezzo), which – if we are to trust the documentary dating of the *Misericordia Altarpiece* – antedate St Sebastian. The best explanation of the aberrant St Sebastian would seem to be that the figure was not actually painted by Piero himself, though it did make use of a drawing supplied by him. The work of several of Piero's later imitators and that of his direct pupil Luca Signorelli, in his less happy moments, shows just this woodenness. Piero's skill in conveying solidity seems to have been found rather hard to copy. The figure of St Sebastian may have been executed by another hand, perhaps that of the painter responsible for the predella panels.[36] The predella panels are not notable for cast shadows, but some passages do show a rather harsh treatment of highlighting; for instance, in the figure of the Risen Christ in the *Noli me tangere*.

36 Banker, 'The Altarpiece of the Confraternity' (full ref. note 5), p.27, gives additional evidence to support his own earlier identification of this painter as Don Giuliano Amidei da Firenze.



3.9 Piero della Francesca (*c.*1412–1492), *St Sebastian* from the *Misericordia Altarpiece*, *c.*1460–62, tempera on panel, 108 × 45 cm, Museo Civico, Sansepolcro.

3.10 Domenico Veneziano (active 1438, died 1461), *Adoration of the Magi*, probably *c.*1440, tempera on panel, diameter 84 cm, Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem.

Learning from Domenico Veneziano

The *Misericordia Altarpiece* confronts us unambiguously with some of what Piero could have learned by looking at works of art in Florence in the late 1430s, but it also warns us that the passage of about twenty years is much less apparent than we might have hoped if we wished to make stylistic datings of Piero's works in terms more subtle than simply saying that they postdate his first visit to Florence.

A rather similar situation would seem to obtain in regard to what Piero learned from Domenico Veneziano. Domenico was probably some years older than Piero, and is believed to have worked in northern Italy.[37] Certain elements in his style apparently owe a debt to the occasionally fussy prettiness of detailing found in, say, Pisanello (1395–*c.*1455) and Gentile da Fabriano (*c.*1370–1427). It may have been from Domenico that Piero adopted the use of the detailed rendering of landscape found in the Berlin *St Jerome* (Fig. 3.2). The landscape is so prominent in this picture that the catalogue of the Berlin collection describes it in rather Netherlandish terms as *Landscape with St Jerome as a Penitent*. Piero's treatment of the landscape, which is seen clearly in the undamaged upper part of the painting, is closely comparable with that in Domenico Veneziano's *Adoration of the Magi* (Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem) (Fig. 3.10), though the landscape itself is slightly simpler, making due allowance for the different sizes of the pictures, and time has been kinder to Piero's greens than it seems to have been to Domenico's, which are now rather uniformly dark. Domenico's landscape is also, like Piero's, notable for its apparently smooth transition from foreground to background. Neither painter betrays the visible awkwardness in dealing with the middle ground that is common in pictures of this period. Like Domenico, Piero has put tiny highlights on the leaves of trees. He has also given a very subtle account of the flow of light over the wall of the house in the background. This house is compositionally important because its wall forms a bright accent near the centre of the picture and its colours pick

37 On Domenico see H. Wohl, *The Paintings of Domenico Veneziano (circa 1410–1461): A Study in Florentine Art of the Early Renaissance*, Oxford: Phaidon Press, 1980.

3.11 Domenico Veneziano (active 1438, died 1461), altarpiece for the church of Santa Lucia dei Magnoli, Florence (the '*St Lucy Altarpiece*'), reassembled, *c.*1445–7, tempera on panel. Central panel: *Madonna and Child Enthroned between Sts Francis and John Baptist and Sts Zenobius and Lucy*, 209 × 216 cm, Galleria degli Uffizi, Florence. Predella, from left: *St Francis Receives the Stigmata*, 26.7 × 30.5 cm, National Gallery of Art, Washington, D.C.; *St John the Baptist in the Desert*, 28.3 × 32.4 cm, National Gallery of Art, Washington, D.C.; *The Annunciation*, 27.3 × 54 cm, Fitzwilliam Museum, Cambridge; *A Miracle of St Zenobius*, 28.6 × 32.5 cm, Fitzwilliam Museum, Cambridge; *The Martyrdom of St Lucy*, 25 × 28.5 cm, Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem.

up those in the figure of the saint. For instance, the pink of his lips, now almost gone because of the damage to this part of the surface, is echoed in the tiles of the roof of the house. Such echoing of colours in depth is highly characteristic of Piero. In this particular case, the echoes would probably have gone some way to mitigating the effects of the strong asymmetry in the linear elements of the composition that dominate what now remains.

Another characteristic of Piero's work, one that is less prominent in *St Jerome as a Penitent* because of the damage it has sustained, is his use of the flow of light to help to define the relationship of bodies one to another within the three-dimensional space of the picture. Like the use of detailed landscape, this too is not something that Piero could have learned only from Domenico Veneziano, but it is highly visible in the *St Lucy Altarpiece*, which Domenico painted for the church of Santa Lucia dei Magnoli, Florence, and which Piero is likely to have known.

This altarpiece is shown reassembled in Figure 3.11. It is usually cited in histories of art as an example of 'Albertian' perspective, presumably in tribute to its having two pavements in perspective: one in the main panel that provides a puzzle for experts, and another, easily readable, in the central predella panel, which shows the *Annunciation* (Fitzwilliam Museum, Cambridge) (Fig. 3.12).[38] However, it is not quite clear what Piero might have learned about perspective from this work. Despite its reputation for being 'Albertian', the floor in the central panel does not make obvious reference to a square grid, and there are several departures from naturalistic perspective in the predella panels. For instance, in the scene of the *Martyrdom of St Lucy* (Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem) (far right in the predella, see Figs 3.12 and 3.13), the figure on the balcony has been made extremely small. In pictorial terms, this allows it to be seen against the trees behind the castellated wall, but it is severely out of scale with the other two figures in the scene, and the balcony, which has been made of a size to fit the figure on it, is consequently also out of scale with the main figures. The balcony is a somewhat fantastic structure, but its archi-

38 Wohl, *The Paintings of Domenico Veneziano* (full ref. note 37), p.100, fig. 15, provides a decoding of the floor of the main panel, whose pattern is based on hexagons, and provides a huge number of lines tracing images of orthogonals to centric points in several other panels.

3.13 (*left*) Domenico Veneziano (active 1438, died 1461), *The Martyrdom of St Lucy*, predella panel from the *St Lucy Altarpiece*, *c.*1445–7, tempera on panel, 25 × 28.5 cm, Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem.

3.12 (*facing page*) Domenico Veneziano (active 1438, died 1461), *The Annunciation*, predella panel from the *St Lucy Altarpiece*, *c.*1445–7, tempera on panel, 27.3 × 54 cm, Fitzwilliam Museum, Cambridge.

tectural style is close to that of the loggia in the *Annunciation*, whose design is apparently rational. However, in this scene, also, the scale of the architecture seems contrived for pictorial purposes rather than naturalistic. The slenderness of the columns makes them look tall in proportion, but they are not much more than half as high again as the Virgin. The style is 'Albertian' in the sense of its being classicizing, but the scale of the structure is small compared with that of Brunelleschian classical architecture, and still more so in comparison with the almost authentically Roman size of some of Alberti's designs.

Perhaps, in Domenico's rendering of the Virgin's loggia, this scaling down may be regarded partly as a domestication of the architecture. Although the ruins of ancient Rome show the huge scale of public buildings, some of the ancient columns reused in churches were not so big, and there appeared to be smaller-scale examples of ancient buildings since, for instance, the church of San Miniato and the Baptistery in Florence were believed to be ancient. Thus it would seem that in the fifteenth century 'classical' architecture could look convincing even when, to today's archaeological eye, it seems to be portrayed as too small. This possibility needs to be borne in mind when we look at the architecture Piero showed in *The Story of the True Cross* and in *The Flagellation of Christ*.

Notwithstanding its 'classical' style, the small loggia shown in Domenico Veneziano's predella panel of the *Annunciation* can surely also trace its ancestry back to the stage-set architecture supplied, for example, by Giotto in the frescos of the Bardi and Peruzzi chapels in Santa Croce, Florence. It seems that, since it was conventional, a certain amount of scaling down was not regarded as liable to detract from the overall effect of optical correctness. A similar, though more uniform, scaling down of both buildings and landscape is found in the predella panels that Filippo Lippi (*c.*1406–1469) painted in 1459–60 when completing the *Trinity Altarpiece* (National Gallery, London) that had been left unfinished by Pesellino (*c.*1422–1459).[39] In contrast to Lippi's treatment of architecture and landscape, that of the

39 The commission dates from 1455. Four of the predella panels are in the National Gallery, London, where they are displayed in a frame with the main panel. The fifth predella panel, showing St Augustine and the Christ Child, was identified in 1995 in the Hermitage Museum, St Petersburg. Its width suggests that some parts may have been lost from the left and right of the present main panel.

human figures is highly naturalistic. In the main panel, the landscape backdrop is handled naturalistically, though there is a notable failure to cope with the middle ground.

The changes of size imposed by photography tend to falsify the effect of perspective pictures, since the viewing distance, built into the picture by the construction, can become utterly unrealistic – that is, a distance too short for the focus of a normal human eye. However, close-up inspection of the original picture shows that the patterning in the floor as it appears in the main panel of the *St Lucy Altarpiece* does not change much in size as it recedes, so the viewing distance must be relatively large. Since the pattern is based on hexagons, it is not easy to see squares in it and on-the-spot guesstimation of a distance point, and hence a viewing distance, is not feasible.[40]

Unfortunately – and here I can, of course, speak only for myself – moving around in front of the picture, as displayed in the Uffizi, I have never managed to find a position from which the painted floor looked natural, or, more importantly, from which I felt able to read the architecture as providing a three-dimensional framing related to the figures. Comic relief, of a kind, has repeatedly been provided by hearing guides telling their attentive charges that the picture was a touchstone of the effectiveness of Albertian perspective. This, I suspect, is probably true: the three-dimensional effect that the picture gives is due not to Domenico's possibly 'Albertian' mathematics but to his convincing rendering of the flow of light. What this proves is that mathematical perspective is not enough. Matters improved considerably when the altarpiece was displayed in a temporary exhibition complete with its predella panels.[41] I think my then finding the central panel more satisfactory was due to the colour of the pink framing elements being repeated in the predella panels, which had the effect of emphasizing these elements in the main panel and making me more aware of the position of the picture plane.

This autobiographical report is to serve as apology for comments on the *St Lucy Altarpiece* that may not seem warranted by what appears in photographs. It is no doubt a subjective judgement how far a sense of the third dimension in any particular image is mediated by the perception of lines and how far it depends upon changes of colour that are readable as light and shade. The judgement is made at the brute level of the visual cortex, with no conscious consent by the supposedly well-educated mind. Perhaps what appears to me to be an essentially decorative display of perspective in the main panel of the *St Lucy Altarpiece* seemed in the 1440s, or seems to others now, like a serious attempt at constructing a mathematically correct pictorial space. However, I find it significant that Domenico was careful about his lighting of the scene. In any case, if Piero learned from this altarpiece, he could well have noticed the light as well as the perspective construction. His own practice was to show careful regard for both, and it seems that he never found it difficult to construct a convincing pictorial space. It is to this characteristic that we shall turn in the next chapter.

40 The lines drawn across the floor in Wohl, *The Paintings of Domenico Veneziano* (full ref. note 37), p.98, fig.12, are the edges of rhombs with angles 60° and 120° and one diagonal parallel to the base line of the panel. (This is clear from Wohl's reconstruction of the pavement pattern in his fig.15, p.100.) As these lines are parallel in reality their images do indeed meet at points on the horizon (one set of images of parallels to one side of the panel and one to the other) but these meeting points are not distance points. If, as Wohl suggests, such lines formed the basis for Domenico's construction, then he used a procedure for which we have no written evidence before 1600 (see Chapter 2 above) – unless, of course, he mistakenly thought he was using the distance point method.

41 The exhibition was held in Florence in 1992. The catalogue is Luciano Bellosi, ed., *Una scuola per Piero: Luce, colore e prospettiva nella formazione fiorentina di Piero della Francesca*, Venice: Marsilio, 1992.

4

A Sense of Space

In Piero della Francesca's time, space did not exist. In saying that his paintings convey a sense of space, we mean that each picture conveys a sense of its own pictorial space, that is a particular space. But Piero himself would not have thought in these terms. In his day, space was not considered to be an entity with a separate existence. Space was defined as extension, and measured by body. This definition, which comes from the work of Aristotle (384–322 B.C.), was universally accepted by philosophers of Piero's time, and was replaced, gradually, only in the seventeenth century. It has the characteristically hands-on, common-sense appeal that partly explains the longevity of Aristotle's ideas about the natural world. It was also characteristically rational, internally consistent and integrated into the complete system of Aristotelian natural philosophy.

One significant element in this natural philosophy was its avoidance of the notion of infinity. For instance, since the Universe, which is spherical, rotates on its axis once in a day-plus-night (the Greeks had a word for that period), it was clear that it must be finite, for if not the furthest bits would be moving infinitely fast, which was clearly absurd.[1] Christians sometimes found problems accommodating the infinity of God within this sort of scheme, and a certain amount of tinkering went on. The most notable fifteenth-century contributions were those of Nicolaus Cusanus (properly Nicholas Khrypffs of Kues/Cusa, 1401–1464), who used mathematics to convey some of his ideas. In fact, Cusanus' methods of getting at the idea of infinity, for instance by considering an isosceles triangle whose apex becomes steadily more distant from its base (and whose equal sides thus make an ever smaller angle with one another), are of interest in their own right as mathematics. However, they were not proposed as such. Mathematics for metaphysicians made no inroads into mathematics proper.[2]

Piero's sense of space is a sense of the metrical relations that establish the three-dimensional shapes of bodies, and those that establish the relations between the positions of bodies. In this, everything is finite and related to something material.

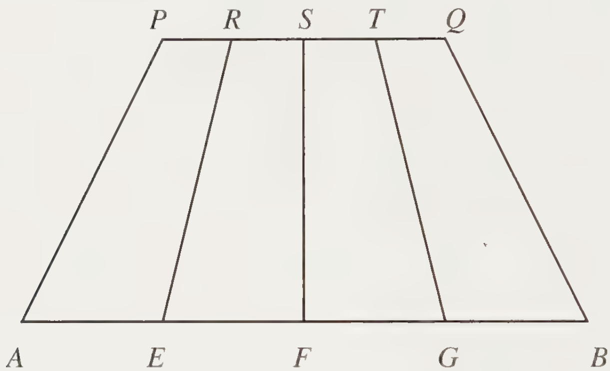
Working with finite mathematical entities

The avoidance of infinity was built into learned mathematics. This was not made explicit because Euclid, whose work had provided the model for learned mathematics, had set the

1 Aristotle, *On the Heavens*, Book 1, Chapter 8, 277a. Nicolaus Copernicus (1473–1543) noticed that by abolishing the daily rotation of the Universe (ascribing the motion to the Earth instead), he had opened up the possibility of the Universe being infinite (*De revolutionibus orbium cœlestium*, Nuremberg, 1543, Book 1, Chapter 8, p.6 verso).

2 Some parts of Cusanus' mathematics will be discussed in Chapter 7.

pattern for a non-discursive style of presentation that worked through series of definitions, axioms and theorems. Linking text was minimal. Euclid leaves it as an exercise to the reader to decide why his results are presented in a particular order. His avoidance of discussion of entities involving the notion of infinity is simply a matter we find by observation. However, it was in accord with the accepted Aristotelian notion of space as extension. In Euclid the concept of measurement by some fixed measure is thus always a finite procedure. With this concept of measure, we also have a concept of line that defines it as a magnitude, that is as a possible measure. To Euclid and his contemporaries, as to Piero and his, the line *AB* is the line from the point *A* to the point *B*. In today's terms, this would be called 'the line segment *AB*', whereas 'the line *AB*' would signify the infinite straight line passing through *A* and *B*. Like his definition of a line, Euclid's definition of a plane is also finite: a plane is what we should now call an area, that is, something that has edges to it.



4.1 Diagram obtained in the course of employing the hypothetical method of constructing the image of a square-tiled pavement by means of equal division of the front and back edges and the use of a diagonal. For the remainder of the set of diagrams see Figure 2.13 above. Drawing by JVF.

The fact that Piero and his contemporaries use a finite definition of a line has some consequences in their mathematics. One rather simple one relates to the 'centric point'. In the course of the hypothetical construction of the image of a square-tiled pavement by the use of a diagonal, described in Chapter 2, one of the diagrams looked as in Figure 4.1 (compare Fig. 2.13). Armed with foreknowledge of the existence of the centric point, we may have a mathematical insight that it looks as though *AP*, *BQ*, *ER*, *FS* and *GT* should meet. That is, *ER*, *FS*, *GT* will all pass through the point of intersection of *AP*, *BQ*. But in fifteenth-century terminology it is clear that the lines *AP*, *BQ* and so on do not meet. So our supposed insight would have to be rephrased in the form: can *AP*, *BQ* and so on all be extended to meet at a point? Or, more formally: if *AP* were extended to C, a point such that *BQ* could also be extended to it, would it be possible to extend *ER*, *FS* and *GT* to C as well? The intervention of the notion of the lines having to be extended has made the idea more cumbersome and probably also less natural.

There is, however, an advantage – at least it looks like one at a certain level – namely that no one is about to ask whether *AB* and *PQ* meet. Once all lines are infinite, any two lines do meet, if they are in the same plane, and we have to start to think rather harder about what we mean by lines being parallel. This problem was dealt with at the time infinite lines were first treated as standard, namely in Girard Desargues' *Brouillon project d'une atteinte aux evenemens des rencontres du cone avec un plan*, published in Paris in 1639.[3] Desargues (1591–1661) had contrived a usable definition of what he meant by infinity, which included the notion that parallel lines could be said to meet. However, while the line *AB* still means the line from point *A* to point *B*, we have no problem with parallels. And there is no problem with infinity either: it is merely an attribute of God.

The standard Aristotelian concept of space as extension and measured by body is implicit in Leon Battista Alberti's perspective construction. He constructs a pavement, that is the image of a body. Piero's della Francesca's perspective treatise is similarly concerned with the construction of images of bodies. It falsifies his reasoning to recast his procedures as leading to the construction of perspective space. Such words would have had no meaning. What he is doing in each case is entirely specific. He is considering a finite object with finite relations to the observer and the picture plane (also finite, simply an area with edges to it). What we are to find is the 'degraded', two-dimensional, perspective image of that particular object. The images of the objects provide a kind of measure of the simulated space shown in each complete picture, but it is the objects that are the prior entities, the ones that matter. This, of course, fits rather well with the ethos of a naturalistic style of representational art that allows even God to be shown in human form.

The Baptism of Christ

The painter will have to show God not as infinite but as perfect, which means that convention decrees He will be shown as 3 *braccia* tall. It is perfectly possible that Piero used this measure in setting out the design of his *Baptism of Christ* (National Gallery, London) (Fig. 4.2). Indeed, measurement shows that the height of the figure of Christ, as painted, is slightly under 1.5 *braccia*.[4] So perhaps Piero made Christ half life-size? That is not to say that we can use this hypothesis to attempt to recover Piero's method of constructing the relations of objects one to another in his picture. However, the disposition of the images on the panel does convey a strong sense that we are seeing a real scene, so it seems reasonable to suppose that Piero did have in mind some precise arrangement of figures, trees, and so on, in a real world at which we are supposedly looking.

Traces of perspective construction of an Albertian kind, that is the drawing of images of rectilinear structures, are nil. Accordingly, when art historians exercise their penchant for drawing lines on Piero's pictures, this particular one attracts a tissue of assertions about its surface organization. This surface organization will be discussed later. We shall begin with what is relatively new in Piero's *Baptism of Christ*, namely that its style, and indeed its format, also demand an organization in depth.[5] The absence of visible construction of the

3 See J. V. Field and J. J. Gray, *The Geometrical Work of Girard Desargues*, New York: Springer-Verlag, 1987; and J. V. Field, *The Invention of Infinity: Mathematics and Art in the Renaissance*, Oxford: Oxford University Press, 1997.

4 One and a half *braccia* is 87.5 cm, the height of the figure of Christ is 87.2 cm. The complete height of the panel is 167.3 cm, so the height of Christ is rather greater than half of it.

5 For comments on the novelty of the vertical format for a *Baptism of Christ* and a discussion of Piero's solution to the problems it posed, see Michael Baxandall, *Patterns of Intention: On the Historical Explanation of Pictures*, New Haven and London: Yale University Press, 1985, Chapter 4.

kind described in *De pictura/Della pittura* or in Piero's own treatise on perspective does not mean that the work is lacking perspective in the wider sense in which the idea was understood in the early fifteenth century. There is plenty of natural optics in the picture. And this too has its mathematical rules.

There is no doubt where one should begin: the figure of Christ dominates the composition. His is the largest figure and it is in the middle of the picture. The pale colours of Christ's flesh and drapery have their counterparts in two lateral figures: the middle one of the three angels on the left of the picture and the man halfway out of his shirt on the right. As can be seen from the position of its feet, the angel is standing a little further away from us than Christ, but is surely meant to be, in reality, smaller than Him and thus does not provide a measure of recession. The half-naked man does provide a measure. His pose is different from that of Christ, but most markedly so above the waist. If we measure the figures to their waist-height only, we find that the man is about half the size of Christ. The fully clothed men standing beyond him across the river are smaller still, about three-eighths his size.

These proportions are almost certainly not precisely the simple ones just given, but we are bound to notice them and situate the figures at appropriately proportioned depths behind the picture plane. At least, it is always assumed that the eye will notice proportions. The pictorial problem is rather to present the eye with objects it can recognize as of a sufficiently well-defined size to allow the sense of proportions to be converted into a sense of distances behind the picture plane. Piero has done that for the half-naked man by making him look as if he were identifiable as a re-posed version of Christ. The figures behind the half-naked man relate to him because their outlines on the picture plane actually run into his. That is, they are partly occluded by him, and therefore manifestly further from us than he is. For three of these clothed figures we have not only the figures themselves but also their reflections in the river, which, being obviously related to the figures themselves, give them a larger emphasis on the picture plane than would otherwise be their due.

These human figures, which seem to be used to create a pattern of proportions that will be read in terms of distances, are not closely similar, but they invite comparison because their true size must be roughly the same. Trees do not have the same degree of uniformity. At least, they do not generally have it in reality unless we are looking at managed woodland in which felling or coppicing is carried out at regular intervals. It has in fact been suggested that the trees shown in Piero's Baptism scene belong to a walnut grove.[6] In any case, the right-hand part of the background includes two isolated trees that do look as if they are of an age and of a kind, so that they too provide items that suggest distance. The height of the crown of the nearer one, that is, the measure from the first fork to the top of the foliage, is about three times that of the similar measure on the more distant tree. Again, we have no reason to take the proportion as numerically precise, but it seems designed to be observed and to play a part in establishing depths. The trees themselves are given additional emphasis by there being a tree that is prominent in the left foreground, though we are given no way of comparing sizes with this one.

6 M. A. Lavin, *Piero della Francesca's 'Baptism of Christ'*, New Haven and London: Yale University Press, 1981.

4.2 (*facing page*) Piero della Francesca (*c.*1412–1492), *The Baptism of Christ*, tempera on panel, 167.3 × 116.2 cm, National Gallery, London.

The boldest use of perceived size as a measure of depth is the small portrait of Borgo San Sepolcro as Jerusalem between Christ and the innermost angel, about on a level with St John's left hand. The city is shown in pale colours that tie it in with the figure of Christ, but these colours do not pull it forward because they also relate to the patches of similarly pale colour in the background landscape.

The proportions we have mentioned are probably not exactly expressible as ratios of small numbers. The estimates were taken from photographs by means of an ordinary ruler and make no claim to great precision on the scale of the actual picture. However, since the sizes of objects are Piero's chief means of establishing relationships in depth in his picture, it seems likely that he did actually work out how big certain elements should be made in relation to others. Despite its resemblance to the landscape between Borgo San Sepolcro and Monterchi, with its small hills and little patches of different kinds of cultivation, there is no question of Piero's having constructed the background to the *Baptism* by actually sketching a real landscape, a procedure that would no doubt have introduced correct series of proportions between the sizes of the objects it presents. The landscape Piero shows is naturalistic but it is not a portrait. There is, in fact, a discursive passage in his perspective treatise in which Piero considers the kind of problem he seems to have solved here, namely that if objects are really the same size they will appear in various proportions in the picture, depending on the distance of the eye. The passage in question follows Book 1, Proposition 11, and appears to be an introduction to Proposition 12, which is: 'From the given eye [position] and on the determined limit to degrade the assigned surface.'[7] The words translated as 'the determined limit' are, in the vernacular text, 'nel termine posto'. As has been made clear in the earlier part of Piero's text, these words signify the picture plane, whose position has been decided. As throughout the treatise, 'to degrade' means to render in perspective.

By way of introduction to this proposition, Piero first refers back to his earlier consideration of dividing areas by means of parallel lines:

> So far what I have said here [is] about non-degraded lines and surfaces, and how the diagonals divide quadrilateral surfaces into two equal parts, and [how] all the divisions are made in these surfaces by parallel lines [the resulting divisions being] in proportion. And now, because I want to talk about degraded lines and surfaces, it is necessary to show that proportion, so that when I say 'proportionally' it will be understood what proportion it is that I mean, because the proportions are not expressible in numbers; and this [proportion] is not double as is .2. to .4. and .4. to .8., and . . . but it is according to the distance from the eye to the limit where the degraded things will be put[8] and the distance from the limit to the thing that is seen.

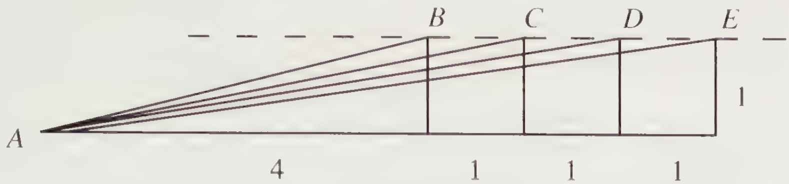
He then gives the example of magnitudes seen at different distances:

> That is, thus: there are four parallel lines and they are one *braccio* apart, and they are one *braccio* long, and they are contained between two parallel lines, and from the first line which is the limit [that is, the picture plane], to the eye [the distance] is four *bracci*; I say

7 Piero della Francesca, *De prospectiva pingendi*: Parma MS, p.6 recto; BL MS, p.6 recto; Piero ed. Nicco Fasola, p.75. Where differences arise, my translation generally follows the Latin text, which I have found more reliable (see Chapter 5 and J. V. Field, 'Piero della Francesca and the "Distance Point Method" of Perspective Construction', *Nuncius* 10.2, 1995, pp.509–30).

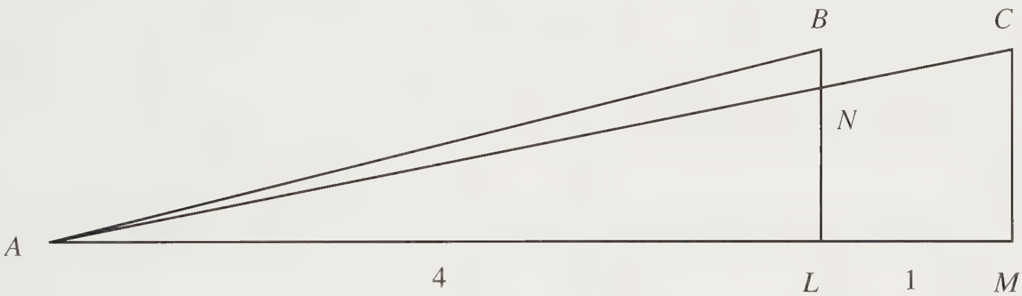
8 That is, the picture plane.

that the second to the first will be sesquiquartic [5 to 4], and the third to the second will be sesquiquintic [6 to 5], and the fourth to the third at the limit is sesquisextic [7 to 6].



4.3 Magnitudes spaced at even intervals and seen from point *A* at 4 *braccia* from the picture. Figure to illustrate the paragraph preceding Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 12. No corresponding figure is supplied in manuscripts of the treatise. Drawing by JVF.

One possible form of the diagram is given in Figure 4.3. The manuscripts of *De prospectiva pingendi* do not supply diagrams for this part of the text. The proportions Piero mentions are derived from pairs of similar triangles, one formed by the line itself and lines joining each of its ends to *A*, and the other the triangle formed when this one is cut by the line next to it on the left, as triangles *ACM*, *ANL* in Figure 4.4. *BL* and *CM* are both of length 1 *braccio*, and the ratio we require is that of *NL* to *BL*.



4.4 Enlarged version of part of Figure 4.3, with additional lettering. Drawing by JVF.

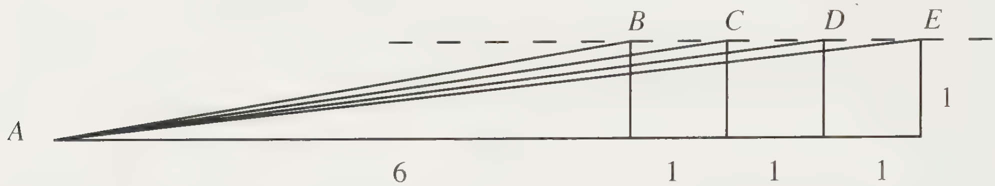
Following the style of abacus books, Piero proceeds to make his point clearer by giving the numerical results:

> To help you understand me better, the proportion among those four lines is as of the four numbers .105. .84. .70. .60.

This is followed by changing the position of the eye:

> but if we change the distance from the eye to the limit [that is, the picture], the proportion changes. That is, if you move away two *bracci* back, so that it will be six [*bracci*] from the eye to the limit, these four lines will change [their] proportion and will be like the four numbers .84. .72. .63. .56. which are not in the same proportion as the first [four], because the distance of the eye from the first limit is not in the same proportion as the distance from the thing for the second limit.

The corresponding figure, not supplied in manuscripts of *De prospectiva pingendi*, might look as in Figure 4.5. The proportions of the intercepts on the vertical through *B* (which represents the picture plane) are noticeably different from those obtained in Figure 4.3.



4.5 Magnitudes spaced at even intervals and seen from point *A* at 6 *braccia* from the picture. Figure to illustrate the paragraph preceding Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 12. As in the previous figure, the proportions are derived from the pairs of similar triangles formed by the lines joining *A* to the ends of each vertical line and either the line itself or the length cut off on the first vertical line (which represents the picture plane). The proportions are visibly different from those in the previous diagram. No corresponding Figure is supplied in manuscripts of the treatise. Drawing by JVF.

Piero then states a general rule, in a form that makes one wish for a lettered diagram:

> Therefore, by changing limit [that is, the position of the picture] the proportion changes. And there is always the same proportion between the second line and the first as between [the distance] from the eye to the limit, which is the first [line], and that from the second to the eye, that is the proportion there is between the line that starts from the eye and ends at the first line, and the line that starts from the eye and ends at the second line; and because one cannot demonstrate the changes in these proportions sufficiently clearly with numbers, I shall demonstrate it with lines, as in degrading surfaces.

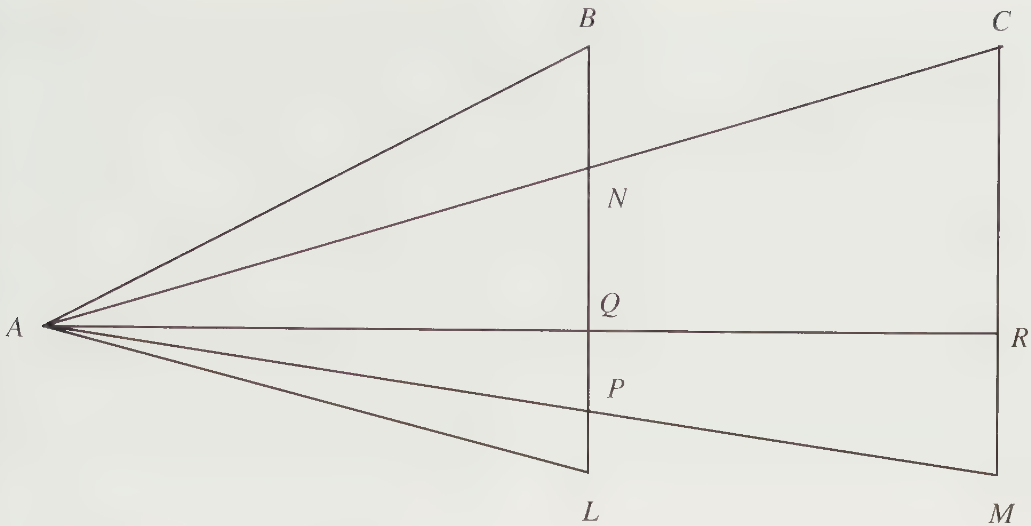
That is, Piero, having shown the inadequacy of arithmetic for this purpose, proposes to rely on geometry. His trying arithmetic is an indication of his commitment to the habit of giving numerical examples, which clearly derives from the abacus tradition.

For simplicity, our figures have been drawn as if we were to look at vertical magnitudes with our eye on a level with their lower ends. Piero has in fact specified only that the magnitudes be parallel and equal.[9] Changing the height of the eye makes no difference to the mathematics, as can be seen from Figure 4.6, in which, for the sake of clarity, we have made the proportions slightly different from what Piero prescribes. In Figure 4.6 we are finding the ratio of *NP* to *BL*. We may note that both the part of Figure 4.6 above the line *AQR* and the part below it have the same form as Figure 4.3, so things will clearly work as before, or we may give a more formal proof as follows.

Triangles *ACM*, *ANP* are similar (as in Fig. 4.3), so we have

$$\frac{NP}{CM} = \frac{AN}{AC}.$$

9 We have made the lengths vertical because we are considering vertical measurements in the *Baptism*. It would make no difference to the mathematics if all the lengths were horizontal, or indeed had any other orientation, provided they remained parallel to one another.



4.6 Looking at equal parallel magnitudes from a point that is not in line with their lowest ends. Compare Figure 4.3 above. Drawing by JVF.

Triangles ACR, ANQ are similar, so we have

$$\frac{AN}{AC} = \frac{AQ}{AR}.$$

Therefore

$$\frac{NP}{CN} = \frac{AQ}{AR}.$$

Comparing with Figure 4.4, we see this is the same ratio as before, since AQ now measures the length previously designated as AL, and AR the length that was AM.

The fact that Piero considers the images of things that are one unit in size and spaced at one-unit intervals relates this problem to a square grid, that is a square-tiled pavement, which at this stage has not yet been constructed, but will appear from the next four propositions, numbers 12 to 15. However, the placing of this short discussion before Proposition 12 does also show that Piero did not think that constructing the grid was a necessary preliminary for other constructions. It seems unlikely that all painters would have agreed with Piero on this matter. For example, the plant life in Uccello's *St George and the Dragon* (National Gallery, London) shows a curious tendency to assemble itself into square patterns (Fig. 4.7). And the discarded lances in two of the panels of his *Rout of San Romano* (National Gallery, London; Galleria degli Uffizi, Florence) are somehow disposed to fall at right angles to one another, with one set parallel to the picture plane. The remaining panel in the series (Louvre, Paris), which may have been painted some time earlier than the other two, has no fallen lances, but includes vegetation like that in the *St George* panel. Nonetheless, Uccello clearly knew, some years later, that the repetition of equal objects would give a sense of distance, since he used this technique to good effect in his *Hunt in the Forest* (*c*.1460) (Ashmolean Museum, Oxford), repeating figures of horsemen at various depths

4.7 Paolo Uccello (1397–1475), *St George and the Dragon*, *c.*1460, tempera on canvas, 56.5 × 74.3 cm, National Gallery, London.

(Fig. 4.8).[10] In this picture the trees show considerable uniformity, like that of managed woodland, a characteristic that is also noticeable in some works by Piero della Francesca, for instance the *St Jerome* in the Gemäldegalerie, Berlin (formerly Staatliche Museen, Dahlem) (see Fig. 3.2).

It may appear that too much stress is being laid upon an extremely simple matter of proportions determined by the well-known laws of geometrical optics. However, there are a number of reasons for taking a serious interest in such considerations. There is, of course, the fact that Piero himself sees fit to discuss them in his treatise about perspective, whose content seems to be entirely concerned with the processes actually used in the construction of pictures (see next chapter). Moreover, what he says is apparently not intended to help one construct appropriate magnitudes – a matter he no doubt regarded as too elementary to require attention – but rather to point up the consequences of the choice of viewing distance.

As can be seen from the numbers Piero supplies, and Figures 4.3 and 4.5, a shorter viewing distance leads to a wider range in size of the equal objects as shown in the picture. This is a rather formal way of pointing out that a shorter viewing distance will give greater clarity in relations of depth. Alternatively, to put it in prescriptive terms, we are being warned that if we are going to rely upon proportions as indicators of distance, we had better not choose too great a viewing distance. The point could have been made with reference to different versions of the square-tiled pavement, which Piero constructs in his next four propositions, but he in fact constructs only one pavement and considers this matter of proportions independently. This suggests that he saw it as constituting a method of designing pictures in its own right. His practice suggests the same. Considering that Piero has a reputation for his

10 M. J. Kemp and A. Massing, with N. Christie and A. Groen, 'Paolo Uccello's "Hunt in the Forest"', *Burlington Magazine* 133, no.1056, March 1991, pp.164–78.

.8 Paolo Uccello (1397–1475), *Hunt in the Forest*, *c*.1460, tempera and oil glaze on panel, 73.3 × 177 cm, Ashmolean Museum, Oxford.

mastery of mathematical perspective, it is a little startling to note how many of his pictures show no sign of what in later generations was to pass for 'perspective', namely architectural vistas and other visible pieces of straight-line construction.[11]

In the large-scale frescos of *The Story of the True Cross* (San Francesco, Arezzo) Piero shows that he has taken in the lesson of Masaccio's *Tribute Money* (Brancacci Chapel, Santa Maria del Carmine, Florence) (see Fig. 2.6) which is that the buildings can be used to give a sense of depth that carries through a whole register, even when the register includes more than one scene. In relatively small paintings, on panel, such as the two of St Jerome and *The Baptism of Christ*, Piero seems rather to have taken a hint from the landscape left-hand side of *The Tribute Money*, in which depth is conveyed by a series of trees (now brutally damaged, but still visible as trunks) and the daring inclusion of a single distant human figure. In the *St Jerome* in the Galleria dell'Accademia, Venice (see Fig. 3.3), we have a pattern of trees and a distant city; in the Berlin *St Jerome* (see Fig. 3.2) we have a quantity of orderly woodland and a fairly distant house; and in the *Baptism* we have, as already noted, human figures, trees and a distant city. In all three pictures, care has been taken to give strong modelling to the foreground figures, and both the St Jeromes are appropriately supplied with books in the form of codices in suitably perspectival poses.

Books are convenient objects because – assuming the volume is not maltreated – the outer edges of the cover and the central line between the pages of the opened codex, all of which are parallel to the spine, remain parallel to one another. Closer inspection shows that the

11 It is this usage that lies behind the complaint by Daniele Barbaro (1513–1570), in the introduction to his *Pratica della perspettiva* (Venice, 1568, 1569), that the pictures of his time were lacking in perspective; see J. V. Field, 'Giovanni Battista Benedetti on the Mathematics of Linear Perspective', *Journal of the Warburg and Courtauld Institutes* 48, 1985, pp.71–99; and Thomas Frangenberg, *Der Betrachter: Studien zur florentinischen Kunstliteratur des 16. Jahrhunderts*, Berlin: Gebr. Mann Verlag, 1990, esp. pp.24–8.

Berlin picture does not give us enough lines on any one book to check the perspective, while the three lines in the book open on the saint's knees in the Venice picture do not converge exactly. If the book had been drawn correctly, the lines should have converged, by Guidibaldo del Monte's theorem of 1600. It is not likely that Piero knew this theorem, which means he would not have used it for his construction, but the result would nevertheless hold true for any correct perspective image of sets of lines that were parallel in the imagined real scene.[12]

In both the *St Jerome* panels the landscape is treated in detail, in a manner that Piero could have learned from Domenico Veneziano or some other artist who adopted the style most usually associated with works painted for north Italian courts. However, the style of the landscape of *The Baptism of Christ* may owe something to the example of Masaccio. The cleaning carried out in the 1980s showed that the landscape behind Masaccio's scene of *St Peter Baptising* in the Brancacci Chapel is painted in dabs, short strokes and curves that, like so much of Masaccio's paint, have an appeal of their own to the modern eye but nonetheless – as intended – act as a convincing shorthand for trees and hills. There is similarly broad treatment of other landscapes in Masaccio's frescos.

As can be seen from the contrasts noted above between the *Misericordia Altarpiece* and the two *St Jerome* panels, looking at the work of Masaccio does not seem to have given Piero an instant feeling that he could or should go out and do likewise. Since his whole *œuvre* shows that his admiration for Masaccio was profound, Piero's delay in following Masaccio's example may most reasonably be attributed to a temperament that was characterized by patience rather than by rapid response. This makes it difficult to attempt to prove that any of Piero's works is related to any specific work he had seen. All the same, Masaccio seems a possible source for Piero's unaccustomed use of high-visibility brushstrokes in the landscape of the *Baptism of Christ*. We do not, however, find the same in Piero's frescos: both *Sigismondo Malatesta before St Sigismund* (Tempio Malatestiano, Rimini) and the scenes of *The Story of the True Cross* are finished almost throughout to the same level of detail as the foreground figures in the *Baptism*. The pictures include elements that cannot be read by a normal viewer (that is one standing on the floor of the church) so the purpose of the detailed finish is presumably not simply that of contributing to the naturalistic appearance of the scenes concerned.[13] In contrast, *The Baptism of Christ* could have been seen from close quarters, at least by the clergy for whose church it was painted, so the visibility of the brushwork is certainly a deliberate element that is part of its overall style.

If relatively explicit indebtedness to Masaccio can be used as an indication of date, then the *Baptism of Christ* would seem to be closer to the *Misericordia Altarpiece* than are the *St Jerome* panels. Since the landscape setting of the *Baptism*, and the means used to make it readable in depth, are noticeably similar to those of the two *St Jerome* panels, one may suggest a date relatively close to theirs – which makes *The Baptism of Christ* a work of the 1450s.

The *Baptism* is often, and entirely reasonably, cited as an example of Piero's beautiful rendering of light. Poetic terms such as 'Springlike' do in fact find an objective justification

12 See Chapter 2 and Chapter 5; see also J. V. Field, 'When is a Proof not a Proof? Some Reflections on Piero della Francesca and Guidobaldo del Monte', in *La Prospettiva: Fondamenti teorici ed esperienze figurative dall'Antichità al mondo moderno. Atti del Convegno Internazionale di Studi, Istituto Svizzero di Roma (Roma, 11–14 settembre 1995)*, ed. R. Sinisgalli, Florence: Edizioni Cadmo, 1998, pp.120–32, figs pp.373–5.

13 This invisible realism will be discussed further in Chapter 6.

in the state of the plant life, though the density of the foliage on the trees suggests it is not early spring. However, although the light is in general coming from the right, allowance must also be made for the use of gold. The Albertian gospel censuring the use of real gold in pictures had not reached Borgo San Sepolcro.[14] Despite time and cleaning, Christ still has the remains of a gold halo, shown as if it were a disc seen at an angle and placed between His head and the bowl held by the Baptist. The Baptist himself also has a halo, a disc seen edge on, making an angle of about 45° to the vertical in a position where this makes it neatly tangent to the profile of the back of St John's head. The angels also have the remains of haloes, discs of rays seen in perspective, and the outermost angel has substantial traces of gold on its wings and on the cuff of its rose-coloured robe. There are also traces of gold on the wings of the other two angels and on the hems of their robes. The lower edge of the outermost angel's outstretched hand has been given some emphasis, possibly in gold but more probably in some other pigment. This emphasis serves to detach the hand from the pale background of the robe of the middle angel. There are also clear traces of a sheaf of gold rays fanning out through the dove, though not quite taking in the tips of its wings.[15] These rays seem to come down from an apex close to the upper edge of the picture.

Luckily, although the original frame of the *Baptism* is now missing, we still have the frames of the remainder of the altarpiece from which the panel came, and these indicate that each of the three main panels had a small *cimasa* containing a roundel (Fig. 4.9). That above Christ presumably held God the Father, who would thus be the visible source of these golden rays. Unless they have been completely removed by cleaning, it would appear that the rays did not continue downwards beyond Christ's halo. However, their light seems to do so. Christ's hands are lit from directly above as well as from the right (Fig. 4.10). The gold, and the continuation of the flow of light that it partly represented, make a visual link between what we see on Piero's panel and the gold of its now missing frame. The remains of the altarpiece from which the *Baptism* was detached when it was purchased by the National Gallery in the nineteenth century is shown in Figure 4.9.

Piero's work on the panel is not documented, but there is documentary evidence that suggests the work was commissioned in the 1430s, before Piero visited Florence; and, though there have in the past been dissentient opinions, the evidence in favour of the work having been destined for the high altar of the church of San Giovanni Battista (or d'Afra) in Borgo San Sepolcro is now overwhelming.[16] In the absence of a contract we cannot, strictly speaking, be entirely sure that the altarpiece as completed by Matteo di Giovanni (active 1452, died 1495) follows the original design that was proposed at the time the work was commissioned. However, as we have seen, a small roundel like those over Matteo's panels, placed above the *Baptism*, does seem to make sense in combination with Piero's picture, and the slight insetting of the arched tops of the framing for each panel would explain the shape Piero has given to the top edge of his painting.

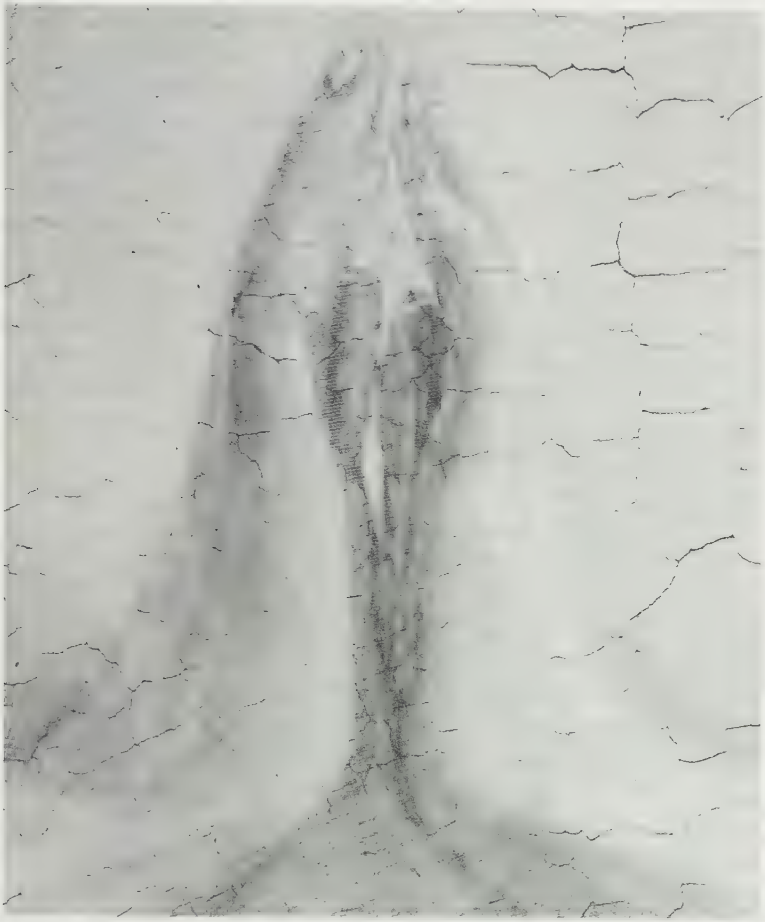
14 Alberti, *De pictura*, Book 2, §49. This passage refers to the use of gold for details such as the quiver worn by Dido. It does not consider solid gold grounds behind figures.

15 I am grateful to Jill Dunkerton of the National Gallery, London, whose close inspection of the surface of the painting corrected and supplemented my own observations. Many of these traces of gold are too faint to show on photographs and cannot easily be seen under normal viewing conditions in the gallery.

16 On both points, see James R. Banker, *The Culture of San Sepolcro during the Youth of Piero della Francesca*, Ann Arbor: University of Michigan Press, 2003, pp.225ff and 253–6.

4.9 Matteo di Giovanni (active 1452, died 1495), *Altarpiece of Sts Peter and Paul*, tempera on panel, 358 × 352 cm, Museo Civico, Sansepolcro.

The panel was no doubt supplied to Piero but, as can be seen in Figure 4.2, the painting does not follow its actual upper edge. Some of what we now see would have been lost behind the frame, since the top of the frame is not semicircular but slightly pointed. We may also note that the gilded tracery on the inside edge of the arch of the frame would have obscured some of the upper parts of the panel, and that the framing also probably cut off a little of the scene on the vertical edges. In any case, it is certain that in its original state, and in its original framing, Piero's *Baptism of Christ* must have looked considerably less chaste than it now does in the National Gallery. The present more discreet framing does have the countervailing advantage of compensating for some of the paint loss: the very careful and intricate highlights to be seen in well-preserved parts of the surface show that Piero had given his painted reality a glitter that would stand up to the gold around it. Also, there is less gold for the now-diminished glitter to stand up to in the present frame. The impression of strength and the sense of three-dimensional structure are surely still fairly close to what Piero intended.



4.10 Piero della Francesca (*c*.1412–1492), *The Baptism of Christ*, detail of Christ's hands, tempera on panel, National Gallery, London. See Fig. 4.2.

Composition in three dimensions was a relatively novel element in the art of Piero's time. Composition in the plane was a traditional one. It is not at all clear how far such composition was ever subjected to explicit mathematical rules. An apprentice copied works by his master and was probably considered capable of recognizing a successful pattern when he saw one. Such a skill seems decidedly basic to the profession. There is, however, one undoubted use for simple mathematical structures in the picture plane: they can help to ensure that various elements are placed as was intended in a preliminary drawing of the design. For instance, Piero's Christ is not only the spiritual centre of the *Baptism*, He is also on its central vertical. Since the painting has an arched top, we can easily find the central axis, which proves to pass down the centre line of Christ's nose and through the big toe of His right foot. The fit is so exact that it seems likely this line was used in setting out the design on the panel. There is a similarly convincing horizontal line: the diameter across the base of the semicircular arc that forms the top of Piero's painting lies close to the line through the tips of the outspread wings of the dove. The bird's beak is, however, slightly to the left of the central vertical. Piero may have come to this idea independently, but he might also have been sharp-eyed enough to notice the slight non-alignment of the central figures in Masaccio's *Trinity* fresco (Santa Maria Novella, Florence).[17]

The vertical and horizontal lines just mentioned would have been useful to Piero in setting out his work on the panel. As they do not appear as actual lines in the finished picture, there is no reason to suppose that he would have regarded it as important that the viewer should notice their existence. The same is true of the verticals that may have been used to divide the panel into three equal parts: one lines up neatly with the inner edge of the tree at the front on the left and the other lies somewhere close to the inner edge of the figure of St John on the right.[18] There are no traces of the transfer of a complete drawing, but it seems possible that some magnitudes were transferred to the gesso in a reasonably precise way, perhaps using dividers (which have left no trace). As we have seen, some of the proportions in this picture are important because they are being used to convey depth.

Beyond such straightforward mathematics, we enter the realm where one has to prove the Moon is not made of green cheese. There seems to be no practical or compositional reason why Piero should have drawn the complicated mathematical diagrams that some scholars have found in the *Baptism of Christ*.[19] In fact, most such diagrams seem to amount to little more than extremely fancy ways of constructing the central vertical. Since the geometrical designs, which imply perfect regularity in the shape of the panel, generally go right to the edge of the painted surface, they inevitably refer to points that would have been hidden once the picture was placed on show in its frame. This implies that Piero was dealing with hidden perfections. If the elaborate geometry is meant to be expressive of something higher than mere practical construction, then the onus is surely on its proponents to prove, for instance, that Piero was a secret convert to Islam and hence made covert use of its five-pointed star in an apparently Christian picture. In any case, there is no doubt that all of Piero's pictures show strong control over the composition in the plane. This element in his style is perhaps to be traced back to the Sienese pictures that he must have known from his earliest years. New ideas about composition in three dimensions did not apparently lessen the significance he accorded to composition in the plane. Indeed, one of Piero's leading characteristics as a painter is his ability to handle both kinds of composition together.

We have already remarked that, like Domenico Veneziano, Piero regularly makes a smooth transition from foreground to middle ground to background. This transition is one that many painters seem to have found awkward; there are abrupt jumps from foreground to backdrop even in the work of painters who are clearly interested in constructed perspective. For instance, the *Martyrdom of St Sebastian* (National Gallery, London) by Antonio Pollaiuolo (*c.*1432–1498) and Piero Pollaiuolo (*c.*1441–1496), while justly famous for the three-dimensional archers in the foreground, is much less convincing in the rendering of increasing distance in the townscape setting of the scene. Paolo Uccello's *Rout of San Romano* (National Gallery, London) also seems to have a backdrop rather than a landscape setting, and some of Uccello's pictures are, to today's eye, more remarkable for the visible

17 The line drawn in *sinopia* down the centre of the Father's nose is 2.6 cm to the left of a *sinopia* line that marks the axis of the figure of Christ. See J. V. Field, R. Lunardi and T. B. Settle, 'The Perspective Scheme of Masaccio's *Trinity* Fresco', *Nuncius* 4.2, 1988, pp.31–118.

18 A scheme of this sort is discussed in Baxandall, *Patterns of Intention* (full ref. note 5) Chapter 4.

19 The nature of these diagrams is discussed in the Introduction above.

presence of a perspective construction than for any sense that the objects presented in the picture are actually solid. For instance, the profile princess that St George is rescuing (Fig. 4.7) seems to me to be bisected, like a figure on a medal, and much less real than the corresponding figure in Donatello's version of the scene (Fig. 2.8). The comparison with the work of a sculptor may seem unfair, but is in fact to the point: sculptors needed to have a strong grasp of three-dimensional form, whereas it was not necessarily of the essence for painters.

Piero della Francesca is, of course, not unique as a painter in having what we might call a sculptor's sense of form, but this sense is a notable part of his personality as a painter. It is also a notable part of his personality as a mathematician. Piero's *Trattato d'abaco* and his *Libellus de quinque corporibus regularibus* are the only abacus-style texts of the time that deal with three-dimensional geometry and, as we shall see, Piero's treatment goes beyond what may be found in the learned tradition. This aspect of Piero's mathematical skills may well be connected with his painting, but before examining it we shall first look at some other paintings in which, as in the *St Jerome* panels and the *Baptism*, the mathematics stays well out of sight.

Other pictures with 'no perspective'

St Jerome as a Penitent (Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem), shown in Figure 3.2, is one of the pictures in which there is 'no perspective' in the sense of there being no visible Albertian construction. It is also one of the two surviving pictures by Piero that is both signed and dated. Its being atypical in the latter respect is not necessarily relevant in attempting to assess whether it is giving us helpful historical information in the former one, for instance by providing reliable evidence that 'early' Piero works lack visible constructed perspective. If the signature were taken as evidence that the work was particularly significant for Piero, one might perhaps regard the painting as having embodied what he thought of as the best of his skill, which would suggest a preference for this type of composition. However, it is by no means obvious that this is the case. As has been mentioned in Chapter 3, it is possible that Piero's reason for signing the two *St Jerome* panels was that he wished to associate himself with the humanist learning that the saint represented. On the other hand, no such explanation will serve for Piero's having also signed and dated his fresco of *Sigismondo Malatesta before St Sigismund*, shown in Figure 6.1. Presumably this might have been signed to make it a memorial to Piero's skill, in which case we must assume that Piero intended us to take notice of the neat perspective pavement constructed in the foreground. Whatever we are to make of the signatures on the two *St Jerome* panels, we clearly cannot reasonably suppose Piero's earliest interest in using perspective construction arose any later than 1451 (the date of the Rimini fresco). So the plain marble slabs under the feet of the saints in the *Misericordia Altarpiece* (painted about a decade later) may be seen as showing a deliberate decision to rely upon modelling of the figures rather than trying to introduce depth by means of constructed perspective.

In the Rimini fresco, the perceived distance of objects behind the picture plane is established both by the use of formal mathematical perspective to convey the regular patterning of the pavement, and by the simulation of a flow of light that mainly follows the direction of the natural lighting. Sizes are difficult to compare since the two human figures are in ostentatiously different poses. However, the pose of each is similar to that of one of the two

4.11 Piero della Francesca (*c.*1412–1492), *Battista Sforza*, oil on panel, 47.7 × 33.6 cm, Galleria degli Uffizi, Florence.

4.12 Piero della Francesca (*c*.1412–1492), *Federigo da Montefeltro*, oil on panel, 47.5 × 33.6 cm, Galleria degli Uffizi, Florence.

figures in *St Jerome with Girolamo Amadi* (Fig. 3.3) – which is to say that Piero may well have made some studies of the relative heights of seated and kneeling figures. The mixture of techniques for conveying depth that is used in the Rimini fresco is found in many other pictures by Piero, but the balance in the importance given to the components varies. This matter will be discussed in Chapter 6. Here we shall consider some pictures in which the balance seems to be very much in favour of what we may call 'non-Albertian' means, that is ones that do not depend upon the construction of a grid or of images of simple geometrical bodies such as pieces of classicizing architecture.

Since Vasari tells us that Piero made many panels in perspective with little figures for the Duke of Urbino – though since he names the duke in question as Guidobaldo it is clear his chronology is confused – it is discomforting that two spectacular examples of the non-perspective style (in Vasari's sense, which is what we are calling 'Albertian') should be provided by Piero's portraits of the rulers of Urbino, Federigo da Montefeltro and his wife Battista Sforza (Figs 4.12 and 4.11). Furthermore, the 'small panels in perspective' are the first of Piero's works that Vasari mentions, whereas these portraits must date from fairly late in Piero's career, since they are in oil rather than tempera. There are, however, some 'little figures' on the reverse sides of the portraits, which show triumphal chariots carrying Federigo and Battista enthroned, and otherwise laden with personifications of the virtues appropriate to each sitter (Figs 4.13 and 4.14).

All four scenes divide sharply into foreground and distant landscape. It was presumably the portraits that set the pattern and they show the sharper transition. The chariots require a causeway, so they have been provided with a rocky ledge whose near side has a curlicue edge. Although Piero certainly knew recent examples also, these ledges closely resemble those often found at the base of pictures of much earlier date, such as some mosaics in Santa Maria in Trastevere, Rome, by Pietro Cavallini (active 1273–1308), and it is possible that Piero thought of them as an ancient motif. The two portraits on the other sides of the panels do not even indicate the existence of a balustrade or a window frame: the sitters simply appear against the distant landscape.

Piero has not relied merely on scale to ensure that we read the landscape as distant. The lighting is from the right in all the pictures, allowing Battista's profile to be seen in softer contrast against the sky and providing highlights on her jewellery. However, it is chiefly in the chariot pictures, in which the lighting is stronger, that cast shadows of landscape features make a substantial contribution to our reading the shapes of hills. In the portraits, there seems to be a cloud shadow on the nearer part of the landscape, decreasing its contrast with the busts in front of it. The sunlight behind it, notably caught on the walls of the city behind Battista, acts as an indication of greater distance. Aerial perspective also plays an important part in all four pictures. This effect is not given anything like such prominence in any other surviving picture by Piero, but we may perhaps detect some traces of it in the Venice *St Jerome* (whose colouring is, however, at present rather dark) and some faint traces in the *Nativity* (National Gallery, London) (Fig. 6.38). The aerial perspective and the patches of mist are naturalistic in the sense that the landscape is almost, but not quite perfectly, identifiable with views near Federigo's home city of Urbino. The total effect is nonetheless Netherlandish, and it would clearly be difficult to assess the priority between the various factors that may have contributed to Piero's use of a suitable naturalistic landscape background in this way.

There may, however, be a hint in the fact that these pictures belong to Urbino. Federigo was an admirer of Netherlandish art, as witness his employment of Justus van Ghent (Joos van Wasserhove, active *c.*1460–80) at his court. Moreover, the Ducal library was eventually to contain a copy of Piero's treatise on perspective. These pictures use everything that Piero tells us, in the introduction to his first book, has not been included in the treatise, and make only scanty use of anything that is (see Chapter 5). In this sense they could perhaps be seen as a complement or supplement to it, though that suggestion does not help us in any way in regard to their date or that of the treatise.

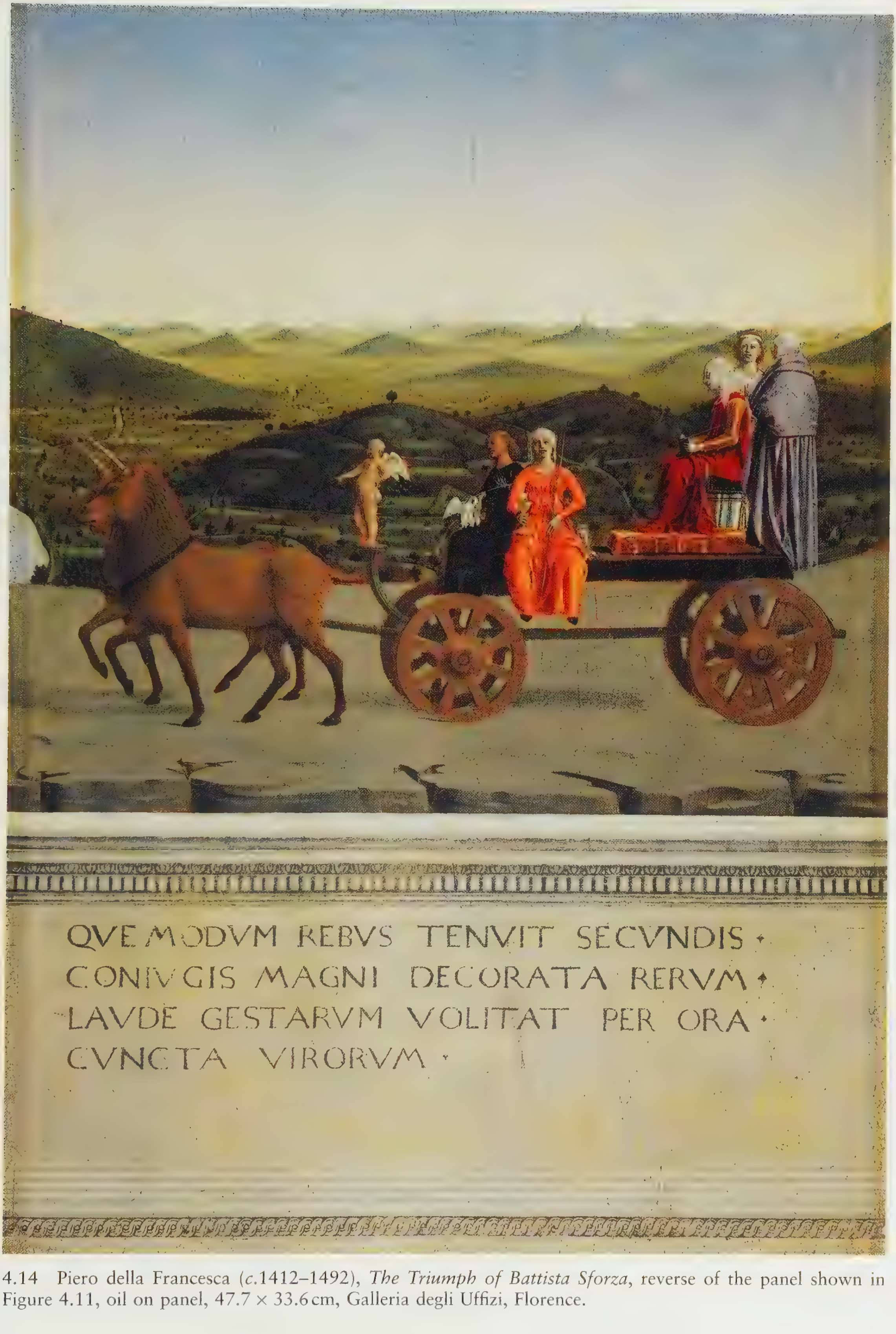
The same could also be said of another picture that has links with Federigo da Montefeltro, namely the *Madonna di Senigallia* (Galleria Nazionale delle Marche, Urbino), shown in Figure 4.15. The picture came to the gallery in Urbino from the church of Santa Maria delle Grazie, outside Senigallia, and, despite its rather domestic air, may have been painted for this church, which was founded by a relation of Federigo's. The picture is generally accepted as relatively late, usually being placed in the 1470s.[20] Piero's use of oil paint is taken as confirmation of a late date. For reasons that may be to do with its delicate colour and exquisite degree of finish, reproductions of this picture manage to look what passes for reasonably like it without looking sufficiently like it to be beautiful. However, no great subtlety is required to note that the composition presents us with figures against a background from which they are essentially disjoint.

The case is a little less dramatic than that of the Montefeltro portraits in that we have a domestic interior, whose elements cannot be a long way behind the figures, rather than a sweep of distant landscape. However, the objects that are shown in the background are all of only roughly determined size, so they cannot be used to establish an exact sense of distances. At least, this cannot be done in an objective way. All the same, as far as I know, no viewer has ever complained about any sense of ambiguity relating to the spatial organization, so one may guess that each (that is, each one's visual cortex) has made an individual decision about sizes and constructed a satisfactory reading in accordance with it. The only objective geometrical clue we are given is that the alcove with its shelves is seen a little from below. Subtle consistencies in the lighting probably play a part in establishing the reading of the spatial structure. However, these, like the motes in the sunbeams coming through the window, get swamped by the process of photography. In any case, there can be no question that while some mathematical means may have been used in drawing the still-life scene of the alcove and shelves, and in putting in the edges of light beams and shadows in the room seen through the doorway on the left, there is no explicit link between these background elements and the figures. It is, moreover, clear that the main lighting of the figures derives from an invisible source. To judge by the squarish highlight on the pendant jewel of the angel at the left, this source is a window like the ones we see in the background.

How one reads the three-dimensional structure of the scene shown in a picture like the *Madonna di Senigallia* is clearly a subjective matter. However, the disjunction of the figures from their setting is not. It is curious that we should also find a similar, though not quite so absolute, effect of disjunction in another picture associated with Urbino, the altarpiece showing Federigo da Montefeltro before an enthroned Madonna and Child with Saints that is now in the Galleria di Brera, Milan (Fig. 6.35). In this example, however, the background is rather obviously formally 'done in perspective', so we are presented with a mixture of means of constructing spatial relations (see Chapter 6).

20 A case has been made out for associating the painting with the marriage of one of Federigo's kinsmen, see R. Lightbown, *Piero della Francesca*, London: Abbeville Press, 1992, pp.257–8. On this basis, Lightbown dates the picture to 1478–80.

4.13 Piero della Francesca (*c.*1412–1492), *The Triumph of Federigo da Montefeltro*, reverse of the panel shown in Figure 4.12, oil on panel, 47.5 × 33.6 cm, Galleria degli Uffizi, Florence.



4.14 Piero della Francesca (*c*.1412–1492), *The Triumph of Battista Sforza*, reverse of the panel shown in Figure 4.11, oil on panel, 47.7 × 33.6 cm, Galleria degli Uffizi, Florence.

4.15 Piero della Francesca (*c.*1412–1492), *Madonna di Senigallia*, oil on panel, 61 × 53.5 cm, Galleria Nazionale delle Marche, Urbino.

One might perhaps wish to regard the few extreme examples of disjunction between figures and setting as in some sense anomalous, but there would seem to be no very secure basis for doing so. They certainly resemble Piero's other pictures in conveying a strong sense of three-dimensional relationships – even if this does seem to have been achieved by invisible means, or at least by means that defy detailed and rational isolation by a normal viewer. It is surely more reasonable simply to accept that Piero had a good grasp of spatial relationships and how to convey them to the viewer, and on occasion was sufficiently confident in the power of this gift to be willing to do without the more obvious means that we find in some of his other pictures. This matter will be discussed further in the next chapter.

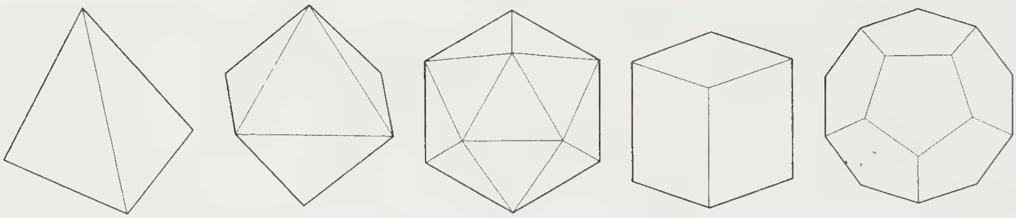
As Piero's firm grasp of three-dimensional relationships is evident in all his pictures, it is not surprising that we can find its counterpart in his mathematics. The absence of formality in the three-dimensional design of the pictures seems to show a capacity for imagining things in a properly three-dimensional way. In Piero's mathematics we find descriptions and drawings of new mathematical shapes in three dimensions. In working out their structure, Piero was presumably indebted to his unusually powerful visual imagination, and specifically to what we have called his sculptor's sense of three-dimensional form. Thus Piero's skill in handling three dimensions led him to make original contributions to geometry.

It seems appropriate to discuss this part of his work here because his having an unusually strong visual imagination is the only clue we have as to how the solids were discovered. From the point of view of the history of mathematics this is, of course, a rather unsatisfactory situation, but we are in any case trying to explain the surely rather unexpected discovery of something that in principle belonged to the tradition of scholarly mathematics by someone whose training seems to have been in the rather different tradition of practical mathematics.

Mathematics with three dimensions

As we saw in Chapter 1, Piero's *Trattato d'abaco* takes its style, and much of its content, from the textbooks used in abacus schools. However, as the work was not actually written for use in a school, Piero was presumably free to exercise his own taste in producing a model text. His deviations from the abacus-book norms are all towards a more learned treatment; for example, by including many algebraic problems that are entirely abstract and highly unlikely to be of practical application. The geometrical parts of the work not only show a notable tendency to refer the reader to Euclid but also tend to deal with matters that are not of obvious practical use. For instance, Piero was not to know that in the following century, thanks to the improved performance of cannon, the problem of drawing a regular pentagon (to Euclid and Piero an 'equilateral and equiangular' pentagon) was to become a real-life exercise for designers of fortifications, who needed to give suitable instructions to workmen.[21] Piero seems to have included the pentagon merely because it is found in Euclid and contributes to the construction of the regular icosahedron and the regular dodecahedron, two of the five regular solids that had been known since ancient times.

21 Surviving documents in the Archivio di Stato in Venice (Savorgnan papers, 'Secreta, Materie miste notabili', reg.6) show how to lay out several polygons. For the pentagon, see Field, *The Invention of Infinity* (full ref. note 3), Chapter 6.



4.16 The five regular polyhedra known since ancient times, the 'Platonic solids'.

These polyhedra were often known as the 'Platonic solids', because Plato mentions them in *Timæus*. They have faces that are regular polygons. Each solid has faces of only one kind, and the faces meet in the same way at every vertex of the solid. (To be technically correct, one should also note that the solids are convex, that is their faces are completely visible on the outside of the solid.) The five Platonic solids are the tetrahedron (four triangular faces, meeting three at each vertex of the solid), the octahedron (eight triangular faces, meeting four at each vertex of the solid), the icosahedron (twenty triangular faces, meeting five at each vertex of the solid), the cube (six square faces, meeting three at each vertex of the solid), and the dodecahedron (twelve pentagonal faces, meeting three at each vertex of the solid). Modern sketches of the solids are given in Figure 4.16.

Since all the vertices of each solid are exactly the same, it is clear that the vertices lie on a sphere (the 'circumsphere' of the solid concerned). In *Elements*, Book 13, Euclid constructs each of the regular solids from its circumsphere. Piero also considers the solids in this way. His style is, however, different from Euclid's. Whereas Euclid prescribes a sphere of known but unstated size and requires the construction of the edges of the solid whose vertices lie on the sphere, Piero gives a numerical value for the diameter of the sphere and asks for the length of the edge of the solid. That is, like the rest of the *Trattato d'abaco*, Piero's treatment of polyhedra is numerical. Diagrams are provided only at the end of each problem, so one is presumably expected to draw intermediate diagrams for oneself. The matter may be elementary in the sense of being derived from Euclid's *Elements*, but like Euclid's Book 13 it is not notably simple.

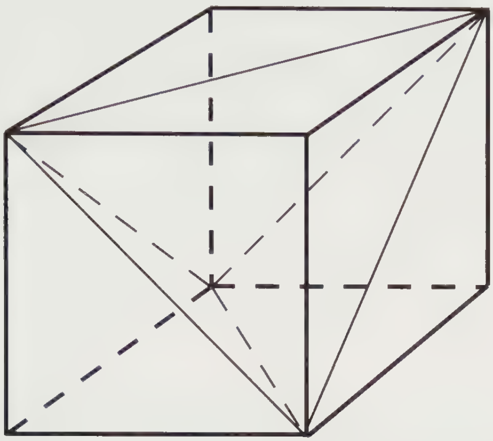
All the same, Piero begins his section on three-dimensional geometry in a completely straightforward manner:

> A body has three dimensions, that is length, width and depth; and they [bodies] are of various kinds, though I propose only to speak about some [of them], such as about 4-faced solids [made of] triangles and about cubes and about round spheres; . . .[22]

The first few problems are, indeed, correspondingly simple: to find the height of a tetrahedron of known edge, given the height to find the edge, given edge and height to find the volume, and so on. As in Piero's earlier geometrical examples, diagrams are supplied. There follow some problems on the cube, not all of them very simple. For instance, we are asked to draw the largest possible tetrahedron inside the cube.

22 Piero della Francesca, *Trattato d'abaco*: BML MS, p.105 recto; Piero ed. Arrighi, p.224.

There is a cube, which is 4 *bracci* along each side; I want to put inside it the greatest solid equilateral triangle figure that can be put [there]. I ask for its sides.[23]



4.17 Cube showing diagonals drawn to mark where corners of the solid are to be removed, in the process of cutting the cube down to make it into a tetrahedron. Drawing by JVF.

The solution to this is, effectively, to be found in Euclid. It involves drawing diagonals of faces of the cube and cutting off corners of the solid figure, as shown in Figure 4.17. Piero in fact letters the vertices of the cube so as to make it clear which diagonals are to be drawn. He then finds the length of the side of the tetrahedron by using Pythagoras' theorem in the isosceles right-angled triangles formed by two sides and a diagonal of the face of the cube. There is no explicit proof that the four-faced figure that has been produced is in fact a regular tetrahedron. The reader was presumably supposed to recognize that everything was symmetrical. The final calculation would, of course, yield equal lengths for the six edges, though Piero, in a rare instance of his avoiding repetition, only calculates the length of one.[24]

There follow some propositions on the sphere, such as finding the surface area of a sphere of diameter 7 *braccia*. Piero silently uses Archimedes' approximate value for π, namely $\frac{22}{7}$, but he correctly credits Archimedes for the result that the surface area of a sphere is four times the area of one of its great circles.[25] After the sphere, we have two numerical versions of problems from Euclid, namely inscribing a tetrahedron in a sphere and inscribing a cube in a sphere. In both, we are merely given techniques for solution, without any explanation. Then, with no indication that what follows is of any particular importance, Piero states his next problem:

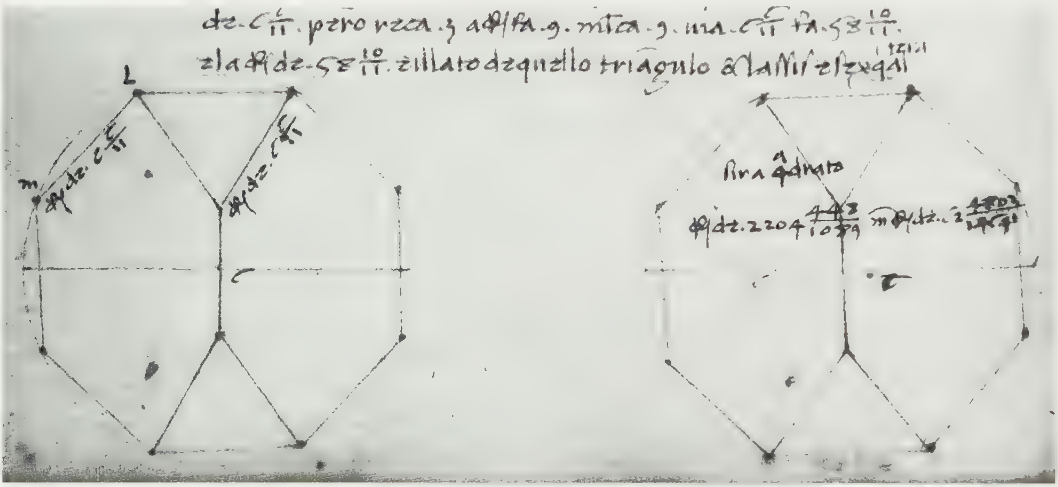
23 Piero della Francesca, *Trattato d'abaco*: BML MS, p.106 recto; Piero ed. Arrighi, p.227.

24 The problem is given in full in Appendix 6.

25 Archimedes gives this value for π, specifically as an approximation, in *On the Measurement of a Circle*. His result for the surface area of the sphere is found in *On the Sphere and Cylinder* (the work Archimedes is said to have considered his best). We now know that Piero knew the second of these works directly. (James R. Banker, private communication; Banker's paper on the matter is forthcoming.) References to Archimedes' results are to be found in many mathematical treatises, practical as well as learned.

There is a spherical body whose diameter is 6; I want to put in it a body with 8 faces, 4 triangular and 4 hexagons. I ask what its edge is.[26]

As far as is known, this polyhedron with eight faces was new to the mathematical literature of Piero's day. It is, however, one of the thirteen polyhedra that Pappus of Alexandria (active A.D. 300–50) describes as having been discovered by Archimedes.[27] Although Pappus is writing about five centuries after Archimedes' time and no text on the subject by Archimedes is known to survive, there is no good reason to doubt the truth of what Pappus says. Unfortunately, it is terse. He merely lists the numbers and types of faces of the bodies concerned. It is extremely unlikely that Piero knew this text, since later writers, such as Pacioli, clearly did not know it.[28] It was first published in Pesaro in 1588 by Federigo Commandino (1509–1575) in his edition of the works of Pappus.[29] In any case, although his initial verbal description of the solid is much the same as that of Pappus, Piero provides diagrams, as shown in Figure 4.18.



4.18 Drawings of the solid now known as the truncated tetrahedron, from Piero della Francesca, *Trattato d'abaco*, BML MS, p.107 verso.

There are two diagrams because we have one to illustrate each of Piero's problems on the body.[30] These diagrams contain information not supplied by the text – which in itself is something new in a work on mathematics. They show the pattern of polygons round each

26 Piero della Francesca, *Trattato d'abaco*: BML MS, p.107 recto; Piero ed. Arrighi, p.230.

27 Pappus, *Collection*, Book V, Theorem 16, Proposition 17 (English translation in Ivor Thomas, *Greek Mathematical Works: Selections Illustrating the History of Greek Mathematics*, 2 vols., London: Heinemann (Loeb Editions), 1939–41).

28 See J. V. Field, 'Rediscovering the Archimedean Polyhedra: Piero della Francesca, Luca Pacioli, Leonardo da Vinci, Albrecht Dürer, Daniele Barbaro, and Johannes Kepler', *Archive for History of Exact Sciences*, 50 (nos 3–4), 1997, pp.241–89.

29 Unfortunately, the typesetters seem to have made a mistake: the reference to Archimedes' work in this edition appears as if it were part of Commandino's commentary, following *Collection*, Book V, Theorem 16, Proposition 17, p.83 verso.

30 An English translation of the first of these problems is given in Appendix 6.

vertex of the solid, and they show that, since all the vertices are equivalent, they lie on a sphere. (The modern definition of the Archimedean bodies is that they are convex uniform polyhedra. That is, for each solid each vertex is surrounded by regular polygons arranged in the same way.) Piero's diagrams may have been intended primarily as an aid to visualizing the polyhedron, but they in fact supply important mathematical information, which is, moreover, not to be found in Pappus. In the course of his first proposition, Piero explains that this body is to be constructed by cutting corners off a regular tetrahedron in such a way that each side is divided equally into three parts. This property is the basis for the name by which the solid is now known: truncated tetrahedron.[31]

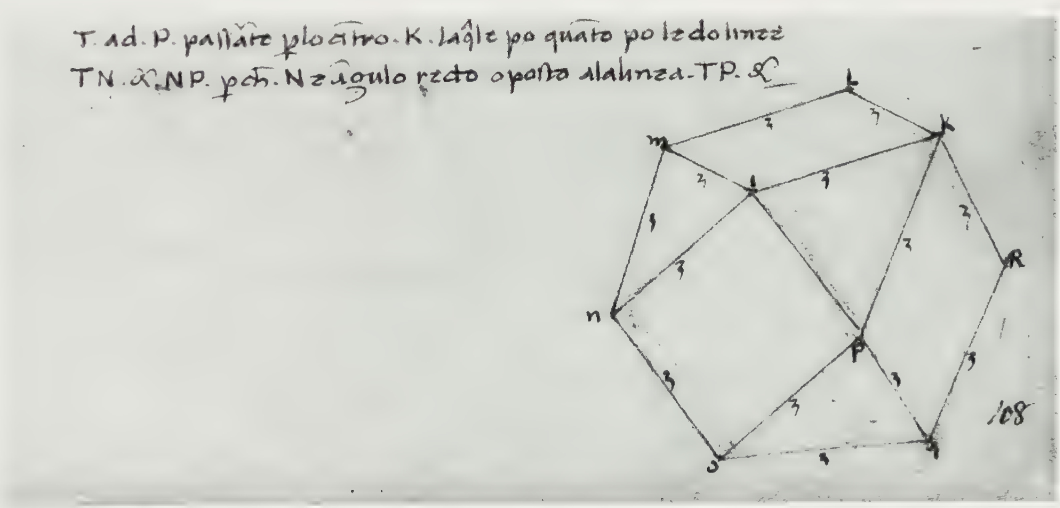
The two problems on the truncated tetrahedron are followed by two on another new solid. The first is:

> There is a spherical body, whose diameter is 6 *bracci*; I want to put in it a figure with fourteen faces, 6 square and 8 triangular, with equal edges. I ask what each edge will be.[32]

This polyhedron with fourteen faces is now known as the cuboctahedron. Piero immediately explains how it is obtained from the cube:

> Such a figure as this is cut out from the cube, because it [the cube] has 6 faces and 8 corners; which, cutting off its 8 corners, makes 14 faces, that is thus. You have the cube ABCD.EFGH, divide each side in half: AB in the point I, and CD in the point L, BD in the point K, . . .

Each of these mid points, *I*, *L*, *K* and so on, will be a vertex of the new solid. Piero also supplies a drawing of the new solid, showing it with its vertices on a sphere (Fig. 4.19).



4.19 Drawing of the solid now known as the cuboctahedron, from Piero della Francesca, *Trattato d'abaco*, BML MS, p.108 recto.

31 Like all the names now given to the individual Archimedean polyhedra, this one derives from the first complete description of all thirteen solids in Johannes Kepler, *Harmonices mundi libri V*, Linz, 1619, Book 2, Proposition 28 (English translation *The Harmony of the World*, trans. E. J. Aiton, A. M. Duncan and J. V. Field, Memoirs of the American Philosophical Society, vol.209, Philadelphia, 1997).

32 Piero della Francesca, *Trattato d'abaco*: BML MS, p.108 recto; Piero ed. Arrighi, p.231.

Like his drawings of the truncated tetrahedron, Piero's illustration of the cuboctahedron shows how faces fit together round each vertex and makes it clear that he has recognized certain properties of symmetry that were not mentioned by Pappus.[33]

Piero goes on to give brief discussions of properties of the remaining three Platonic solids, the dodecahedron, the icosahedron and the octahedron. All the problems are numerical. One figure is provided for each solid, showing it inscribed in a sphere. There follow a few miscellaneous additional problems on the tetrahedron, the cube and the sphere, and finally some more problems in algebra. The last few pages of the *Trattato d'abaco* thus give the appearance of being something of a ragbag. The extra problems might be taken to represent afterthoughts.

There is, in fact, a similarly miscellaneous group of problems at the end of Piero's *Libellus de quinque corporibus regularibus*. Since this work largely consists of problems taken from the *Trattato d'abaco*, some in neater or more developed forms, and we have every reason to believe that the surviving Latin text is a translation, Piero had clearly had time to put things in appropriate order, so afterthought seems an unconvincing explanation for the apparently untidy ending. Moreover, the *Libellus*, which is divided into four parts, each called a 'treatise' (*tractatus*), is a notably orderly work. Perhaps it was acceptable, or even expected in certain circumstances, that an abacus book should end with a selection of extra problems, possibly addressed to particularly able pupils. In any case, the *Libellus*, despite being in Latin in the only known manuscript (Vatican Library Codex urbinas 632), is put together in the style of an ordinary abacus book. As in the *Trattato d'abaco*, the work proceeds by means of numerical examples, with very little linking text.

As already noted, in view of its relationship to the *Trattato d'abaco*, it is clear that the *Libellus* must be the later of the two texts, but it cannot be dated precisely. The letter that serves as a preface is addressed to Guidobaldo da Montefeltro (1472–1508), as duke of Urbino, which proves that the letter was written in or after 1482. However, that does not give us a date for the text, only for this particular copy of it. The contents of the letter are unhelpful in this respect. They leave it open whether the text was written specially for Guidobaldo, or whether it was only the surviving copy that was made for him. Piero does, indeed, refer to the work as a fruit that has come late in his life (see the translation of the letter in Appendix 7), which implies that the book was written specially for the duke. However, it is hardly likely that one would tell a duke – even a young duke whose father one had known a long time – that the work he was being offered was merely a new copy of something that had been lying around in one's study for years.

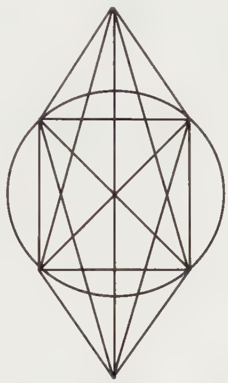
The *Libellus* is the only one of Piero's writings that is known to have been published in the Renaissance. It appears, in the vernacular and without Piero's name, as the final item in Luca Pacioli's *De divina proportione*, published in Venice in 1509.[34] Vasari tells us that Pacioli came into possession of Piero's manuscripts after his death.[35] The illustrations printed in the 1509 edition are not all the same as those that appear in the Latin manuscript, some of the differences being that the printed text has perspectival figures where the Latin one does not. The likeliest source of these up-to-date figures is that they were taken from draw-

33 An English translation of the problem is given in Appendix 6.

34 For Pacioli's dedication of the vernacular text of the *Libellus* to Pietro Soderini, see Appendix 7.

35 This provides a neat explanation for the extensive borrowings from Piero's *Trattato d'abaco* found in Pacioli's *Summa*, published in Venice in 1494. For the unlikelihood of Pacioli's having been Piero's pupil, see Introduction above.

ings by Piero. Pacioli's work is famous for the illustrations of polyhedra that were drawn for it by Leonardo da Vinci – apparently as substitutes for the actual models that had accompanied presentation copies of the manuscript versions – but Leonardo's illustrations are not for Piero's part of the text, and where Pacioli himself has supplied illustrations, in the first part of the book, they are in the conventional style of the time. A copy of Pacioli's illustration of the regular octahedron is given in Figure 4.20. Although Piero's figures in the *Trattato d'abaco* and the manuscript of the *Libellus* are not always in completely naturalistic perspective, they are far more naturalistic than those that we know were designed by Pacioli or by his printer.[36] Since it seems highly unlikely that Pacioli would have arranged for certain of Piero's figures to be redrawn, particularly in an unconventional style, the most natural explanation for the divergences between the illustrations would seem to be that Pacioli was not working from the manuscript now preserved in the Vatican Library. It is, of course, possible that he was working from a vernacular manuscript, and that the text he supplies is actually Piero's original.



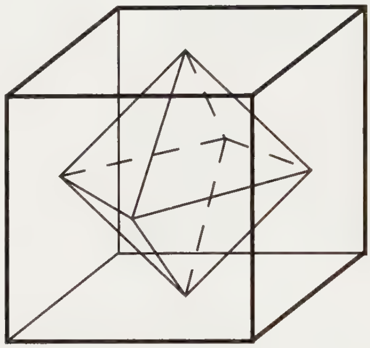
4.20 Copy of the illustration of the regular octahedron (which has eight equilateral triangle faces, meeting four at each vertex of the solid), with its diagonals and circumsphere. After Luca Pacioli, *De divina proportione*, Venice, 1509, Part 1, p.9 verso. Drawing by JVF.

The title of Piero's *Libellus de quinque corporibus regularibus* refers to three-dimensional geometry, but the work starts not in three dimensions but in two, with problems of finding the heights and areas of triangles. There follow some more elaborate problems on triangles (cases 6 to 13). As in Piero's *Trattato d'abaco*, these are in turn followed by problems on the square (cases 14 to 26), then problems on the regular pentagon (cases 27 to 36), on other polygons, and on the circle. Three-dimensional bodies appear in the second 'treatise' (*tractatus*) simply in the form of a first problem on the regular tetrahedron – that is, without any substitute for the brief introductory passage about bodies found in the *Trattato d'abaco*. The move towards an even less discursive style than before is presumably to be taken as an indication that the *Libellus* is the more learned work, and thus in a style closer to that of

36 See Field, 'Rediscovering the Archimedean Polyhedra' (full ref. note 28); and J. V. Field, 'Renaissance mathematics: diagrams for geometry, astronomy and music', *Interdisciplinary Science Reviews*, 29(3), 2004, 259–77.

Euclid. The whole of the second *tractatus* of the *Libellus* is taken up with problems on the five Platonic solids, 36 problems in all.

The third *tractatus* deals with Platonic solids inscribed inside one another, so that, for instance, the centres of edges of the tetrahedron become vertices of an inscribed octahedron (case 1), or a cube is inscribed in a dodecahedron so that vertices coincide (case 7). This latter problem is adapted from the construction given by Euclid in *Elements*, Book 13, Proposition 17, but the remainder of Piero's inscription problems come from the work he knew as *Elements*, Book 15. This book is now generally considered to be late antique, but it was known as part of the *Elements* throughout the medieval period, and was printed in several Renaissance editions of Euclid.[37]



4.21 A regular octahedron inscribed in a cube so that the vertices of the octahedron lie at the centres of the faces of the cube. Drawing by JVF.

The mathematical interest of the matter is that the inscription problems show symmetry properties of the bodies concerned. For instance, the fact that an octahedron can be drawn with its vertices as the centres of faces of a cube (case 3, Fig. 4.21) corresponds to the property that the vertices of either solid can be brought into coincidence with former positions of neighbouring vertices by turning the solid through a right angle around an appropriate axis. Since four such turns would bring the solid back where it started, this property is known as 'four-fold rotational symmetry'.[38]

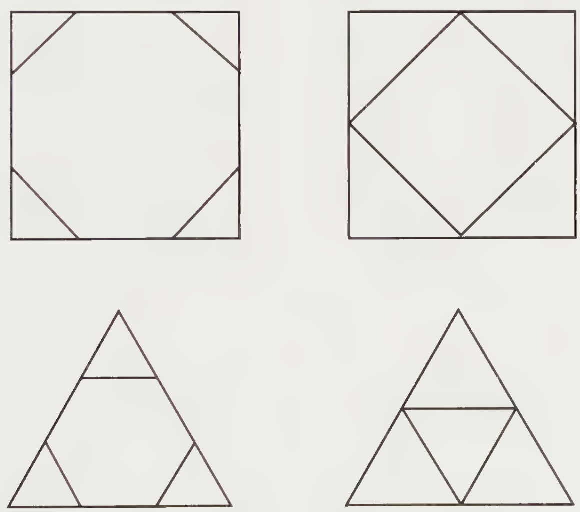
As we have seen, these inscription problems are not original, but they have no counterpart in Piero's *Trattato d'abaco* and their origin is in a work that undoubtedly belonged to the learned tradition of mathematics, namely one of the books accepted as part of Euclid's

37 Many were based on medieval versions. For instance, the *editio princeps* of the *Elements*, published by Erhard Ratdolt (*c*.1443–?1528) in Venice in 1482, printed the Latin translation by Campanus (Giovanni Campano da Novara, died 1296). Zamberti's Euclid (Venice, 1505), a new Latin version translated from a Greek manuscript, gives the author of Book 15 as Hypsicles, whom Zamberti's manuscript source had named (correctly) as the author of Book 14. This seems to be the first time doubt was cast on the authenticity of these last two books. The first printed Greek text of the *Elements* was edited by Simon Grynæus (died 1541) and published by Johannes Hervagius, Basel, 1533.

38 The experts on the mathematics of symmetry in the real world tend to be crystallographers. A crystallographer and a specialist in the relevant branch of mathematics have now written an excellently clear and well-illustrated book on the subject: I. Hargittai and M. Hargittai, *Symmetry: A Unifying Concept*, Bolinas, California: Shelter Publications Inc., 1994. In addition to its other virtues, this book shows a highly commendable tendency to avoid the work of Maurits Cornelis Escher (1898–1972).

Elements. Piero's letter to Guidobaldo in fact mentions that his problems come from learned sources, saying of his work: 'For, despite its novelty, it may not displease. And indeed it deals with things noted by Euclid and other geometers, in this work however newly expressed in arithmetical terms.'[39] The thirteen problems concerning polyhedra are followed by some problems on the sphere, only the first few of which have counterparts in the *Trattato d'abaco*. Further departures from the content of the *Trattato* are found in the fourth *tractatus*, which deals with what Piero somewhat unhelpfully calls 'irregular bodies'. These include truncated versions of all the Platonic solids, giving us five of the Archimedean polyhedra. One of these, the truncated tetrahedron, has already been described in the *Trattato d'abaco*, but the other four are new. The new ones appear in problems that begin by specifying the edge of the solid and ask for its surface area, its volume and the diameter of its circumsphere, but the problem on the truncated tetrahedron begins, as in the *Trattato*, by specifying the diameter of the circumsphere.[40]

Despite the non-uniformity in the way the problems are set, there is an important uniformity in the type of truncation concerned. Piero is applying to all the Platonic solids the form of truncation that in the *Trattato d'abaco* was applied to the tetrahedron, that is, the kind that turns a face with *n* sides into one with 2*n* sides: triangle to hexagon, square to octagon, pentagon to decagon. There is no use of the type of truncation used to make the cuboctahedron in the *Trattato d'abaco*, that is, truncation to the centres of edges, which turns a face with *n* sides into a smaller one of the same shape. Examples of these two types of truncation applied to polygons are shown in Figure 4.22.



4.22 Two types of truncation. On the left truncation turns a regular polygon with *n* sides into one with 2*n* sides. On the right, truncation turns a regular polygon into a smaller figure of the same shape. In the *Libellus de quinque corporibus regularibus* Piero della Francesca uses only the kind of truncation shown on the left. Both kinds appear in his *Trattato d'abaco*. Drawings by JVF.

39 Piero della Francesca, *Libellus de quinque corporibus regularibus*: Piero ed. Mancini, p.562; see also Appendix 7.

40 The problem that introduces the truncated octahedron is translated in full in Appendix 6.

Since the work of the *Trattato d'abaco* shows that Piero had thought about both types of truncation, the fact that he later consistently uses only one type in the *Libellus de quique corporibus regularibus* indicates that he distinguished between the two types. That is to say that he had invented the notion of truncation as a mathematical procedure. Piero's capacity for visualizing things in three dimensions clearly played a very important part in this. Thus one of the determining characteristics of Piero's pictorial style shows up as being one of the determining characteristics of his mathematics as well. As we shall see in the next chapter, it is also important in his work on perspective. The treatise on perspective of course provides the most obvious interface between Piero's abilities as a mathematician and his abilities as a painter.

5

The Force of Lines: *On Perspective for Painting*

The title of Piero's perspective treatise, *De prospectiva pingendi* (On perspective for painting) is partly an indication that by this time the word 'perspectiva' had begun to need qualification. In earlier centuries, it had simply designated the complete science of vision, including matters that properly belonged to natural philosophy, such as considerations of the nature of light. The subject was called a science because it was recognized as partly mathematical, that is as 'mixed'. Thus *perspectiva* was a form of study in which it was acknowledged as legitimate to use mathematics in the pursuit of natural philosophy. The use of mathematical methods to give an illusion of depth in pictures clearly belonged to the mathematical end of the subject. It formed part of the exercise of a craft but, as we shall see, Piero apparently did not see this as diminishing its intellectual credentials. All the same, this craft use probably contributed to the word *perspectiva* being considered to need qualification. The study of natural vision, the older or standard part of *perspectiva*, came to be called *perspectiva communis*, while the new part was called *perspectiva artificialis* or *perspectiva pingendi*.

Historians of art have sometimes felt able to discern a difference between the meanings attached to the Latin terms *perspectiva* and *prospectiva* as used in the Renaissance, and, correspondingly, between the meanings of the vernacular words *prospettiva* and *perspettiva*. Some distinction may perhaps be made among philosophers and more particularly among philologists of the time, but technical literature on the science of sight seems to use the words interchangeably, as if they merely represented variant spellings of the same term. For example, in his treatise *La pratica della perspettiva*, published in Venice in 1568 and 1569, Daniele Barbaro (1513–1570), who has impeccable credentials as a humanist scholar, uses the term 'perspettiva' while taking everything he says from Piero della Francesca's text, often verbatim and *in extenso*, apparently happy to ignore the fact that Piero himself uses the terms 'prospettiva' or 'prospectiva' (which in context are certainly no more than variant spellings). One has the impression that Barbaro regards this exchange of one term for another as no more than a dialect shift from fifteenth-century Tuscan to sixteenth-century Venetian. It takes its place alongside the tiny grammatical adjustments that, page after page, constitute the only differences between Barbaro's text and corresponding passages in Piero's. Barbaro himself does seek to distinguish between the natural appearance of objects, the 'aspect' of the Euclidean *De aspectuum vareiteate*, and the result of the analytical glance of the painter, but he presumably takes this to be a new distinction, since he does not suggest that Piero is misusing the term 'prospectiva'.[1]

1 Daniele Barbaro, *La prattica della perspettiva*, Venice, 1568, 1569, p.6.

We may note, also, that the term used in the title of Barbaro's treatise is 'perspettiva'. The title is certainly his own since it is found in both the manuscripts, now in the Biblioteca Marciana, Venice, one of which is autograph and the other partly so. The fact that he does not even mention the change he has made to Piero's usage in the title of his own work strongly suggests that Barbaro regards the change as anodyne. So indeed it seems to a historian of science. And, as we shall see, Piero presents his treatise as concerned with the learned scientific side of things. Indeed, the Latin title at once suggests such affiliation. As we have already noted in Chapter 3, all known manuscripts of Piero's perspective treatise, vernacular as well as Latin, give the title in Latin. Furthermore, Latin and vernacular manuscripts all have the same incipit: 'Petrus pictor burgensis de prospectiva pingendi'. Signing his text in Latin is, of course, in accord with his also having used Latin when signing his pictures. In any case, as we shall see, the content of the perspective treatise provides ample evidence to support the suggestion that by giving the work a Latin title Piero was making a claim that the subject it deals with is intellectually respectable.

There is no dedicatory letter to *De prospectiva pingendi*. The prefatory letter to the *Libellus de quinque corporibus regularibus* tells us that a copy of the perspective treatise was held in the Ducal library at Urbino but does not say that the work had any other connection with the dukes of Urbino, Federigo or Guidobaldo da Montefeltro (see translation in Appendix 7), so we must surely presume that it did not. Like the huge majority of Piero's surviving works, the perspective treatise is thus datable only relatively, as earlier than the *Libellus* and by inference from internal evidence (see below). As far as we know – and, it would seem, as far as Piero himself knew – *De prospectiva pingendi* was the first work to give a detailed account of the techniques of perspective. The work is in principle addressed to practitioners, who are taken, step by step, through worked examples of perspective construction. It is not entirely clear what readers Piero really has in mind; the matter will be discussed below and in Chapter 7. As in abacus books, the reader is addressed as 'tu' and drawing instructions are mainly delivered in the imperative. Thus, although the subject matter is new, the style has clear antecedents in the abacus tradition. Passages of discursive or explanatory text are short. All the same, as in the *Trattato d'abaco* and the *Libellus de quinque corporibus regularibus*, Piero's work is decidedly orderly, and one can thus reasonably speak of following a line of thought. Piero is also precise about what he will deal with. For example, he makes it clear from the outset that we are not concerned with the whole of painting, but only one part of it.

Piero's introduction to *De prospectiva pingendi*

The introduction that precedes the first book of *De prospectiva pingendi*, and thus serves as an introduction to the complete work, covers about a page (folio) in the manuscript versions. It begins

> Painting has three principal parts, which we say are drawing [*disegno*, *designatio*], proportion [*commensuratio*] and colouring [*colorare*, *coloratio*]. Drawing we understand as meaning outlines and contours contained in things. Proportion we say is these outlines and contours positioned in proportion in their places. Colouring we mean as giving the colours as they are shown in the things, light and dark according as the light makes them vary. Of the three parts I intend to deal only with proportion, which we call perspective, mixing in with it some part of drawing, because without this perspective cannot be shown in action;

colouring we shall leave out, and we shall deal with that part which can be shown by means of lines, angles and proportions, speaking of points, lines, surfaces and bodies.[2]

Piero then goes on to specify the subdivision of this part, incidentally providing definitions of terms:

> Which part itself contains five parts: the first is seeing, that is the eye; second is the form of the thing seen; the third is the distance from the eye to the thing seen; the fourth is the lines which go from the edge [*estremità*, *extremitas*] of the thing and come to the eye; the fifth is the limit [*termine*, *terminus*] that is between the eye and the thing seen, which is where it is intended to put the things.

The definitions of terms that are given here are those used in the remainder of the work. There was no standard vocabulary for discussing perspective at this time, or indeed for about two centuries afterwards, so Piero was essentially free to choose what words he pleased. The one that has sometimes caused confusion among historians, and presents a difficulty for the translator, is *termine/terminus*, which in context clearly signifies what would now be called the picture plane (but bearing in mind that to Piero a plane has edges to it). However, although a modern reader may not find the word *termine* felicitous, it is not a serious impediment to understanding since Piero, in the manner of Euclid and succeeding generations of serious mathematicians, has defined the meaning of the term before proceeding to use it.[3]

Piero goes on to discuss each of the five parts he has mentioned, starting with 'the eye', that is with what we should call natural optics:

> The first I said was the eye, of which I do not intend to treat, except insofar as it is necessary for painting. So I say the eye is the first part, because it is that in which all the things seen present themselves as subtending various angles; that is, when the things seen are equally distant from the eye, the larger thing presents itself as subtending a larger angle than the smaller, and similarly, when the things are equal and are not equally distant from the eye, the nearer presents itself as subtending a larger angle than the further one does, through which differences we account for the degradation of these things.[4]

The results in natural optics to which Piero refers, and to which he will return in his series of propositions, are entirely standard ones, derived from the optical work of Euclid. They are to be found in both the treatises on *perspectiva communis* that circulated most widely in Piero's time, those of John Pecham and of Witelo, both of which date from the thirteenth century (see Chapter 2). As will be clear from the context, the 'degradation' to

2 Piero della Francesca, *De prospectiva pingendi*: Parma MS, p.1 recto; BL MS, p.1 recto; Piero ed. Nicco Fasola, pp.63–4.

Unless otherwise indicated, my translations have been made from the vernacular in the first instance, sometimes using the Latin for clarification. Differences between the texts have been noted when there seems to be a significant shift in the sense. See also note 15.

3 This cold-blooded appeal to logic rather than what usually passes for reason is normal in mathematics. The standard formulation of what is meant by a definition is in Lewis Carroll (Charles Lutwidge Dodgson, 1832–98), *Through the Looking-Glass and What Alice Found There*, London: Macmillan, 1872, Chapter 6: '"But 'Glory' doesn't mean 'a nice knock-down argument'," Alice objected. "When *I* use a word," Humpty Dumpty said in a rather scornful tone, "it means what I choose it to mean – neither more nor less."'

4 Piero della Francesca, *De prospectiva pingendi*: Parma MS, p.1 recto; BL MS, p.1 recto; Piero ed. Nicco Fasola, p.64.

which Piero refers in the last line of this passage is the change of apparent shape caused by these optical effects, that is the shape of an object is 'degraded' when it is seen in perspective. The seemingly pejorative term is presumably considered appropriate because, as we shall see, a neat shape such as a square will become a less neat shape such as a trapezium. In any case, Piero always refers to the original shapes of objects as their 'proper' or 'perfect' form, and the perspective image as the 'degraded' form.[5]

Next comes the geometrical structure of the object:

> The second is the form of the thing, because without that the intellect could not judge or the eye take in that thing.

This may seem neutral enough, but giving a usable description of the form involves plans and sections that can become rather complicated, for instance when we are to describe a human head (in the third book of Piero's treatise, see Fig. 5.25). The next item is the distance between the object and the eye:

> The third is the distance from the eye to the thing, because, if it were not for the distance, the thing would be touching or close to the eye, and when the thing was bigger than the eye, the eye would not be capable of receiving it.

This too seems straightforward, and it is in fact an effect that was discussed briefly by Alberti,[6] but in the present context it will turn out to have consequences: it will lead Piero to discuss the maximum viewing angle that can be prescribed if a picture is to be what he regards as optically correct.

Next we come to the sheaf of rays that define the outline of the object:

> The fourth are [*sic*] the lines which present themselves [as starting] from the edge of the thing and end in the eye, through which the eye receives and discerns them [that is the things seen].

Piero's wording might be taken as implying a belief that sight is by the reception of light rays, which is the theory of vision put forward in detail by Ibn al-Haytham and known in Piero's time mainly through the optical treatise of Witelo. However, the phrase translated as 'present themselves' is also used when Piero means to refer to something subtending an angle at the eye, and it is possible that here his formulation is intended to refer merely to the geometry of the set-up. Parts of the main text of *De prospectiva pingendi* seem to imply that Piero subscribed to the more commonly held optical theory of his time, whereby vision took place through the emission of visual rays from the eye.[7]

The final item included in the mathematical treatment of perspective is the picture itself (in connection with which Piero refers to rays emitted by the eye):

5 There is thus an asymmetry between object ('perfect') and image ('degraded'), which makes this work, and all other works on perspective, entirely different from what is now called projective geometry. In projective geometry the relationship between object and image is symmetrical, so that the object is the projection of the image in exactly the same way as the image is the projection of the object. This symmetry is fundamental to the way results are obtained in projective geometry. See J. V. Field, 'Linear Perspective and the Projective Geometry of Girard Desargues', *Nuncius* 2.2, 1987, pp.3–40; and J. V. Field, *The Invention of Infinity: Mathematics and Art in the Renaissance*, Oxford: Oxford University Press, 1997.

6 Alberti, *De pictura*, §6.

7 In particular Book 1, Section 30, which is discussed below. On the final decision, made in the early seventeenth century, in favour of the theory that vision was through the reception of light rays, see Chapter 2, p.37.

The fifth is a limit [*termine*] on which the eye draws the things in proportion with its rays and in which it can judge their size: if there were no limit it would not be possible to take in how much the things were degraded, for it could not be demonstrated.

The word 'demonstrated' is clearly a pointer towards the mathematical treatment that follows. But first there comes a little practical matter that is, of course, entirely straightforward for Piero:

Furthermore it is necessary to know how to draw all the things that one intends to deal with, in their proper form on the plane.

The 'proper' form clearly refers to the actual shape of the object. From what follows in the main text of Piero's treatise, it will appear that 'on the plane' means that we are to give two-dimensional diagrams to explain the shape of objects, some of them three-dimensional objects, as a preliminary to making a perspective picture of the object concerned. These diagrams are usually what would now be called ground plans and horizontal or vertical sections.[8]

This may seem uncomplicated, but awkward elements appear almost at once. First, it becomes clear that Piero's methods of making a two-dimensional diagram of what is in principle a three-dimensional set-up do not always follow exactly the same conventions as those that prevail in our own time. This should perhaps be taken as obvious by art historians, but misreading of Piero's diagrams, in particular misreading of their relationship to the words of his text, has led to queer misunderstandings of his mathematics.[9] Second, drawing the plans and sections – without the help of a perspective picture, since that is what we want to find – involves a use of the mind's eye that reminds the lay reader that Piero is, one must suppose, addressing himself to people with an unusual degree of skill in this respect. In some cases, a Brunelleschian use of carved turnips as an intermediary procedure may seem to have much to recommend it. In historical terms also, Antonio di Tuccio Manetti's account of Filippo Brunelleschi's method of communicating with his workmen by the use of three-dimensional models serves as due warning that in this period the reading of technical drawings was not yet unproblematic.[10] Piero's usages are thus interesting pieces of evidence, since there can be no doubt of his competence both as mathematician and as draughtsman.

Having dissected perspective into five components and explained each of them, Piero sketches the contents of his treatise:

The above things being understood, let us proceed with the work, making of this part [of painting] called perspective three books. In the first we shall discuss points, lines and plane surfaces. In the second we shall discuss cubic bodies, square pillars, round columns and ones with many faces. In the third we shall discuss heads and capitals, bases, solid rings with many faces and other bodies variously placed.

As we shall see, this is a perfectly accurate account, though Piero is not as clear as he might be about the content of the second book. It deals with right prisms. That is, the plane figures

8 The notion of an 'elevation' begs questions in this context, since it implies a projection on a vertical plane. As already mentioned in note 5, the notion of projection is foreign to Piero's treatment of perspective.

9 These include the implausible claim that Piero failed to prove the correctness of his perspective construction, in James Elkins, 'Piero della Francesca and the Renaissance proof of Linear Perspective', *Art Bulletin* 69.2, 1987, pp.220–30. As will be seen below, Piero's proof is perfectly correct but uncomfortably concise. Elkins has simply misunderstood the mathematics.

10 Antonio di Tuccio Manetti, *Life of Brunelleschi*, ed. H. Saalman, trans. C. Engass, University Park and London: University of Pennsylvania Press, 1970, p.93 (lines 1011ff).

of the first book are used as ground plans from which bodies are constructed by putting in vertical plane faces, thus, for instance, converting a square into a cube.

In view of Piero's vigorous defence of perspective in his introduction to his third book, we may note that this introduction to the work as a whole is apparently taking it for granted that perspective is a useful tool for the painter. Piero seems to be assuming that his readers share the opinions expressed by Alberti in *De pictura/Della pittura*. This is, of course, not exactly the same as assuming they had read Alberti. Nor is it strong evidence that Piero himself had read Alberti. Whether he read him or not, Piero is, nevertheless, effectively going through dotting the 'i's and crossing the 't's in the mathematical part of what Alberti said. He is making the contribution that his mathematical competence entitles him to make. One possible explanation for his deferring a defence of perspective until later is that it is only in the third book of his work that he goes beyond the kind of material Alberti had dealt with. Perhaps by the time Piero wrote his treatise such material was regarded as a normal use of perspective – 'the ordinary rules' as Egnazio Danti (1536–1586) was to call them[11] – whereas the work of the third book went beyond habitual practice and was thus seen as in need of defence. The technique described in the third book is also decidedly laborious, which might have been another reason for its requiring defence.

Mathematical preliminaries

Since the whole of *De prospectiva pingendi* is mathematical, it is not unexpected that at the end of the introduction there are some remarks about mathematics. Piero considers the relation between the point and the line (when defined as ideal objects of mathematics) and the definitions of point and line that apply in the diagrams one can draw to represent such entities:

> A point is that which has no parts, accordingly the geometers say it is [merely] imagined; the line they say has length without width. And because these are not seen except by the intellect and I say I am dealing with perspective in demonstrations that are to be taken in by the eye, on this account it is necessary to give a different definition. So I shall say the point is a thing as tiny as it is possible for the eye to take in; the line I say is extension from one point to another, its width being of the same nature as that of the point. A surface I say is width and length enclosed by lines.[12]

Apart from its greater length and formality, this is essentially the same comment that Alberti had made about representing a point in *De pictura/Della pittura* (in the first sentence of §2). In itself the remark is of no great importance. However, in the Latin version of Piero's treatise there is a slight modification. The first two sentences become:

> A point is that which has no parts so we understand it by our imagination. A line is length without breadth and it has the same property as the point when we perceive it too in the mind.[13]

11 Egnazio Danti, in his introduction to his edition of the treatise *Le Due regole della prospettiva prattica* (Rome, 1583) by Giacomo Barozzi called da Vignola (1507–73). See J. V. Field, 'Giovanni Battista Benedetti on the Mathematics of Linear Perspective', *Journal of the Warburg and Courtauld Institutes* 48, 1985, pp.71–99.

12 Piero della Francesca, *De prospectiva pingendi*, Introduction: Parma MS, p.1 verso; Piero ed. Nicco Fasola, p.65.

13 'Punctum est cuius pars non est quoniam id imaginando complectimur linea est longi tudo sine latitudine que natura puncti obtinet quando ea quoque mente percepitur.' Piero della Francesca, *De prospectiva pingendi*, Introduction: BL MS, p.1 verso, ll. 6–7.

As we have already noted in Chapter 3, there is good external evidence that the Latin text of *De prospectiva pingendi* is a translation from a vernacular original. There is also good internal evidence for the Latin's being a translation, for instance, in inevitable infelicities in dealing with things unknown to the Romans, such as the use of 'testudinem sciue fornicem' (literally, a tortoise or arched entrance) to translate 'una volta in crociera' (a cross-vault) (Book 2, Proposition 11).[14] However, in this passage at the end of the introduction to Book 1, the omission of the reference to 'geometers' would have been rather high-handed if the decision were taken by the translator. It is, in fact, only one of several changes of this kind that occur in the Latin text. There are also a number of small changes that make proofs more formal, and sometimes a few words are added by way of further explanation (for instance, in Book 1, Proposition 6). All the changes tend in the same direction: we are presented with a text that apparently aspires to be naturalized into the learned tradition, rather than something that is merely a Latin rendering of some possibly exotic material from the practical one. Moreover, the nature of many of the changes suggests that a competent mathematician must have been consulted, for instance to correct errors in the lettering given in the vernacular text. Since the translator, Maestro Matteo, was a friend of Piero's (see Chapter 3), the simplest explanation is that Piero himself took a hand in the revisions that were made to the text in the course of the translation process. This has implications for our assessment of Piero's possible engagement with other Latin texts, and for our interpretation of the omission from the Latin text of the perspective treatise of the passage that refers to the 'distance point' construction (Book 1, Proposition 23).[15]

Having discussed the point and the line, Piero goes on to surfaces (all finite areas, of course):

> Surfaces are of many kinds, such as a triangle, a quadrangle, a tetragon, a pentagon, a hexagon, an octagon and with more and different edges, as will be shown in the figures.[16]

Although he does not seem to be entirely consistent in the matter, Piero usually uses quadrangle to mean any four-sided figure, whereas the Greek-derived name tetragon, which should mean the same as quadrangle, is used to mean a square. However, as in the *Trattato d'abaco*, we sometimes find curious awkwardnesses that seem to arise from the absence of a recognized word that is equivalent to the modern 'rectangle'.

After this giving of names we move on to the series of propositions. Unlike those of Piero's *Trattato d'abaco* and *Libellus*, the first of these propositions in *De prospectiva pingendi* are

14 Piero della Francesca, *De prospectiva pingendi*, Book 2, Section 11: Parma MS, p.29 recto, and Piero ed. Nicco Fasola, p.122 for the vernacular; BL MS, p.33 verso for the Latin.

15 See below and J. V. Field, 'Piero della Francesca and the "Distance Point Method" of Perspective Construction', *Nuncius* 10.2, 1995, pp.509–30.

In clearing up difficulties of mathematical interpretation, I found it was often easier to use the Latin manuscript in the British Library (Add. MS 10366) than the vernacular manuscript from Parma (Biblioteca Palatina, MS no. 1576), which forms the basis for Nicco Fasola's printed edition of Piero's work. It is perhaps unfortunate that the latter has been given authority by the mere fact of being available in print. As appears from her notes, Nicco Fasola herself quite often had recourse to the Latin manuscript in the Biblioteca Ambrosiana, Milan (C307 inf), to make emendations to obviously faulty lettering found in mathematical passages in the Parma text.

My own use of the British Library manuscript was purely a matter of convenience. The relationship of the various Latin manuscripts is not yet well understood; see Giovanna Derenzini, 'Note autografe di Piero della Francesca nel codice 616 della Bibliothèque Municipale di Bordeaux. Per la storia testuale del *De prospectiva pingendi*', *Filologia Antica e Moderna* 9, 1995, pp.29–55.

16 Piero della Francesca, *De prospectiva pingendi*: Parma MS, p.1 verso; BL MS, p.1 verso; Piero ed. Nicco Fasola, pp.65–6.

not problems posed in numerical terms but theorems, that is general results given in an abstract form like that found in works by Euclid. This is partly a reflection of the fact that some of the theorems do indeed come more or less straight out of Euclid, that is, from his optical work. However this is not all the story. Another part of it is that throughout *De prospectiva pingendi* the answer to a problem, the usable solution, is not a number but a drawing, something one can do on the picture surface or transfer to it.

Proceeding by worked examples would have reminded the apprentice not only of classroom lessons in the abacus school but also of more recent hours in the painter's workshop when he copied drawings by his master. Piero's treatise does indeed seem to be conceived more or less as an equivalent, for perspective, to the manuals of drawings that are known to have been in use in painters' workshops. Whereas working through a manual of drawings, making a copy of each one, taught the apprentice how to draw, working through *De prospectiva pingendi*, following the drawing instructions given for each problem, taught an apprentice how to draw things in perspective. Perhaps Melozzo da Forlì (1438–1495) and Luca Signorelli, both excellent draughtsmen in later life, really were put through a course of instruction like that contained in Piero's treatise.[17] The detailed and hugely repetitious drawing instructions are mind-numbing to a mere reader.[18] They seem to be ready-made for the electronic computers invented about five centuries later. However, they do tell the reader everything that would be required by someone reading pencil in hand and drawing while proceeding through the text. This makes a pleasant change from the guessing and adjusting that has to be done in attempting to follow Alberti's work in detail, and it is presumably indicative of a difference in the expected primary readership of the works.

The bulk of *De prospectiva pingendi* consists of worked examples, giving instructions for how to construct the perspective image of some given figure. Piero's choice of examples was endorsed by posterity: every later treatise on perspective up to about 1630 borrows them and presents them in his order. What later treatises omit is the preliminary mathematics, the series of theorems with which Piero begins his first book. These are theorems that will be needed in the remainder of the work. Some of them are well known, either extracted from standard works on optics or adapted from theorems found there. For instance, as Book 1, Proposition 1, Piero gives the elementary optical statement that: 'Every quantity presents itself as subtending an angle at the eye.'[19] However, as Proposition 4 we have the purely geometrical result: 'If from a point there start lines to two equal bases and one [base] is nearer than the other, the nearer will make a larger angle at the said point.'[20] And in

17 As we have seen in Chapter 3, there is good evidence that Signorelli was a pupil of Piero. The evidence for Melozzo is more tenuous, being largely stylistic, though it is generally accepted; see Nicholas Clark, *Melozzo da Forlì*, ed. Amanda Lillie *et al.*, London: Sothebys, 1990; for a contrary opinion about Melozzo, see Creighton E. Gilbert, 'Melozzo: his Status, his Drawings', in *Arte d'Occidente: Studi in Onore di Gloria Maria Romanini*, Rome: Sintesi d' Informazione, 1999, pp.1043–50.

18 Daniele Barbaro's sense that Hell was freezing over as he read *De prospectiva pingendi* (probably in the vernacular, since he came across the distance point construction; see Field, 'Piero della Francesca and the "Distance Point Method"' (full ref. note 15)) is unkindly encapsulated in a comment that Piero's treatise was written for idiots: Barbaro, *La prattica della perspettiva* (full ref. note 1), Preface.

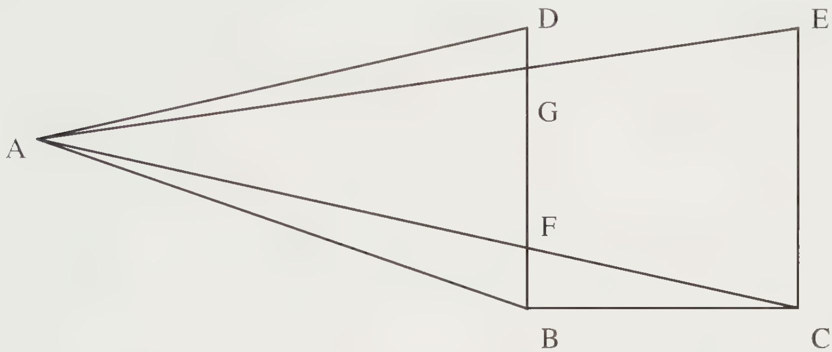
19 Piero della Francesca, *De prospectiva pingendi*, Book 1, Section 1: Parma MS, p.1 verso; BL MS, p.1 verso; Piero ed. Nicco Fasola, p.66.

20 Piero della Francesca, *De prospectiva pingendi*, Book 1, Section 4: Parma MS, p.2 recto; BL MS, p.2 verso; Piero ed. Nicco Fasola, p.67.

Proposition 6 we have a result that combines two results from Euclid's *Optics* (his Propositions 10 and 11):

> If there are two lines perpendicular to a line and parallel to one another, and from a point there start two lines which go to the ends of the nearer [line] and two others which go to the ends of the further [one], I say that the lower end of the further [line] will be represented at a point higher than the lower end of the nearer [line] and, if the upper ends [of the lines] are above the point that of the further [line] will be represented lower down.[21]

A copy of Piero's diagram is given in Figure 5.1. Piero has not only amalgamated two of Euclid's results to make one more general one, he has also given a proof that is slightly different from Euclid's. However, all Piero's proofs use entirely standard mathematical methods. As is usual in fifteenth-century mathematics, he relies heavily on similar triangles.



5.1 Copy of the illustration to Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 6. Drawing by JVF.

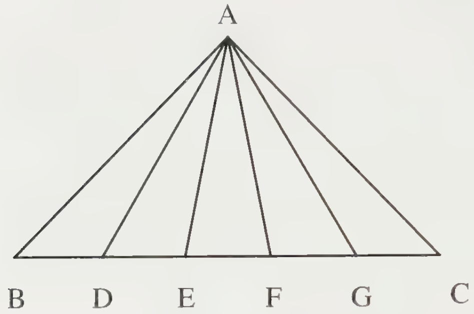
The theorems up to Book 1, Proposition 6 are effectively a series of elementary geometrical results that are relevant to optics. With hindsight assisted by some knowledge of perspective constructions, Proposition 7 can be seen as marking the beginning of the emergence of something new. It is:

> If the straight line be divided into several equal parts and from these divisions there start several lines which end at a [single] point, they will make at the said point unequal angles, the shorter lines will make a greater angle than the longer ones.[22]

A copy of the accompanying diagram is shown in Figure 5.2. The fact that the angles enclosed between the shorter lines are larger than those enclosed between the longer ones (nearer the outside edges of the figure) turns out to be important in Piero's discussion of the problem of representing a set of columns (Book 2, Proposition 12), in which the lines

21 Piero della Francesca, *De prospectiva pingendi*, Book 1, Section 6: Parma MS, p.3 verso; BL MS, p.3 verso; Piero ed. Nicco Fasola, p.69.

22 Piero della Francesca, *De prospectiva pingendi*, Book 1, Section 7: Parma MS, p.3 verso; BL MS, p.3 verso; Piero ed. Nicco Fasola, p.69.



5.2 Copy of the illustration to Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 7. Drawing by JVF.

BA, *CA*, *DA* and so on meet in the eye. Piero's proof is based on theorems from Euclid, and ends, in both vernacular and Latin texts, with an explicit reference to *Elements*, Book 1, Proposition 24.

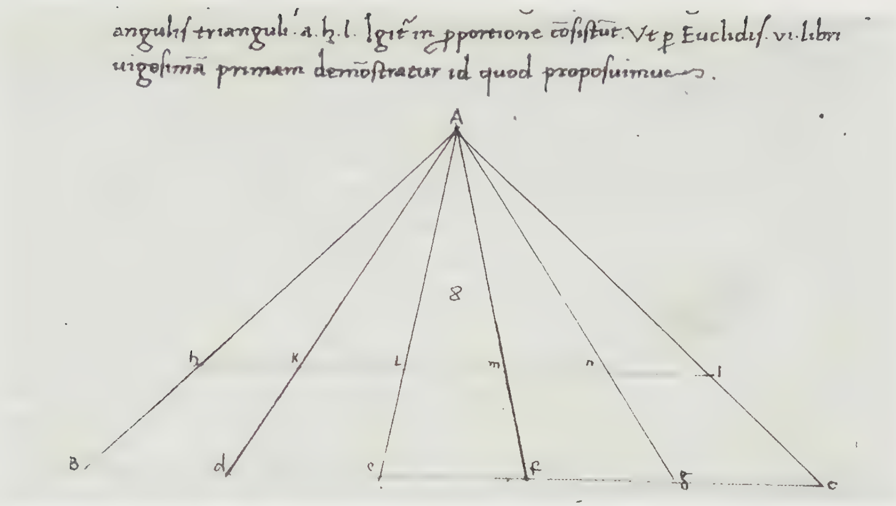
The diagram for Proposition 7 also resembles that for the following one, Proposition 8, in which the lines to *A* are reminiscent of the orthogonals converging to a 'centric point'. Proposition 8 is:

> If above a given straight line divided into several parts another line be constructed parallel to it and from the points dividing the first line there be drawn lines which end at one point [i.e. are concurrent], they will divide the parallel line in the same proportion as the given line'[23]

Figure 5.3 shows the diagram that accompanies this proposition in the British Library Latin manuscript of *De prospectiva pingendi*. The result Piero proves here is an important one, and should properly be known as 'Piero's theorem'. What Piero shows, for the first time, is that the perspective images of orthogonals converge to a point. In *De pictura/Della pittura* Alberti merely assumed the truth of this result.

However, Piero proves his theorem without any reference to its significance, that is, to the significance it will acquire once he comes to use it, in Proposition 14. For the moment, all he has is a set of convergent lines and two parallel transversals. His proof that the pattern of ratios found on the lower line is reproduced exactly on the upper one depends entirely on the use of properties of pairs of similar triangles (for a translation of the complete proof, see Appendix 8). This method of proof has the important property of allowing the proof itself to be reversible. Thus Piero has not only proved that if *BH*, *DK*, *EL*, *FM*, *GN* and *CI* can all be extended to meet at *A*, then the pattern of ratios defined by the points *HKLMNI* on the second transversal, *HI*, is the same as that defined by *BDEFGC* on the first one, *BC*. He has also proved the converse theorem, namely that if the patterns of ratios are the same then the lines can all be extended to meet at *A*. It is this converse theorem that establishes the convergence of the perspective images of orthogonals. Piero does not state this explicitly, but he presumably expects that the attentive reader will have noticed it, or

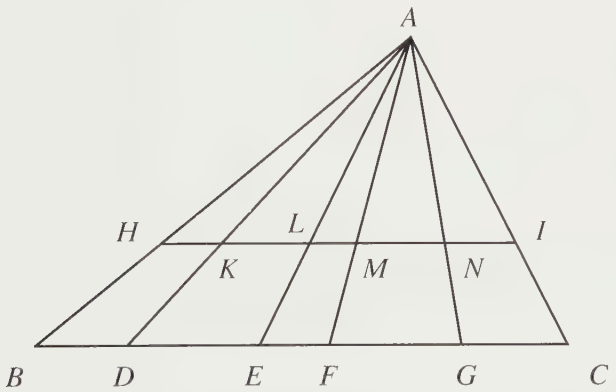
23 Piero della Francesca, *De prospectiva pingendi*, Book 1, Section 8: Parma MS, p.4 recto; BL MS, p.4 recto; Piero ed. Nicco Fasola , p.70.



5.3 Illustration to Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 8. From the Latin manuscript, BL MS, p.4 recto.

there was no point in his proving Proposition 8 in the first place. We may also note that Piero has in fact proved more than is immediately apparent from the diagram.

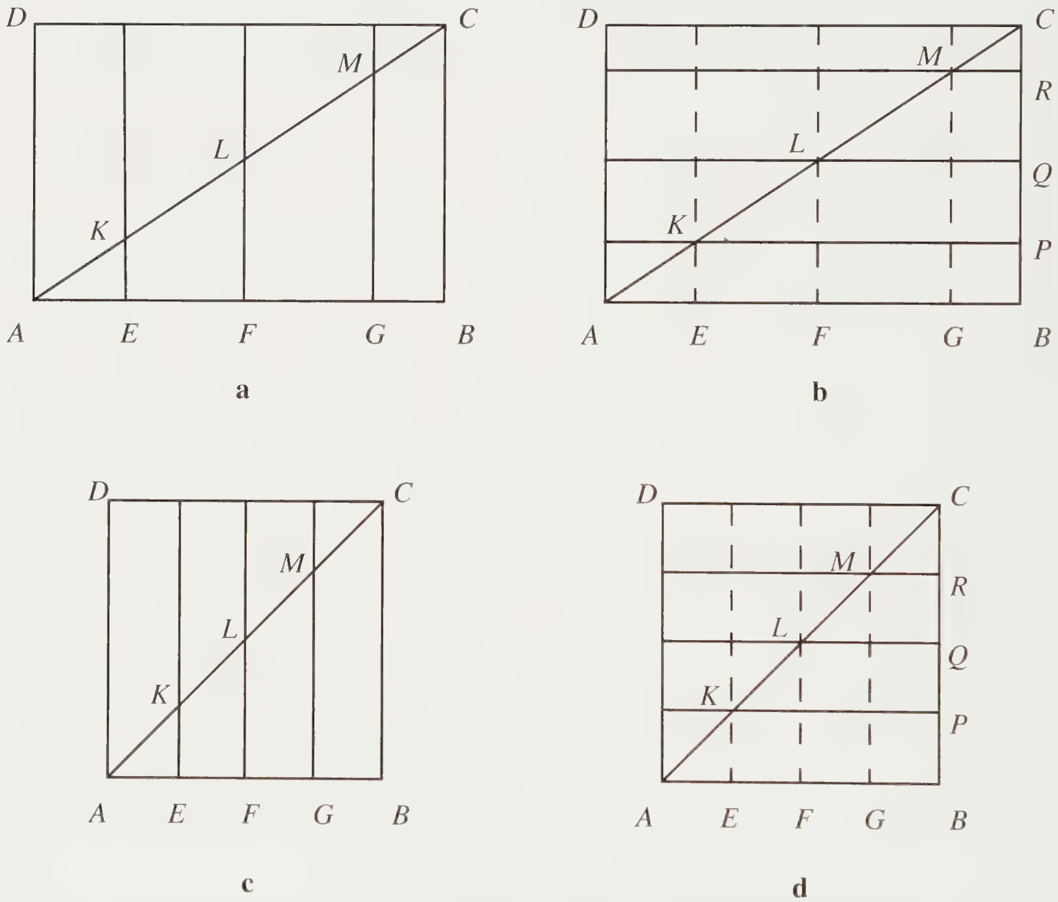
This diagram, which appears in much the same form in all manuscripts, is presumably drawn by Piero himself in the autograph manuscript now in Parma (Biblioteca Palatina 1576). The diagram is symmetrical, that is *A* is directly above the mid point of the line *BC*, and the divisions along *BC* are even. The diagram is so tidy that the theorem itself may seem obvious, merely from considerations of symmetry. However, the word 'obvious' makes mathematicians bristle at any time, and Euclid makes a habit of avoiding the notion of symmetry (which can, indeed, be misleading on occasion). Piero's proof makes no appeal to symmetry; it merely uses similar triangles. The proof would accordingly work just as well if the diagram looked as in Figure 5.4. As we shall see in considering Proposition 14, where Piero's point *A* corresponds to what Alberti called the 'centric point', this possible asym-



5.4 Diagram showing a general case of the theorem Piero has proved in *De prospectiva pingendi*, Book 1, Proposition 8. Drawing by JVF.

metry in the diagram has some interesting mathematical consequences. It is, of course, a separate, historical, question whether Piero himself was aware of them.

The following three sections, Propositions 9, 10 and 11, deal with dividing up a quadrilateral surface, rectangular or square, by means of lines parallel to its sides and by the use of diagonals.[24] These results enable Piero to show that one can use a diagonal to transfer a pattern of proportions from one side of a rectangle to another side at right angles. That is, in Figure 5.5a, let *ABCD* be a rectangle. If we divide the side *AB* in the points *EFG*, and draw through each of these points a parallel to *BC*, then the points *KLM* in which these parallels meet the diagonal *AC* will define the same pattern of ratios on *AC* as the points *E F G* defined on *AB* (because triangles *AKE*, *ALF*, *AMG*, *ACB* are all similar). If, as in Figure 5.5b, we now draw through the points *K L M* lines parallel to *AB*, the points *P Q R* in which these parallels meet *BC* will define on *BC* the same set of ratios as *K L M* defined



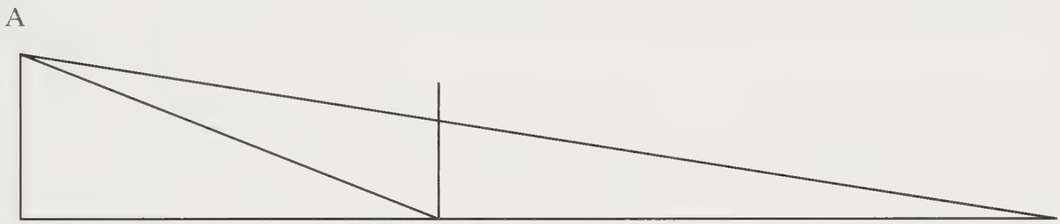
5.5 Diagrams to show the division of a rectangle or square using the diagonal to transfer patterns of ratios. Drawings by JVF.

24 Propositions 9 and 10 are translated in Appendix 8.

on *AC* (because triangles *CAB*, *CKP*, *CLQ*, *CMR* are similar). So working through the diagonal has enabled us to transfer a pattern of ratios on the line *AB* to the line *BC*.

In the simplest case, shown in Figures 5.5c and 5.5d, *ABCD* is square and the original division was into equal parts, so we have divided a square up into smaller squares. This simple case is the one needed for the simplest of Piero's paving patterns (completed in Book 1, Proposition 15). More complicated ratios can be used to construct more elaborate paving patterns, such as that in the black and white parts of the flooring shown in Piero's *Flagellation of Christ* (Galleria Nazionale delle Marche, Urbino, Fig. 5.28) (for which the front side of the square needs to be divided in the ratios $\sqrt{2}:1:\sqrt{2}:1:\sqrt{2}$).[25]

Proposition 12 shows how to 'degrade' a surface, of undefined shape, which is shown in profile as a straight line, making the problem that of finding the image of a line perpendicular to the picture plane. As can be seen in the copy of Piero's diagram in Figure 5.6, the representational convention used is that of showing a vertical section of the set-up.[26] However, the following proposition, in which Piero proves the correctness of his construction of the perspective image of a square, apparently makes use of more complicated representational conventions.



5.6 Copy after the diagrams for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 12, Parma MS, p.6 recto and BL MS, p.6 recto. The proportions have not been copied exactly and most of the lettering has been omitted. Drawing by JVF.

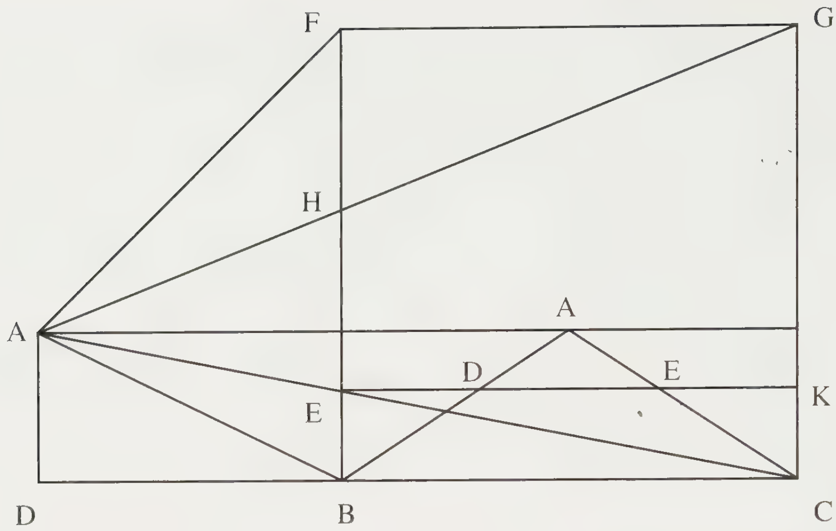
In the British Library Latin manuscript, Book 1, Proposition 13 takes up two sides and a diagram is provided on each (BL MS, pp. 6 verso and 7 recto).[27] The proportions shown in the diagrams are drastically different: the first diagram shows a viewing distance of a little more than twice the width of the square, while the second shows a viewing distance of about half the width of the square, and the height of the eye is markedly lower in the second diagram than it is in the first. However, the lines shown are the same in each figure, except that the line *AB* (outside the square) does not appear in the first figure. The figures do not, however, include all the lines that the reader is instructed to draw, and in some manuscripts the lines supplied are not sufficient to allow one to follow Piero's proof. In this case, the text clearly has priority over the drawings.

As can be seen from the compromise copy of Piero's diagrams in Figure 5.7, they bear a notable resemblance to the figures one draws for the Albertian construction (compare Fig.

25 See Appendix 5.

26 Proposition 12 is translated in Appendix 8.

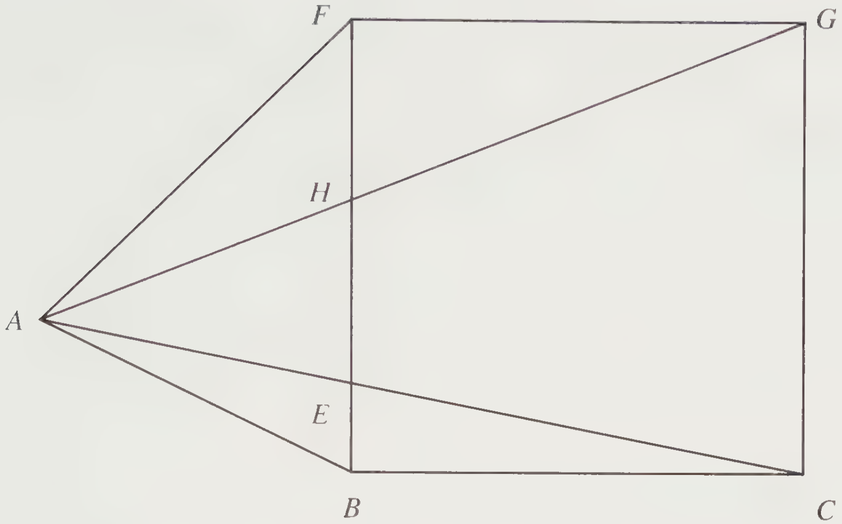
27 Proposition 13 is translated in full in Appendix 8.



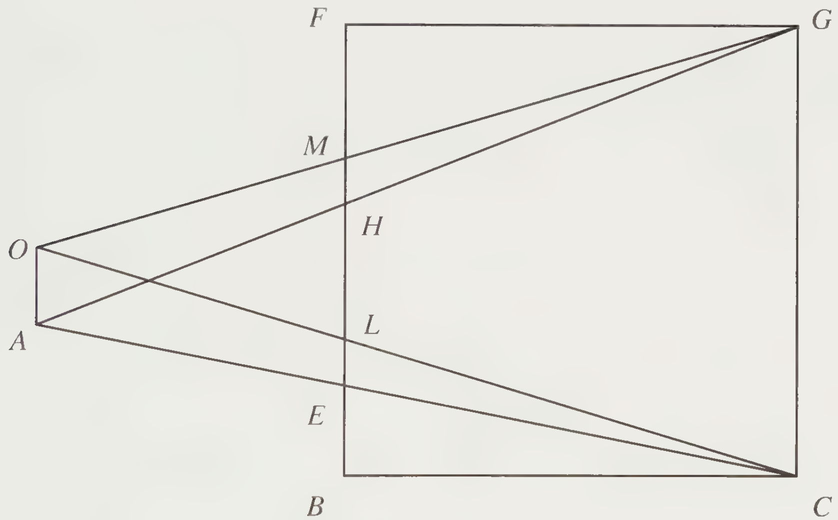
5.7 Copy of diagrams illustrating *De prospectiva pingendi*, Book 1, Proposition 13. Proportions have been altered. The duplication of lettering is found in the originals. Drawing by JVF.

2.3). However, in Piero's figures the square *BCGF* represents the object that is to be drawn (which in reality is in a horizontal plane) and there is no representation of the edges of the actual picture, though the picture plane is represented by the vertical line *BF* (compare the diagram for Proposition 12 shown in Fig. 5.6). In Piero's figure, the *A* outside the square represents the position of the eye, but this is a straightforward matter only in relation to the parts of the diagram that correspond to lines in the diagram for the previous proposition. It is made clear in the text that *A* also represents the position of the eye in relation to the square, but it seems that here Piero is either trusting the reader to do some mathematics without prompting, or (which seems more likely) relying on drawing conventions that allow him to flatten out the three-dimensional set-up without further explanation.

Piero gives drawing instructions that produce a figure like that shown in Figure 5.8. Here the eye is at *A* and it is looking at the square *BCGF*. However, it has been stated that the eye is directly opposite the centre of one side of the square. What we want is the length *EH* and moving the eye off-centre does not affect this length. A reader who is good at drawing (Piero seems to have no idea that he might have readers who are not good at drawing) can check this by drawing a few diagrams of extreme cases. Otherwise, one can prove the result by putting in the ideal position of the eye, at *O*, and considering the pairs of similar triangles *OLM*, *OCG*; *GHM*, *GAO* and *AEH*, *ACG* as shown in Figure 5.9. We need to use the fact that *OA* is parallel to *FB*, so the distance of the eye from the nearer side of the square must be fixed, that is, the eye can only move parallel to the side. However, this deals only with part of the difficulty. The other part is that the eye has been shown in the plane of the square. A three-dimensional diagram of the actual set-up, drawn according to today's conventions, is shown in Figure 5.10. When the eye moves from *A* to *X*, a point vertically above *A*, the intercept of the extreme rays (the line in the picture plane cut off between the lines from the eye to the ends of the line *BG*) becomes *PQ* instead of *EH*. One can prove that $PQ = EH$ by considering the pairs of similar triangles *AEH*, *ACG*; *CEP*, *CAX* and *XPQ*, *XCG*.

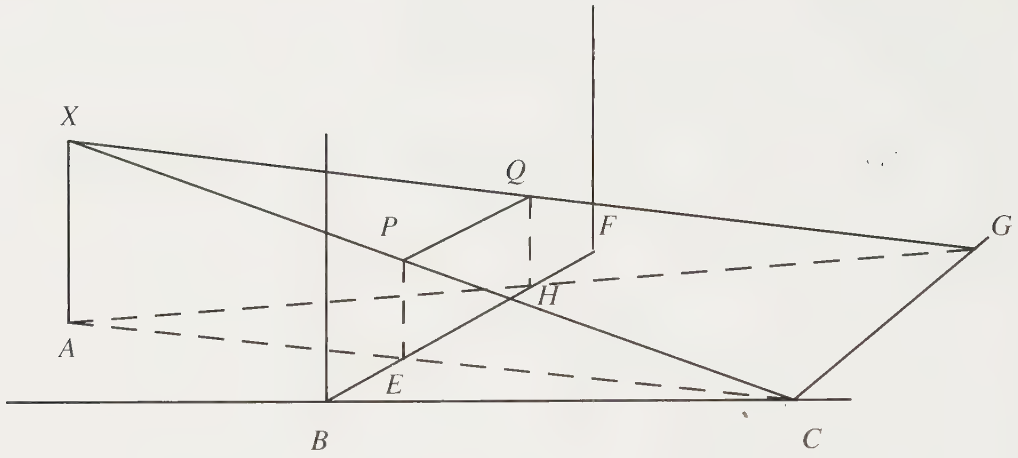


5.8 The first stage of the diagram for Piero's perspective construction, in *De prospectiva pingendi*, Book 1, Proposition 13. The point *A* represents the position of the eye, and the square is the object that is to be shown in perspective. Drawing by JVF.



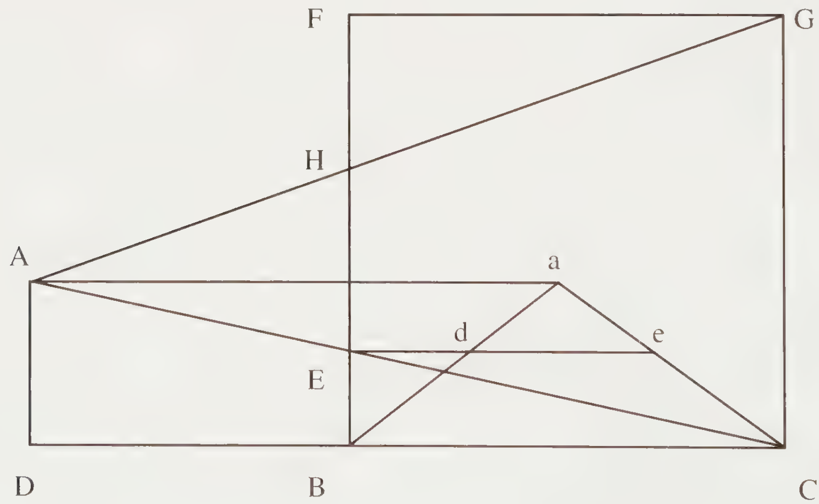
5.9 Moving the eye. The eye is moved from *A* to *O*, that is, parallel to the near edge of the square *BCGF*. It can be proved that *EH* = *LM*. Drawing by JVF.

The final diagram, drawn according to Piero's instructions, cannot be read as a simple combination of elements from different planes but, as we have shown, it is in fact true that *CG*, the far side of the square, will be seen as of length equal to that shown as *EH* in the diagram (copied in Fig. 5.7). Since this diagram uses each of the letters *A*, *D* and *E* twice, it will be easier to discuss it if we partly adopt Piero's habit of giving lettering in lower case,



5.10 The effect of moving the eye vertically, from *A* to *X*. The picture plane stands vertically on the line *BF*. It can be proved that *EH* = *PQ*. Drawing by JVF.

and rename the second occurrences of *A*, *D*, *E* as points *a*, *d*, *e*, as shown in Figure 5.11. Following the detailed drawing instructions has given us *DB* as the distance of the eye from the picture plane, the line *Aa* as horizontal (parallel to *BC*) and the point *a* as on the perpendicular bisector of *BC* (the central axis of the picture of the square). We then construct the line *AC*, to cut *BF* in *E*, and through *E* we construct a line parallel to *BC* to cut *aB*, *aC* in *d* and *e*. Piero asserts that *de* is the required image of the far side of the square. There follows an extremely brief proof, which uses similar triangles.



5.11 A slightly modified version of the diagram for proving the correctness of Piero's perspective construction, after *De prospectiva pingendi*, Book 1, Proposition 13. The second uses of *A*, *D*, *E* have been marked with the lower-case letters *a*, *d*, *e*. Drawing by JVF.

Since some letters have been used twice in the diagram, some of what Piero says is ambiguous. It turns out, however, that when both possible versions make sense, both are true. The duplication of lettering has made it possible for Piero to shorten his proof by allowing him to take it for granted that certain triangles are similar: in his notation they look as if they are actually the same triangle. This sleight of hand may have seemed elegant to mathematicians of the time. Accustomed as they were to handling pairs of similar triangles, it was probably possible for them to handle two pairs simultaneously. It is much more doubtful whether the proof would have made sense to an apprentice painter. Piero may simply have put it in because he regarded it as important and was content to let his readers ignore it if they chose. The proof is intrinsically rather simple.

As we have seen, Piero has already established that the far side of the square *BCGF* will appear to an eye at *A* as of length *HE*. So what he needs to prove is that

$$de = HE \qquad (1).$$

His proof is, of course, presented in continuous prose. What follows here is a version of the proof that stays as close as possible to Piero's line of reasoning but follows today's conventions of exposition.

We first note that the triangles *ade* and *aBC* are similar (since *de* and *BC* are parallel).

Triangles *CEe*, *CAa* are similar (since *Ee* is parallel to *Aa*), so we have

$$\frac{aC}{ae} = \frac{AC}{AE} \qquad (2).$$

Triangles *AEH* and *ACG* are similar (since *He* is parallel to *GC*), so we have

$$\frac{AE}{AC} = \frac{EH}{CG} \qquad (3).$$

The right side of (2) is the inverse of the left side of (3), so the equations may be combined to give

$$\frac{ae}{aC} = \frac{EH}{CG} \qquad (4).$$

As already noted, triangles *ade* and *aBC* are similar, so we have

$$\frac{de}{BC} = \frac{ae}{aC} \qquad (5).$$

The left side of (4) is equal to the right side of (5), so the equations may be combined to give

$$\frac{de}{BC} = \frac{EH}{CG} \qquad (6).$$

But *BCGF* is a square, therefore *BC* = *CG*, and (6) reduces to

$$de = HE,$$

which is what is required.

The style of this is like what we find in Piero's *Trattato d'abaco* and in the *Libellus*, but with the difference that here we are dealing with geometry in purely geometrical terms, not in numerical ones. For those who are dazzled out of their wits by such a line of reasoning, Piero ends with a numerical example. He presumably expects the draughtsman reader to construct a figure and check that the numbers come out as he says. There are several propositions in *De prospectiva pingendi* that end in this way, with a numerical example. Piero was presumably aware that, even when it came to theorems, some of his readers would prefer to think in terms of numerical examples, as was usual in works that belonged to the practical tradition. However, by this stage in the work, we are more or less finished with theorems. Piero has proved the preliminary results he needs and can go on to make use of the construction techniques that they have established as correct. With the exception of the final sections of Books 1 and 2 (Propositions 30 and 12 respectively), the remainder of Piero's propositions would be described by a mathematician as being problems.

Drawing plane figures

The first two problems are to turn the square into a square-tiled pavement. Piero does this in two stages. In Proposition 14, we divide up an edge parallel to the picture plane to put in the lines between tiles that in reality run perpendicular to the picture plane (the orthogonals); and in Proposition 15 we use the diagonal to transfer this division to an orthogonal edge so that we can put in the edges of tiles that are parallel to the picture plane (transversals).[28] What is happening in the 'perfect' figure can be seen in Figures 5.5c and 5.5d. Piero's diagrams put the 'perfect' and 'degraded' figures together, joined along the line *BC*, so that one can see how an operation carried out in the perfect figure corresponds to one carried out in the degraded one. Since this is clearly a case in which a picture is worth any number of words, simplified copies of the diagrams for Propositions 14 and 15 are shown in Figure 5.12.

The diagram supplied for Proposition 14 is in fact the same as that for Proposition 8 (Fig. 5.3). The lines that divided the two parallel transversals in the earlier proposition have become images of orthogonals in the perspective picture of the pavement in the later one. (We may note that until we get to Proposition 14, there are only two lines through *A*, so there is no need to consider whether they meet there.)

5.12 Simplified diagrams showing Piero's procedures for dividing a square into smaller squares. Piero supplies the 'perfect' figure under the 'degraded' one as shown here. Drawings by JVF.

28 Propositions 14 and 15 are translated in Appendix 8.

As we have seen, the theorem of Proposition 8 was proved in such a way that *A* did not need to be directly above the mid point of *BC* (the ground line). At the end of Proposition 13, in characteristically conversational tone, Piero does in fact mention the possibility of *A*'s not being in the centre:

> But if you were to say: why do you put the eye in the middle?[29] Because it seems to me more convenient for seeing the work; all the same, it can be put wherever one wants, provided you do not go beyond the limits that will be shown in the final figure [of this book],[30] and, wherever you put it, it will see in the same proportion.[31]

If we look closely at Piero's work, it becomes clear that the two-dimensional theorem in Proposition 8, and the use made of it in Proposition 14, where it applies to a three-dimensional set-up, add up to Piero's having actually proved that any set of lines that are parallel in reality will, when shown in perspective, become a set of lines converging to a point on the horizon (a line through *A* parallel to *BC*). However, Piero nowhere states this general theorem and, while the practical context of the perspective treatise might explain this omission, it is more naturally explained by Piero's simply not having noticed the more general result. The fact that he has nonetheless proved this more general result is a mathematical consequence of his having followed Euclid's example and proved all his results in the most general possible form.[32] The theorem about the convergence of images of parallels was first stated, and proved, by Guidobaldo del Monte (1545–1607) in his *Perspectivæ libri sex*, published in Pesaro in 1600.[33]

Piero next presents two further problems that start from the square: to turn it into an octagon (Proposition 16) and then into a figure with sixteen sides (Proposition 17). The square has one side along the line *BC*, that is, the line in which the picture plane meets the horizontal plane (in modern terms the 'ground line' of the picture). Since the octagon is derived by cutting corners off the square – though Piero does not explain how this is to be done so as to make the resultant octagon regular – it too has an edge along the ground line, and the same holds for the sixteen-sided figure. Apart from the figure having one edge actually on the ground line instead of merely parallel to it, Piero has solved the problem of drawing the ground plan of the Baptistery as shown in Brunelleschi's demonstration panel.

In the next three problems, the 'degraded square' effectively acts simply as a reference frame against which the positions of points are defined. The problems are to draw poly-

29 That is, directly above the mid point of *BC*.

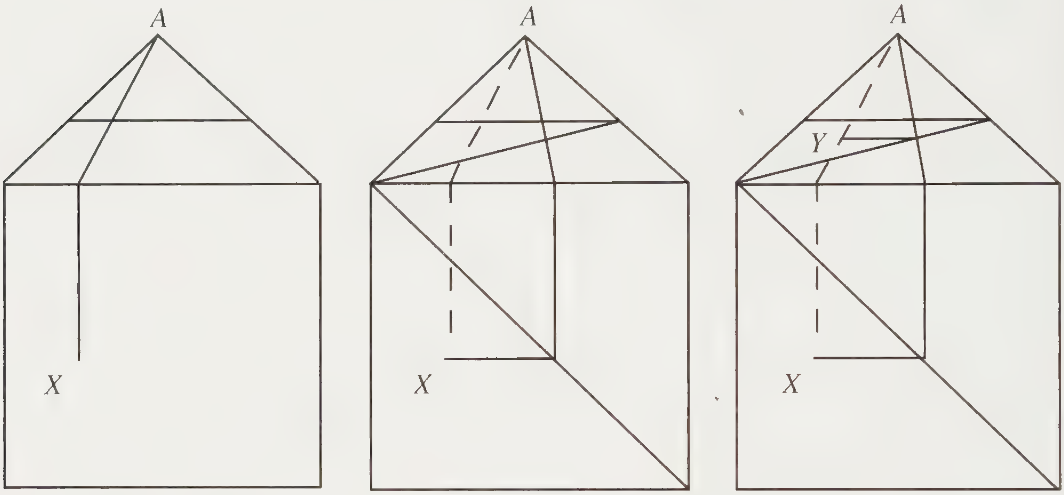
30 That is, Book 1, Section 30, where Piero does not, however, discuss a non-central *A*, but merely the minimum viewing distance for a picture of given width. See below and J. V. Field, 'Piero della Francesca's Treatment of Edge Distortion', *Journal of the Warburg and Courtauld Institutes* 49, 1986, pp.66–99 and plate 21c.

31 Piero della Francesca, *De prospectiva pingendi*, Book 1, Section 13, Parma MS, p.6 verso; Piero ed. Nicco Fasola, p.77: 'Ma se tu dicesse perche mecti tu l'ochio nel mezzo[?] perche me pare piu conveniente a vedere il lavoro[.] nientedimeno se po mectare dove al omo piaci non passando i termini che nell' ultima figura se mostrara & dove tu il mecterai verra in quella medesima proportione.' In context the word *proportione* clearly has the meaning given it in Piero's introduction to Book 1, namely 'perspective'.

32 This result is discussed in detail in J. V. Field, 'When is a Proof not a Proof? Some Reflections on Piero della Francesca and Guidobaldo del Monte', in *La Prospettiva: Fondamenti teorici ed esperienze figurative dall'Antichità al mondo moderno. Atti del Convegno Internazionale di Studi, Istituto Svizzero di Roma (Roma, 11–14 settembre 1995)*, ed. R. Sinisgalli, Florence: Edizioni Cadmo, 1998, pp.120–32, figs pp.373–5.

33 On Guidobaldo's work, see Field, 'When is a Proof not a Proof? (full ref. note 32) and Field, *The Invention of Infinity* (full ref. note 5), esp. pp.171–7.

gons: an equilateral triangle (Proposition 18), a regular hexagon (Proposition 19), and a regular pentagon (Proposition 20). None of these figures has any of its sides parallel to the ground line. Each figure is drawn by constructing its vertices one by one, and at each stage the drawing instructions are repeated completely for each vertex. That is, as in an abacus book, instruction is by worked examples. No general rule is stated. However, one may imagine that, after fourteen repetitions, the apprentice had presumably grasped how to construct the perspective image of a general point in the ground plane.

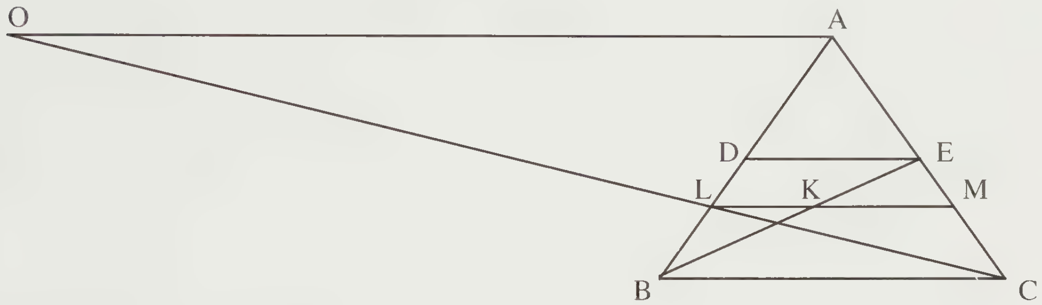


5.13 Method of finding the perspective image of a point in the ground plane. Piero does not give these diagrams, since he finds images of polygons. However, his method of finding the image of each vertex is that shown here. Drawings by JVF.

The stages of the procedure are shown in Figure 5.13. In the 'perfect' figure one first drops a perpendicular from the point, *X*, to the ground line; the equivalent to this in the 'degraded' figure is to join the foot of the perpendicular to *A*. In the 'perfect' figure one draws the diagonal, and a line through *X* parallel to the ground line to meet the diagonal, and from the point in which it meets the diagonal one drops a perpendicular to the ground line; in the degraded figure one joins the foot of this perpendicular to *A*, and through the point in which this line meets the diagonal one draws a transversal. The point of intersection of this transversal and the line representing the perpendicular to the ground line from *X* (shown dashed), the point *Y*, is the image of *X*. As can be seen from the diagrams, it is all more easily done than said.

The next four propositions return to the degraded square as a shape in its own right. Equal rectangular pieces are removed so as to make it a smaller square (Proposition 21), equal rectangular pieces are added to make it a larger square (Proposition 22), a square is made from a rectangle (Proposition 23), and equal squares are added on to a square to make a rectangle (Proposition 24). All these propositions use the diagonal, in the way we find it used in Piero's previous propositions. However, a rider at the end of Proposition 23 uses the diagonal in a different way: as part of using the distance point method to construct the required square. Piero does not give any warning that we have a different construction. He simply says how to carry it out:

> But if the size is not known of either the length or the width [of the given area], I shall draw from the point .A. a line parallel to .BC. of the size [that is length] I put for the distance from the limit to the given [position for the] eye, and its end point will be .O., and from that I shall draw .OC. which will divide the line .BD. in the point .L.. I say that .BL. has taken from the degraded plane .BCDE. the quantity of .BC., which is .BL. Let there be drawn [through] .L. a parallel to .BC., which will cut the diagonal .BE. in the point .K. and .CE. in the point .M.; I say that .BLCM. is a square cut from the plane .BCDE., which is not square, because the line starts from the eye .O. and ends in .C., and divides .BD. in the point .L. so that .C. appears at the eye higher than .B. by the quantity .BL. [*sic*], as was proved by the 12th [proposition of this book].[34]



5.14 After Piero della Francesca, *De prospectiva pingendi*, Book 1, Section 23, Parma MS p.11 recto. Cutting a square from a given rectangle of unknown dimensions. The 'perfect' rectangle, shown below *BC*, has been omitted in this copy. Drawing by JVF.

A copy of the relevant part of the accompanying diagram is shown in Figure 5.14. Piero's reference to his Proposition 12 is not helpful, since that theorem seems to have nothing to do with the case. Moreover, what he says here does not make mathematical sense. All that is clear is that he is using the distance point method (compare Fig. 2.14 and see Appendix 3). Since his distance point has been called *O*, whereas the point representing the eye is otherwise called *A*, we must assume Piero did know what he was about. The distance point method is applied correctly, so the construction would indeed work as Piero says. However, his explanation is a muddle and, to confuse matters further, this is the first we have heard of the distance point method. (It is also the earliest known description of it, but that is not relevant here.)

Up to this point, Piero has carefully been proving all the theorems he needs to use. This is, in fact, the only occurrence of the distance point method in *De prospectiva pingendi*. It

34 Piero della Francesca, *De prospectiva pingendi*, Book 1, Section 23: Parma MS, p.11 recto; Piero ed. Nicco Fasola, p.87.

shows that Piero knew the method, a fact that should probably be read as an indication that the distance point construction was generally known as a workshop practice in his milieu. Presumably Piero also knew, or at least believed, the construction to be correct, or he would not have used it here. However, the passage in question does not appear in the Latin version of his treatise. As we have already seen, there are other changes in the text that suggest that Piero played an active part in the translation process. In connection with this particular passage, he had presumably recognized the anomaly of having used the distance point method without proving it was correct, and had decided that rather than add a proof he should omit the reference to the alternative construction.[35]

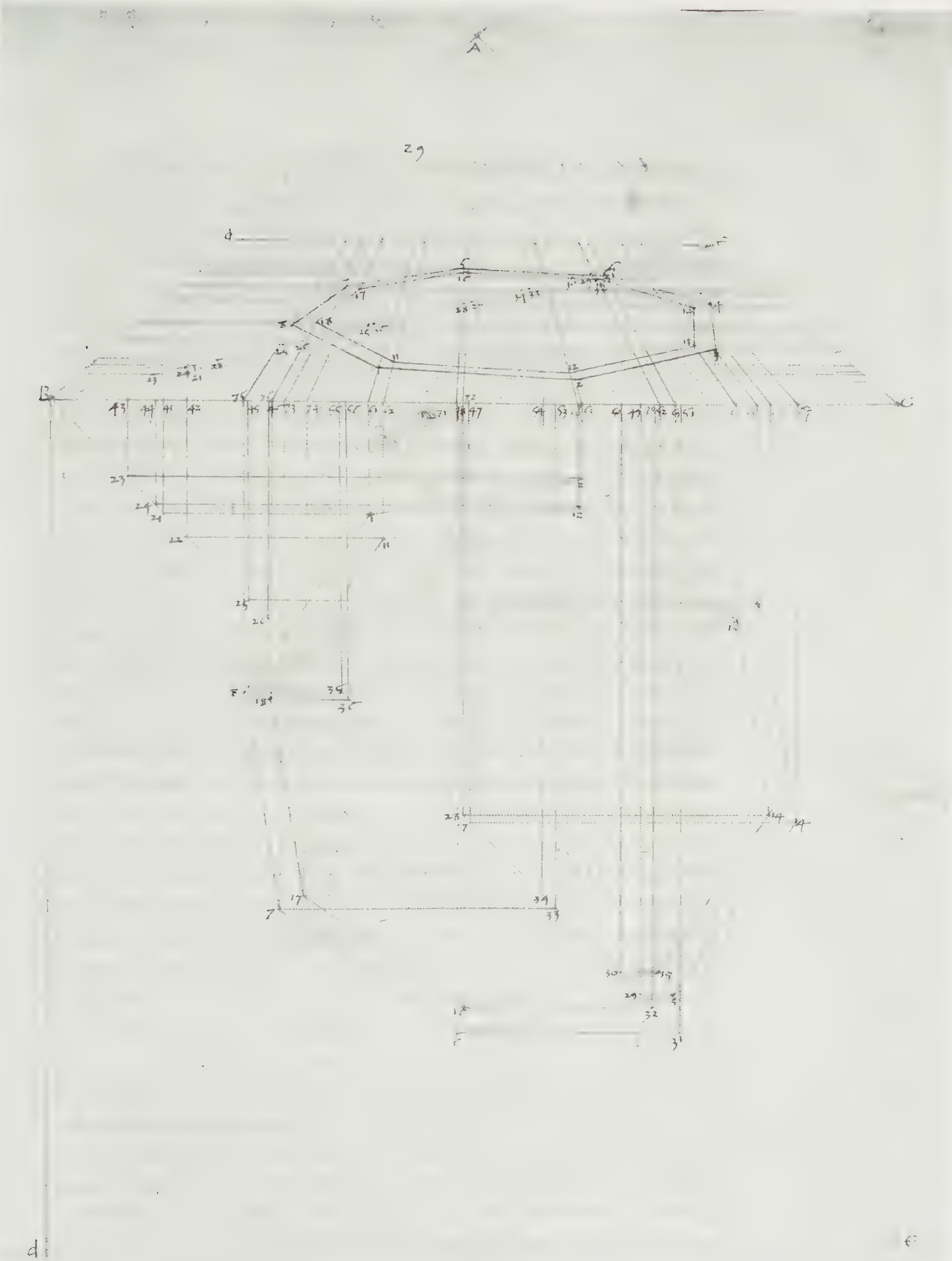
After these various exercises on squares and rectangles, we return to point by point constructions of the vertices of polygons lying in the ground plane. First we have a square (Proposition 25), then an octagon (Proposition 26), then two squares (Proposition 27), then a double square outline that makes a ground plan of a square building (Proposition 28), then a double octagonal outline that makes the ground plan of an octagonal building (Proposition 29, Fig. 5.15). All these figures are placed on the ground plane in completely general orientations, that is so that none of the sides of the figure is parallel or perpendicular to the ground line. The method of solving each problem is consequently completely general, and thus applicable in all possible cases. In principle this is useful. However, the work is difficult for a mere reader. The drawing instructions, given in full, are extremely repetitious. Moreover, in the knowledge that he will run out of alphabet, Piero has chosen to give numbers rather than letters to the points in this set of propositions, so we are faced with page after page of prose sprinkled with numbers.

The repeated drawing instructions – in which Piero introduces slight variations in the wording, apparently to relieve the monotony – suggest rather strongly that the reader, 'tu', is meant to be working through these examples. He is learning how to carry out the practical tasks of drawing the degraded forms of figures. Moreover, the method prescribed has a practical aspect: once we have the 'degraded' square, the remaining part of all the constructions is carried out entirely in an area that will be part of the picture surface (except, of course, for the rider to Book 1, Proposition 23, which, as we have seen, uses the distance point method). That is, one could simply transfer the degraded square to the prepared gesso surface and then carry out the rest of the construction directly on the surface that was to be painted. Such a procedure would avoid the errors introduced in transferring drawings – errors that might become noticeable for small and complicated features. Since Piero's practice shows strong tendencies to include such delicate passages, it is clear that he personally would have found it helpful to be able to carry out constructions on the actual picture surface.[36] Apprentices were presumably expected to follow his example in this respect.

The final section of Piero's first Book, Proposition 30, is not a problem but a theorem. The general result that Piero proposes to prove is, however, introduced in a rather chatty style:

35 The matter is discussed more fully in Field, 'Piero della Francesca and the "Distance Point Method"' (full ref. note 15).

36 For instance, some of the perspective details of the pavements in the *Flagellation* are of the order of a millimetre in size. Piero did indeed transfer a drawing for the turban worn by the figure with his back to us (see M. A. Lavin, *Piero della Francesca: The Flagellation of Christ*, New York: Viking Press, 1972), but there is no reason why he should have tempted fate by taking the unnecessary risk of transferring complete drawings for the pavement. This picture will be discussed further below.



5.15 Constructing the perspective image of the plan of an octagonal building. Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 29, Parma MS, p.16 recto. The 'perfect' shape is shown in the lower part of the diagram and the 'degraded' one in the upper part.

> To remove the error made by some who are not very experienced in this science [*scienza*], who say that often when they divide the degraded surface into units [*bracci/ulnas*], the foreshortened one [*lo scurto/decurtam*], comes out longer than the one that has not been foreshortened; and this happens by not understanding the distance there should be from the eye to the limit where the things are put [i.e. the picture plane], nor how wide the eye can spread the angle of its rays; so they [the inexperienced] suspect perspective is not a true science, judging falsely because of ignorance. It is accordingly necessary to demonstrate the true distance and how far the angle at the eye can increase, so as to end their doubts.[37]

To today's mathematical eye, the difficulty Piero has described is spurious. There is no reason why the perspective construction should not legitimately yield a line whose length is greater than that of the original line of which it is the image. However, we may note that instead of referring to the image as the 'degraded' form of the unit line, Piero is here using the word *scurto*, translated into Latin as [*ulna*] *decurta*, which implies the line has become shorter. Luckily, there is a close English equivalent in 'foreshortened'. Thus the paradox is susceptible of translation into English. However, in today's mathematical terms, it is purely verbal.

On the other hand, to some of Piero's contemporaries, the unexpected result of the *scurto* line coming out longer was clearly important – important enough to cast doubt upon the validity of the perspective construction itself. Piero appears to have agreed with this opinion, for, as he promises, he sets out to show that such an eventuality cannot arise. As is suggested by the reference to the rays emitted by the eye, the proof will involve matters that are the concern of *perspectiva* proper. By going beyond the bounds of *prospectiva pingendi* for his proof, Piero is making it clear that he regards perspective construction as an extension of the larger old-established science. The proof that the construction procedure is a 'true science' is thus also a proof that it is an intellectually legitimate extension, and consequently invests the new construction with the intellectual respectability that attaches to the established science. A craft procedure it may be, but it can make a claim to truth.

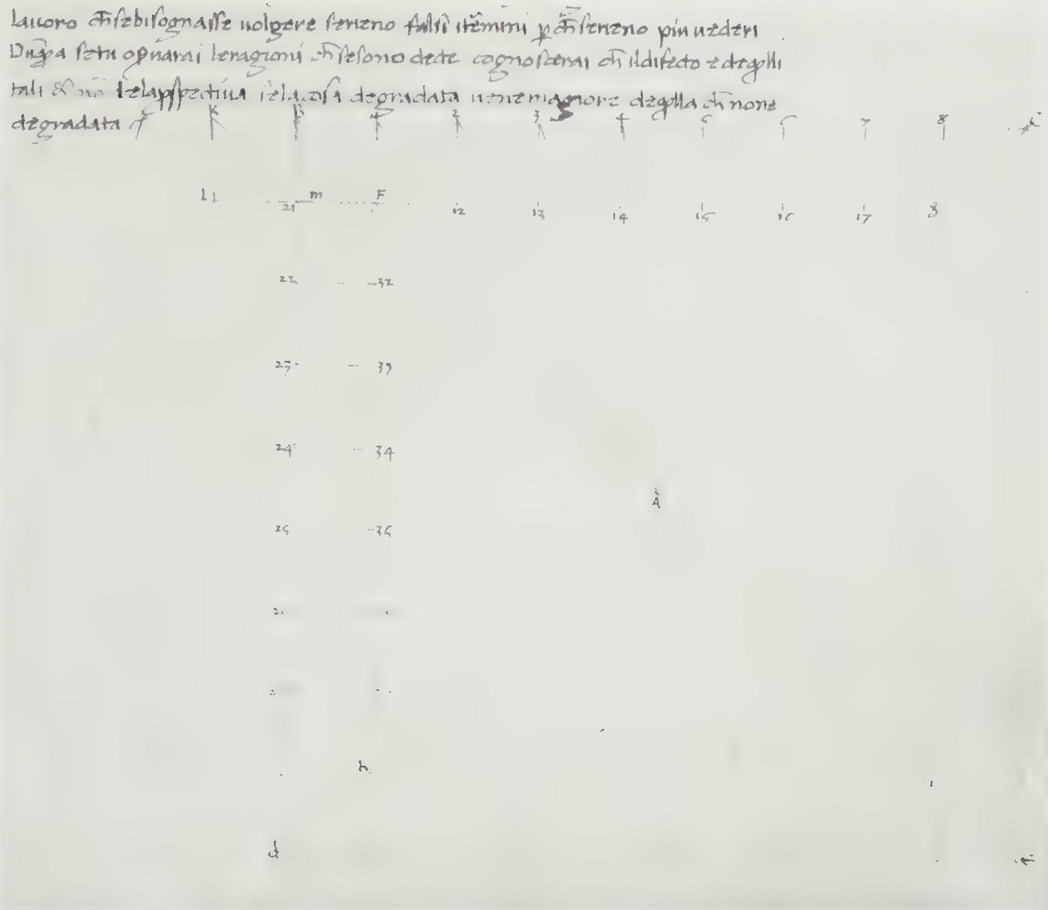
The way Piero sets up his theorem is as follows. The eye can spread its rays only so far as to make a visual cone whose vertical angle is a right angle.[38] It is, of course, assumed that the eye will not move: perspective deals with the geometry of a single glance. Moreover, as was normal at the time, it is taken for granted that looking with two eyes is exactly equivalent to looking with one. The two eyes were believed to combine their information before the mind made use of it.[39] Thus what we need to prove is that if the viewing angle for the picture – that is, the angle it subtends at the eye – is less than a right angle, the *scurto* line does not come out longer than the 'perfect' line was. Unfortunately, this putative theorem is untrue. It is consequently inevitable that Piero's proof is flawed.[40]

37 Piero della Francesca, *De prospectiva pingendi*, Book I, Section 30: Parma MS p.16 verso; BL MS, p.18 recto; Piero ed. Nicco Fasola, p.96.

38 In fact, the width of the visual field of the eye is usually about 135°, but this is not at all easy to measure. The belief that the width was 90° goes back to ancient times and seems not to have been questioned until long after Piero's day.

39 Usually the combining process was believed to take place at the optical chiasmus, the part of the brain where the nerves from the eyes cross. Since it was agreed that nature did nothing in vain, animals' generally having two eyes was explained as divine providence guarding against the loss of sight by supplying an extra part.

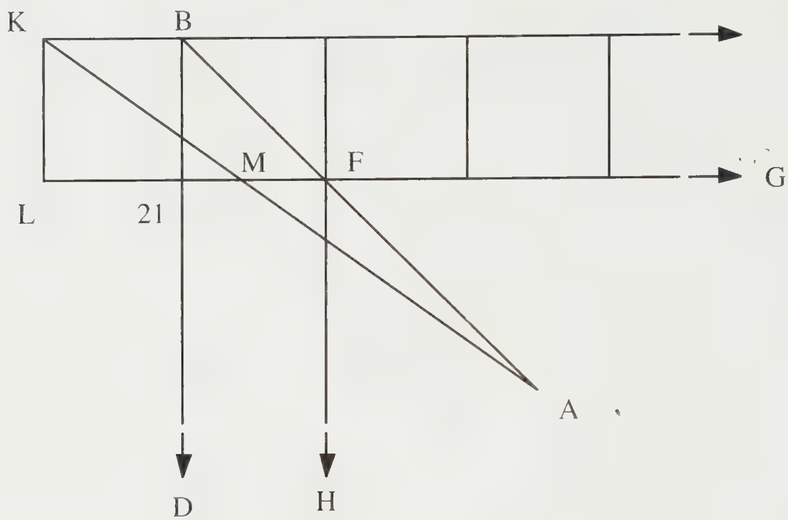
40 The flaw escaped detection by Daniele Barbaro, who reprints Piero's proof almost word for word in his own treatise on perspective (of 1568, 1569). See the more detailed discussion of the proof in Field, 'Piero della Francesca's Treatment of Edge Distortion' (full ref. note 30).



5.16 Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 30, Parma MS, p.17 recto. Piero is concerned with the angle the picture subtends at the eye.

As usual, Piero's text consists almost entirely of drawing instructions, and the diagram we are told to construct is, as usual, supplied at the end of the proposition (Fig. 5.16). The eye is at *A*, in the centre of the array of squares. Moreover, as for Proposition 13, what we have is a quasi-ground plan, in which we are assumed to accept that nothing important is changed if the sight lines are imagined as coming from a point vertically below the eye. The picture plane is represented by the line *FG*. To follow Piero's proof we need to examine only part of the figure, which is shown in an enlarged copy in Figure 5.17. For the sake of clarity, in this copy the small squares have also been made relatively larger in relation to the large square. Piero's drawing instructions make the square .BKL21. the same size as all the other small squares surrounding the large one, and .AK. has been drawn to cut .F21. in *M*. The position of the picture plane is given by the line *FG*, but the picture is now imagined as extended so as to include *L*.

Characteristically, Piero's proof is introduced by more drawing instructions, which give us the points shown in Figure 5.17:



5.17 Modified copy of one part of the diagram Piero supplied for *De prospectiva pingendi*, Book 1, Proposition 30 (see Fig. 5.16). Some of Piero's lettering and numbering has been omitted. Drawing by JVF.

> The proof: let the points .B. .1. .2. .3. .4. .5. .6. .7. .8. .C. be joined to .A.. I say that the line [through] .B. will be the diagonal through [the point] .F. of the line .FG., and if there is added to the line .BC. the quantity [of the line from] .1. to .B., and let this continuation be .BK., and at .21. join on the quantity from .F. to .21., and let it be [the line between] .21. and .L.; then draw .KL., which makes a square which is .BKL21., if .K. is joined to the point .A. [the line] will divide [that between] .21. and .F. in the point .M..[41]

The proof itself is disconcertingly short:

> I say that .KL. when it is foreshortened is greater than [the line between] .21. and .L. [which is] not degraded, by the quantity [of the line between] .21. and .M., because .KL. is represented equal to .LM., which is greater than [the line] from .L. to .21.; as I have said the foreshortened line is greater than the one which is not foreshortened, which cannot happen, because on this limit [the picture plane] the eye cannot see .K., which belongs to [the field of view of] the eye facing the line .FH.

That is, Piero merely says that as seen from *A*, the point *K* will be seen at *M*, so the length *KL* will be represented in the picture plane by *LM*, which is larger than .L21., so the foreshortened length has come out longer, but in fact the eye at *A* cannot see *K* because the limit of its vision is defined by the diagonal *AB*, so all is well. Unfortunately, what Piero has proved, though true, is not what he needs to prove. He has proved that if *K* lies beyond the diagonal *AFB* then the length of the segment of the orthogonal through *K* cut off by the first transversal will be greater than *KB*. What he really needs to prove is that this is so only if *K* lies beyond the diagonal.

41 Piero della Francesca, *De prospectiva pingendi*, Book 1, Section 30: Parma MS, p.17 recto; BL MS, p.18 verso, l.7ff.; Piero ed. Nicco Fasola, p.98.

It turns out that this result Piero requires is not true. At least, it is not true unless the eye is truly imagined as being in the ground plane of the picture. That is, the result is true if Piero's diagram is interpreted as an exact representation of the three-dimensional set-up. As we have seen in connection with Proposition 13, placing *A* in the ground plane had no effect on the validity of Piero's proof of the correctness of his perspective construction. It is possible that Piero himself believed that the proof of Proposition 30 was similarly unaffected by the simplification introduced by putting *A* in the ground plane. However, this is not in fact so, as can be seen by considering an unacceptable case in which the foreshortened length comes out longer than the 'perfect' one, and doing a certain amount of algebra on the three-dimensional geometrical problem it presents. The mathematics concerned need not go beyond the methods we find in Piero's *Trattato d'abaco*.[42]

It is consequently rather difficult to believe that Piero was satisfied with his own proof. His putting it forward may perhaps be partly explained by the fact that he sees his construction procedures as an extension of *perspectiva* proper. Thus the angle the eye can make with its rays seems to be conceivable as a limiting factor on the mathematics. Later developments in natural philosophy have accustomed us to regarding mathematics as the surer guide, so it would seem more reasonable, to us, to use the mathematics to find out what angle the eye can make.[43] However, Piero is apparently prepared to allow his understanding of what would now be called physiological optics to inform his treatment of a mathematical problem. He may thus have been inclined to believe that his theorem was true, and that the difficulty with his proof, as given in Book 1, Proposition 30, lay in some remediable error in his mathematics, such as an oversimplification that could eventually be removed. That would, however, suggest that he had not carried out a detailed analysis of the problem in a correct three-dimensional diagram.

In any case, Piero does not simply leave the matter there once he has given the proof. He goes on to give a rule for avoiding the unacceptable excessive length. This rule is not, as we might have expected, that the viewing distance (used in the construction) must be made such that the picture subtends less than a right angle at the eye. What Piero actually prescribes is that it must make an angle of 60° or less. So perhaps he did have suspicions about the correctness of his proof. In which case, he may have put it in because it was the best proof he could provide, and he considered the matter to be sufficiently important that some kind of proof was required. As it happens, the proof would withstand the practical test. Drawing a few diagrams shows that for eye heights like those used in Book 1, the theorem is very nearly true, and the rule Piero finally recommends will work for all the eye heights he uses in his problems.

Drawing prisms

The second book of *De prospectiva pingendi* is concerned with drawing right prisms. That is, some of the plane figures drawn in Book 1 become ground plans, and vertical planes are constructed on them so as to form a solid shape. Piero provides a brief introduction:

42 The mathematics is given in full in Field, 'Piero della Francesca's Treatment of Edge Distortion' (full ref. note 30).

43 It was precisely by allowing this priority to mathematics and assuming – as a thoroughgoing Platonist – that the physics must follow the mathematics, that Johannes Kepler was able to prove, in 1604, that sight was by the intromission of light rays. See J. V. Field, 'Two Mathematical Inventions in Kepler's *Ad Vitellionem paralipomena*', *Studies in History and Philosophy of Science* 17(4), 1986, pp.449–68.

> A body has in itself three dimensions: length, width and height; its limits are surfaces. Such bodies are of various shapes, for instance a cube is a body, as is [a figure whose ground plan is] a tetragon which does not have equal sides,[44] or a round [thing], so is one with sides, such as a pyramid with sides,[45] or one with many different sides, as are to be seen in natural and accidental things. In this second [book] I intend to treat of the degradations of these [bodies], which subtend angles at the eye on the given limits [picture planes], making some of the surfaces degraded in the first book their bases.[46]

Inevitably, the first example is a cube, first one whose base is the complete degraded square, then (the first case shown in Piero's figures) a cube whose plan is a square with sides parallel to those of the original degraded square, then, still as part of the first proposition, one whose plan is a square whose edges are not parallel to those of the original square (again shown in a figure). The third case explicitly refers back to the corresponding proposition for drawing the square ground plan in Book 1, Proposition 25. This reference back to Book 1 sets the style for the following problems, in which we construct an octagonal prism (Proposition 2), a pentagonal prism (Proposition 3), a hexagonal prism (Proposition 4), and a 16-sided prism (Proposition 5). As in the constructions of images of polygons in the first book, Piero simply reels off drawing instructions. The instructions are repeated, in detail, for each solid. Effectively we have instructions for constructing vertical edges from points in the ground plane, but there is no treatment of this as a problem in its own right.

The principles of Piero's method for constructing a vertical edge are as follows. Let us assume we have drawn a square ground plan, *FGIH*, on the degraded plane, *BCED*, as shown in the first diagram in Figure 5.18, and that we are to construct the vertical edge through the point *G*. We draw through *G* a parallel to *BC*, to cut *BA* in *P*. Then we construct through *P* a vertical line *PT* (the position of *T* is not yet fixed). Next, we draw up from *B* a vertical line *B*℞ whose length is equal to the 'perfect' length of the edge whose 'degraded' form is to be constructed.[47] The point ℞ is joined to *A*, and where ℞*A* cuts the vertical through *P*, we mark the point *T*. We then draw up from *G* a vertical line, parallel to *PT*, and of indefinite length. Next, we draw through *T* a line parallel to *PG*, to cut the vertical through *G* in the point *L*. The line *GL* is the required vertical edge of the prism. Following Piero's drawing instructions does not generate diagrams like those shown in Figure 5.18, since he carries out each operation for all the vertices of the ground plan before passing on to the next operation. In general he supplies only figures of the completed drawing, and these appear at the end of each proposition. However, for the problems of drawing the cube (Book 2, Proposition 1) we have figures only for the last two cases.

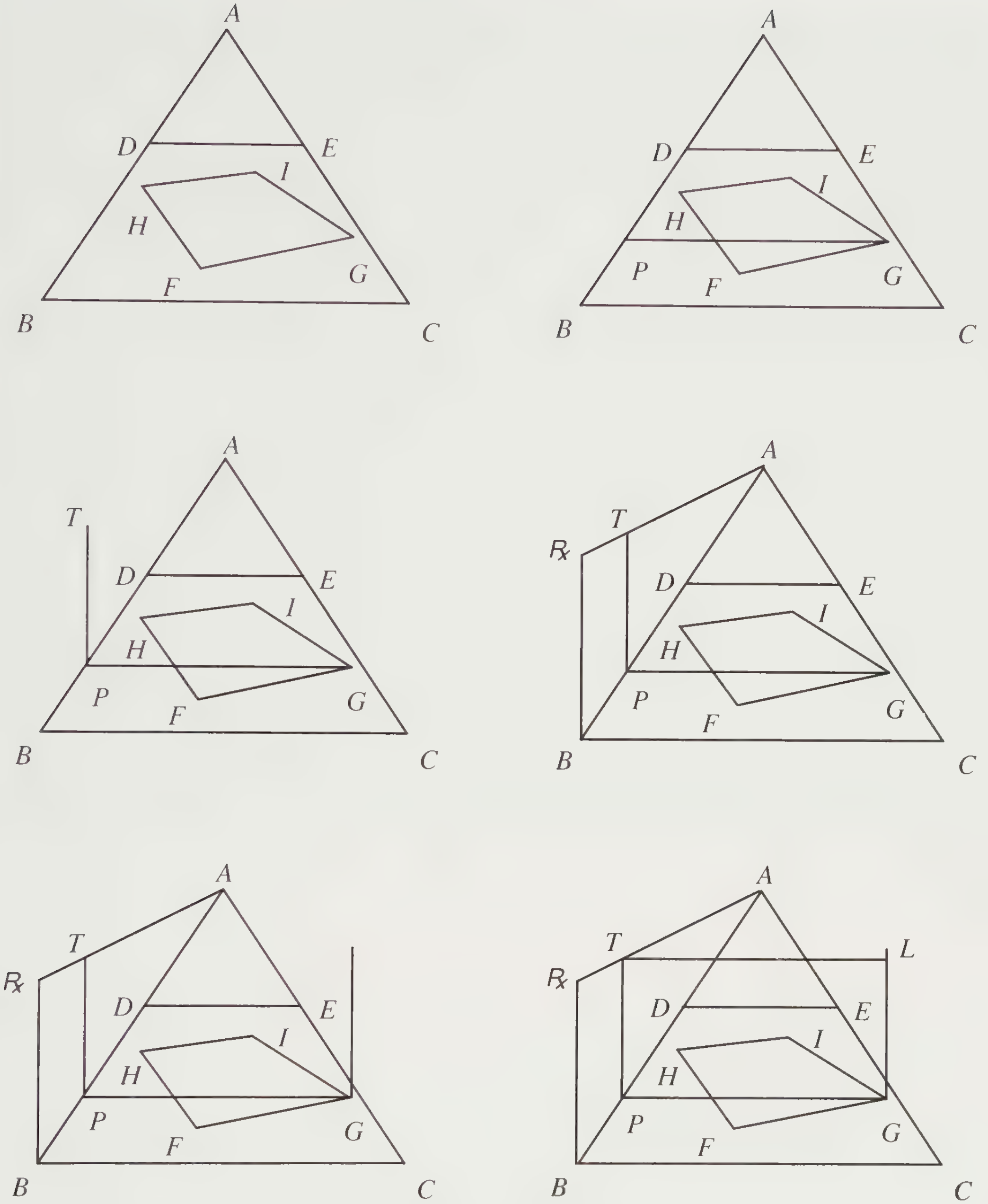
After the sixteen-sided column, we turn to a combination of prisms (Proposition 6):

44 The phrasing is the same in the Latin text as in the vernacular one. Piero does not seem to have a word for what we would now call a rectangle.

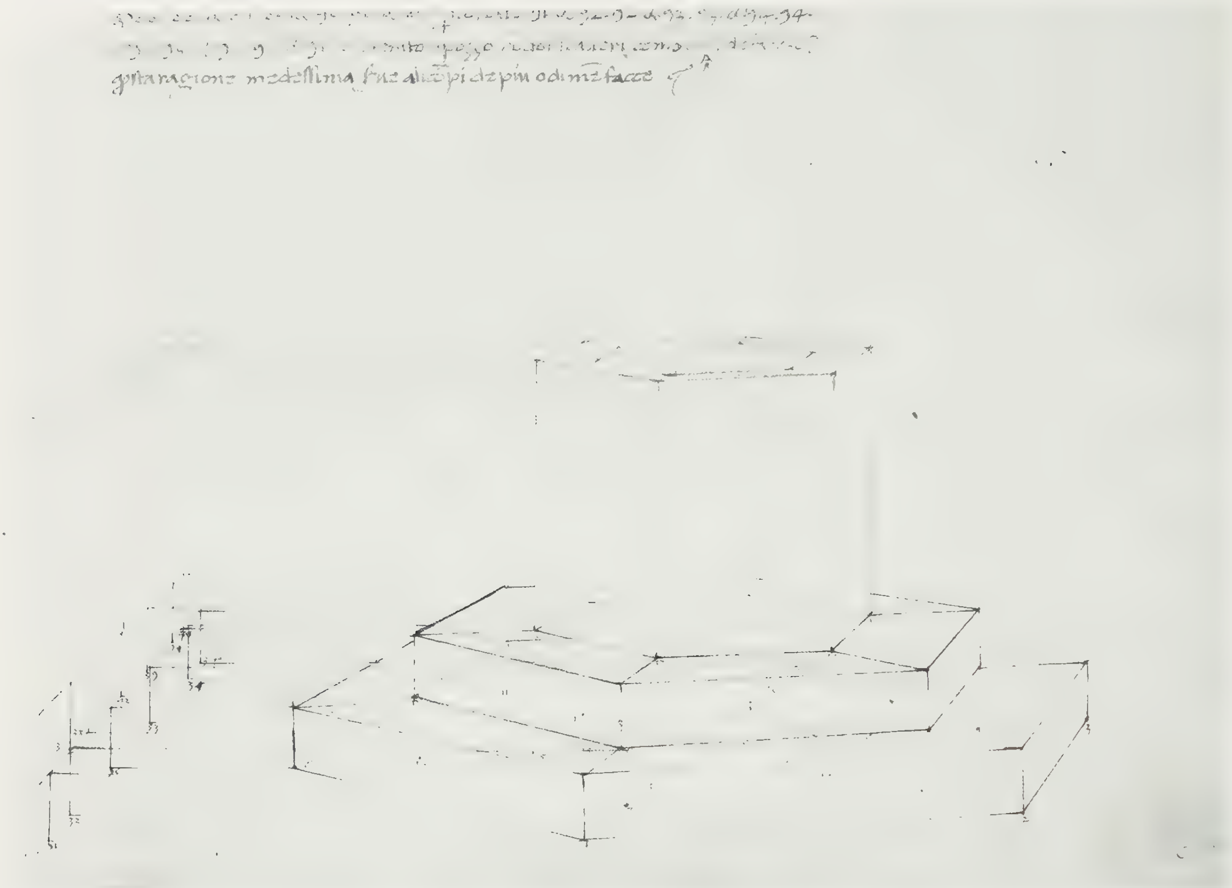
45 That is, with flat faces. He is distinguishing it from what we would now call a cone. The words cone and pyramid were more or less interchangeable in this period.

46 Piero della Francesca, *De prospectiva pingendi*, Book 2, Introduction: Parma MS, p.17 verso; BL MS, p.19 verso; Piero ed. Nicco Fasola, p.100.

47 As we have seen, Piero has no qualms about numbering points rather than lettering them. Presumably his use of ℞ (the usual algebraists' symbol for a square root, an abbreviation for the Latin word *radix*) was also acceptable as a name for a point. In the following century, printers would sometimes merely use characters they had to hand, such as planetary symbols, to designate points in diagrams – a habit that can be grossly misleading to the unwary.



5.18 Method of drawing a vertical edge of a prism. Piero does not give these diagrams. He carries out each operation for all the points in the ground plane before moving on to the next operation. Drawings by JVF.



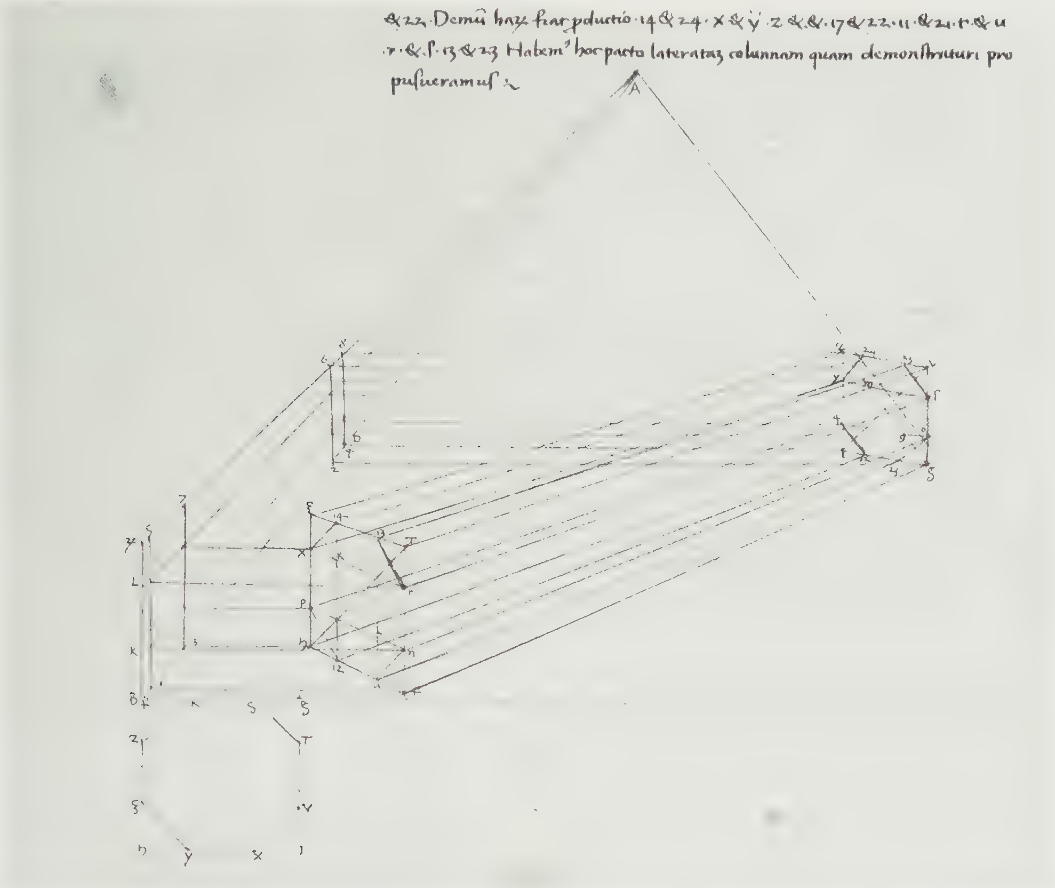
5.19 Drawing a hexagonal well-head in perspective by treating it as superimposed hexagonal prisms. From Piero della Francesca, *De prospectiva pingendi*, Book 2, Proposition 6, Parma MS, p.22 recto.

> On the degraded plane to degrade a well head with six equal faces, and with steps around it, following round [the lines of] the sides.[48]

The accompanying diagram is shown in Figure 5.19. As can be seen from the verticals in the middle of the picture, Piero has been true to reality (albeit invisible reality) in taking the inner edges of the opening of the well right down to the ground plane, though parts of some of the corresponding verticals on the left (between the lines *BA* and *℞A*) seem to be missing. It is possible that some lines were scribed into the paper and that ink was added later. This would help to explain the slight differences found when one compares photographs of illustrations taken from various manuscripts. It is, however, clear that all diagrams are at least derived from figures supplied by Piero.[49]

48 Piero della Francesca, *De prospectiva pingendi*, Book 2, Section 6: Parma MS, p.21 recto; BL MS, p.23 verso; Piero ed. Nicco Fasola, p.106.

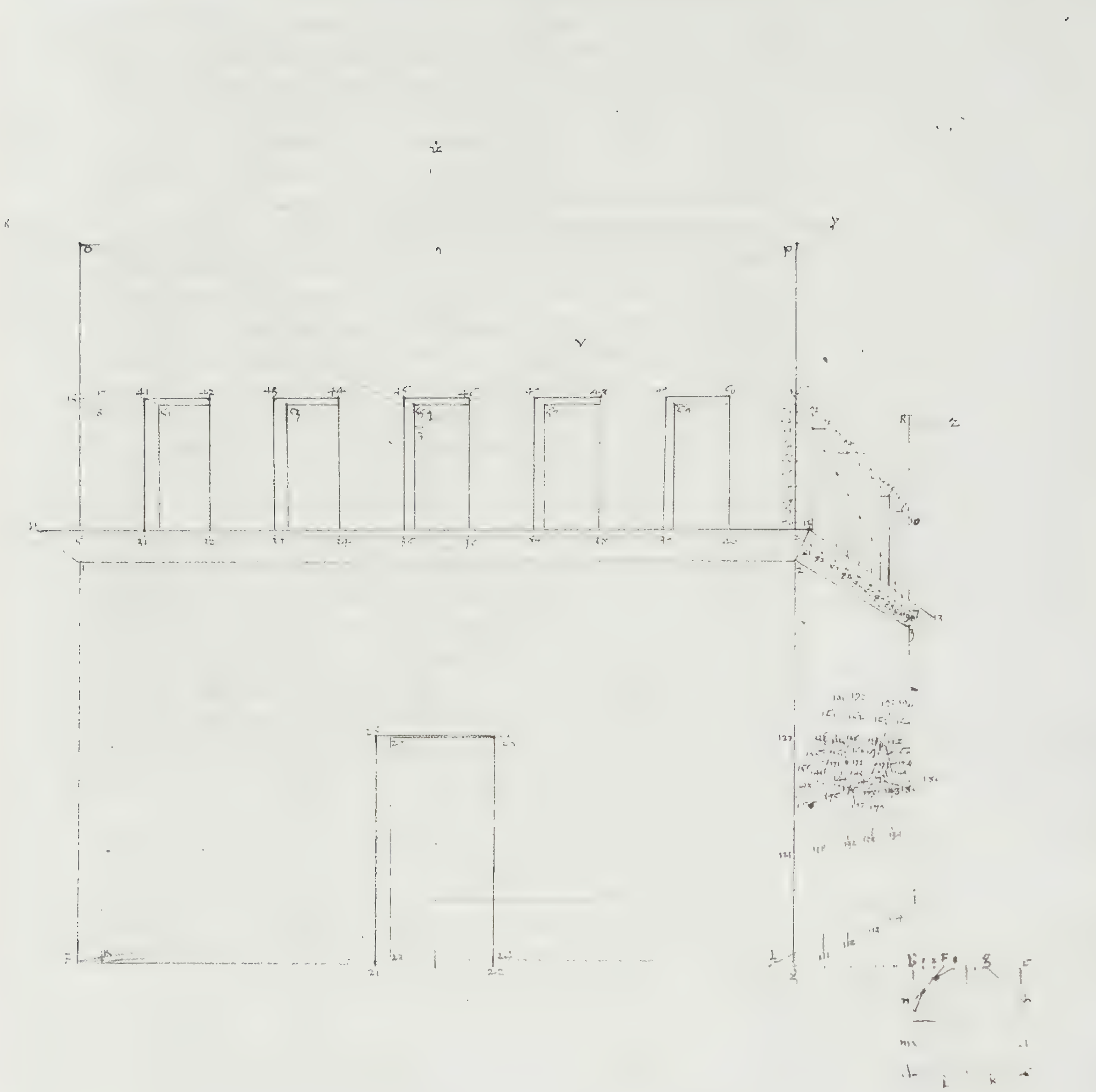
49 In some manuscripts the series of figures is incomplete.



5.20 Drawing a beam of octagonal cross-section in perspective, by treating it as a truncation of a rectangular prism. From Piero della Francesca, *De prospectiva pingendi*, Book 2, Proposition 8, BL MS, p.27 verso.

The hexagonal well head is followed by another probably realistic or useful example: to add a moulding at the base and the top of a cube (Proposition 7). This makes a shape like that of an altar. The following problem, Proposition 8, may also be intended to be realistic. It is to draw a beam of octagonal cross-section, lying on the ground plane (Fig. 5.20). Piero treats this as the problem of drawing a rectangular prism, and then cuts off the corners of the cross-section, as shown in the smaller diagram at the bottom left.[50] Next we have a house (Proposition 9), drawn as a cube, but with a double outline in the ground plan to allow the thickness of the wall to appear in window openings and so on (Fig. 5.21). The small figure of an octagon at the bottom of the drawing of the house refers to the next problem, Proposition 10, which is that of drawing a 'temple' (*templo/templum*) with eight

50 This problem might be a suitable one in which to use the general theorem that the perspective images of sets of parallels converge, but the theorem does not appear. See Field, 'When is a Proof not a Proof? (full ref. note 32).



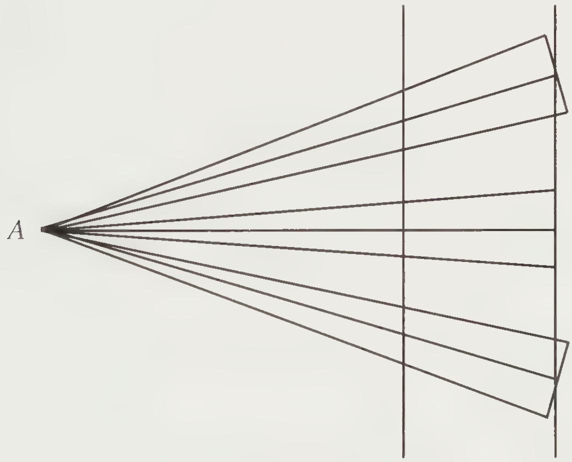
5.21 Drawing a house in perspective, by treating it as an elaboration of a cube. From Piero della Francesca, *De prospectiva pingendi*, Book 2, Proposition 9, Parma MS, p.27 recto.

faces.[51] The final problem, Proposition 11, is that of drawing a cross-vaulted structure with a square ground plan. The curving lines of the vault and the arches are drawn by taking eight points across each arc. 'Tu' was no doubt a good enough draughtsman to join these up with a convincing smooth curve, though the diagrams supplied show series of straight lines, no doubt to emphasize the method of construction. At the very end of the problem, Piero does, in fact, recommend the use of compasses (*sexto/circinus*) to draw the circular shapes of the arches that are parallel to the picture plane.[52]

The final section of *De prospectiva pingendi*, Book 2, Proposition 12, is not a problem but a theorem:

> If in the degraded plane there is drawn a parallel to the limit [that is, the picture plane], and this line is divided into several equal parts, and in these equal divisions there are placed equal bases, each facing orthogonally to the eye, the one that is further away will be represented on the limit [picture] larger than the nearer, all the same it will present itself to the eye as subtending a smaller angle than the nearer one.[53]

Piero divides the line into five parts. For simplicity, Figure 5.22 shows it divided into only three parts. Before going on to prove this theorem, Piero explains why it is of interest to the painter:



5.22 Simplified diagram to show the principle of Piero della Francesca, *De prospectiva pingendi*, Book 2, Proposition 12. Drawing by JVF.

51 Piero della Francesca, *De prospectiva pingendi*, Book 2, Section 10: Parma MS, p.27 recto; BL MS, p.35 recto (*sic*, the pages have been bound in incorrect order, so the text runs from 35 recto and verso to 32 recto and verso, and the diagram appears on 33 recto); Piero ed. Nicco Fasola, p.118.

52 Piero della Francesca, *De prospectiva pingendi*, Book 2, Section 11: Parma MS, p.30 recto; BL MS, p.34 verso (*sic*, because of the incorrect binding mentioned in the previous note, the text of Proposition 11 runs from 33 recto to 34 verso, and the diagram appears on 31 recto), Piero ed. Nicco Fasola, p.125.

53 Piero della Francesca, *De prospectiva pingendi*, Book 2, Section 12: Parma MS, p.30 verso; BL MS, p.31 verso (because of the incorrect binding mentioned in note 51, the text of Proposition 12 starts on 31 verso then runs from 40 recto to verso), Piero ed. Nicco Fasola, p.125.

> This [proposition] is not less necessary than the final one of the first [book],[54] to show the extent of the angle at the eye and the correct size of the base facing it. Because in buildings it happens that we need to make round columns and fluted ones, as in loggias and porticos, where many columns are required, and because [when we are] proceeding according to the correct rules it is surprising that the columns further from the eye come out with greater thickness than those [that are] nearer, [all] having equal bases. Accordingly, I intend to show how it is and what should be done.[55]

Unlike the theorem to which he has referred, the one at the end of Book 1, the theorem at the end of Book 2 appears to be correct. However, Piero's proof is not completely rigorous, apparently because he has chosen to omit reference to some inconvenient sines and cosines. Sines and cosines, defined slightly differently from how they are today, were well known to learned geometers, for instance astronomers, of Piero's time. However, as far as I know, they are not to be found in the 'practical' works of the abacus tradition. Thus, even if Piero did know about them, it may have seemed appropriate to omit them in this context.

Defending perspective

The third book of *De prospectiva pingendi* introduces and employs a method of constructing perspective images that is different from the methods used in the first two books. This new method is essentially a point by point construction using lines of sight (eyebeams or light rays). Although almost everything else in Piero's perspective treatise was to reappear in later ones, the method of the third book does not – unless we recognize an attenuated version of it in the use of sighting instruments, various forms of which are recommended in many works on perspective addressed to painters. It may very well be that the method of Piero's Book 3 was not widely used even in his own day. However, if, as has sometimes been suggested, even Piero himself did not use it, we are somewhat short of an explanation for its being included in his treatise, unless we are to question the author's good faith, which we have no other reason to doubt.

Piero does not tell us why he chose to begin Book 3 with a defence of perspective. This might, perhaps, more naturally have found its place at the beginning of the work as a whole, rather than at the start of its final book. However, if the method used in Book 3 was not only unusual but also unusually elaborate, that fact might go some way to explaining the placing of Piero's defence of perspective. Both the placing and the content of this defence are the same in the vernacular and Latin versions of the treatise.

Piero begins, as in Book 1, Proposition 30, with some unidentified detractors:

> Many painters disparage [*biasimano*/*vituperunt*] [*sic*] perspective, because they do not understand the force of the lines and angles which are obtained from it; with which [lines and angles] every outline and delineation is drawn in correct proportion.[56]

54 *De prospectiva pingendi*, Book 1, Proposition 30, see above.

55 Piero della Francesca, *De prospectiva pingendi*, Book 2, Section 12: Parma MS, p.30 verso, BL MS, p.31 verso (because of the incorrect binding mentioned in note 51, the text of Proposition 12 starts on 31 verso then runs from 40 recto to verso); Piero ed. Nicco Fasola, pp.125–6.

56 That is, in perspective – as in the definition of proportion in the introduction to Book 1, see above. Piero della Francesca, *De prospectiva pingendi*, Book 3, Introduction: Parma MS, p.32 recto; BL MS, p.37 recto; Piero ed. Nicco Fasola, p.128.

This is presumably to say that these detractors do not understand that the constructions used by painters, or at least those recommended by Piero, are rigorously derived from the science of *perspectiva*, and that painters' perspective is consequently itself a 'true science', as Piero had said in Book 1, Proposition 30. He does not, however, propose to defend the scientific rigour of perspective – presumably because, entirely reasonably, he sees himself as having already done so. Instead, he proposes to show that an understanding of perspective (which these detractors lack) is necessary for painting – that is for good painting:

> Therefore it seems to me that I should show how much this science is necessary to painting. I say that perspective literally means, so to say, things seen at a distance, represented as enclosed within given limits [that is, on the picture plane] and in proportion, according to the quantity of their distances, without which [that is, without knowing the distances] nothing can be degraded correctly. And because painting is nothing if not demonstrations of surfaces and bodies degraded or magnified [*acresciuti*][57] on the limit [that is, the picture], placed like the real things seen by the eye as subtending different angles on the said limit, and because for any quantity some part of it is nearer the eye than another, and the nearer part always presents itself as subtending a greater angle than the further one at the assigned limits, and since it is not possible for the intellect to judge for itself of their size, that is the size of the nearer part and the size of the further one, so I say it is necessary [to employ] perspective, which distinguishes all quantities proportionately, as a true science, demonstrating the degradation and magnification of all quantities by means of lines.

Like Alberti, and Pliny, Piero is here taking it for granted that a good painting is accurate in its representation of appearances. It is accordingly not surprising that this essentially practical defence of perspective is immediately followed by an appeal to classical precedents, with the names of painters (other than Apelles) apparently taken from Vitruvius:

> By following this practice [that of perspective] many ancient painters acquired lasting fame. Such as Aristomenes, Thasius, Polides, Apello, Andramides, Nitheo, Zeusis, and many others.[58]

Next, painters who did not use perspective, and those who admired their works, get short shrift:

> And although many have received praise without perspective, it is given by those who have not taken account of what is proper to the art, with mistaken judgement.[59]

Piero then gives a justification for his present enterprise, following it with an accurate summary of the contents of the first two books of the treatise:

> And I lay down rules as one zealous to promote the good name of art in this [our own] time also, and as one who presumes to dare to write this little piece on perspective as it regards painting, making it, as I said in the first, into three books. In the first I showed

57 This, and the following reference to magnification, have no counterpart in the Latin text in the British Library manuscript (BL MS).

58 Piero's list of painters and its possible origins are discussed in Chapter 3, p.75.

59 Translated from the vernacular text. The Latin is slightly wordier, but to the same effect.

> the degradation of [plane] surfaces of various kinds; in the second I have shown the degradation of square bodies and ones with more faces [that is, prisms whose bases are higher polygons], placed perpendicularly on the planes.

The summary may perhaps be an indication that Piero expects some readers will not have already worked their way through the books in question. Together with his humanist references, this may suggest that, even in writing his vernacular text, he was mindful of the possibility of its having learned readers. In any case, he at once goes on to explain that the third book will be different from its predecessors:

> But because now in this third book I intend to treat of the degradation of bodies contained by differing surfaces and differently placed,[60] since I have to deal with more difficult bodies, I shall take a different approach and [use] another method for degrading them, unlike what I did in the previous demonstrations; but the result obtained will be the same thing, and what the one [method] achieves is what the other achieves.[61]

One might suppose this would suffice. However, Piero apparently feels the need to explain more fully:

> There are two reasons why I shall depart from the previous order; one [reason] is that it will be easier in the demonstrations and in understanding [them]; the other [reason] is on account of the great number of lines which it would be necessary to make for these bodies in following the first method, so that the eye and the mind would be confused by these lines, without which such bodies cannot be degraded perfectly, nor [can it be done] without great difficulty. So I shall take to this other method, with which I shall be able to show the degradations bit by bit, in which method, as I said at the beginning of the first [book], it is necessary to understand what one wants to do, and to know how to put it [the body concerned] on the plane in its proper form,[62] because as they [the bodies] are placed in their proper form on the plane, so the force of lines, in accordance with [the rules of] the art, will produce them [the bodies] in degraded form, just as they are represented on the limit [picture plane] by the sight lines. Therefore it is necessary to know how to make, to scale, all the outlines of what one wants to make [a picture of], and to place it on the plane [with all parts] in their positions in their proper form, of which method I shall give an account in the demonstrations that follow.[63]

Following this, Piero at once presents his first proposition. Here, too, instruction is to proceed by worked examples.

Drawing 'more difficult' shapes

Piero's preliminary justification for using a new method suggests some nervousness about the matter. In mathematics at this time, originality was not admired for its own sake – which

60 The nature of Piero's examples suggests this must mean that the objects in question will not be positioned as if they were simply standing on the ground plane, see below.

61 Piero della Francesca, *De prospectiva pingendi*, Book 3, Introduction: Parma MS, p.32 recto; BL MS, p.37 recto; Piero ed. Nicco Fasola, p.129.

62 The words are 'propria forma', which indicates that one is required to draw the ground plan – in principle, a surveying skill.

63 Piero della Francesca, *De prospectiva pingendi*, Book 3, Introduction: Parma MS, pp.32 recto–32 verso; BL MS, p.37 verso; Piero ed. Nicco Fasola, pp.129–30.

no doubt partly explains why Piero made no claim to originality when he wrote about the truncated polyhedra.[64] Nervousness about the new perspective method may also underlie Piero's helpful pedagogical decision to start the third book with some examples that have already been dealt with by the earlier method. Thus in Book 3, Proposition 1, we find ourselves once again drawing the perspective image of a square (placed a little behind the picture plane, but with one edge parallel to the ground line), and in Proposition 2 we draw a regular octagon. The first proposition begins:

> Now, to demonstrate the method which I intend to follow, I shall give two or three demonstrations for plane surfaces, so that through them we can arrive more easily at finding out the degradation of bodies. So, a square surface is made in its proper form, and let it be .BCDE.; then the point .A. is chosen, which will be the [position of the] eye, and let it be as far away as one wants to stand to view the said surface. In the point .A. is fixed the nail, or if you want a needle with a very fine silk thread, it would be good to have a hair from the tail of a horse, particularly where it has to rest against the ruler; then there is drawn a line parallel to .BC., let this be .FG., which will be the limit [picture plane] between the eye and the surface, on which surface, make a point, let it be .M., which must be made for every surface and every body. It does not matter where it is made, because it is a defined limit as you will understand as the work proceeds.[65]

A copy of the diagrams supplied by Piero is given in Figure 5.23. In the first of Piero's diagrams the square is shown in section, but in order to make it more visible it has been folded upwards into the vertical plane (a normal practice in the fifteenth century).

As we shall see, once the bodies to be drawn truly are 'more difficult' the method is laborious in its application, but Piero's description is nonetheless severely practical, involving as it does the use of a nail or a needle and recommending the longer and straighter kind of horsehair. This practical tone is sustained in Piero's suggesting the use of different materials for the two 'rulers' we are to construct, presumably to avoid confusing them at a later stage. The first ruler, for the horizontal measurements, is made of wood:

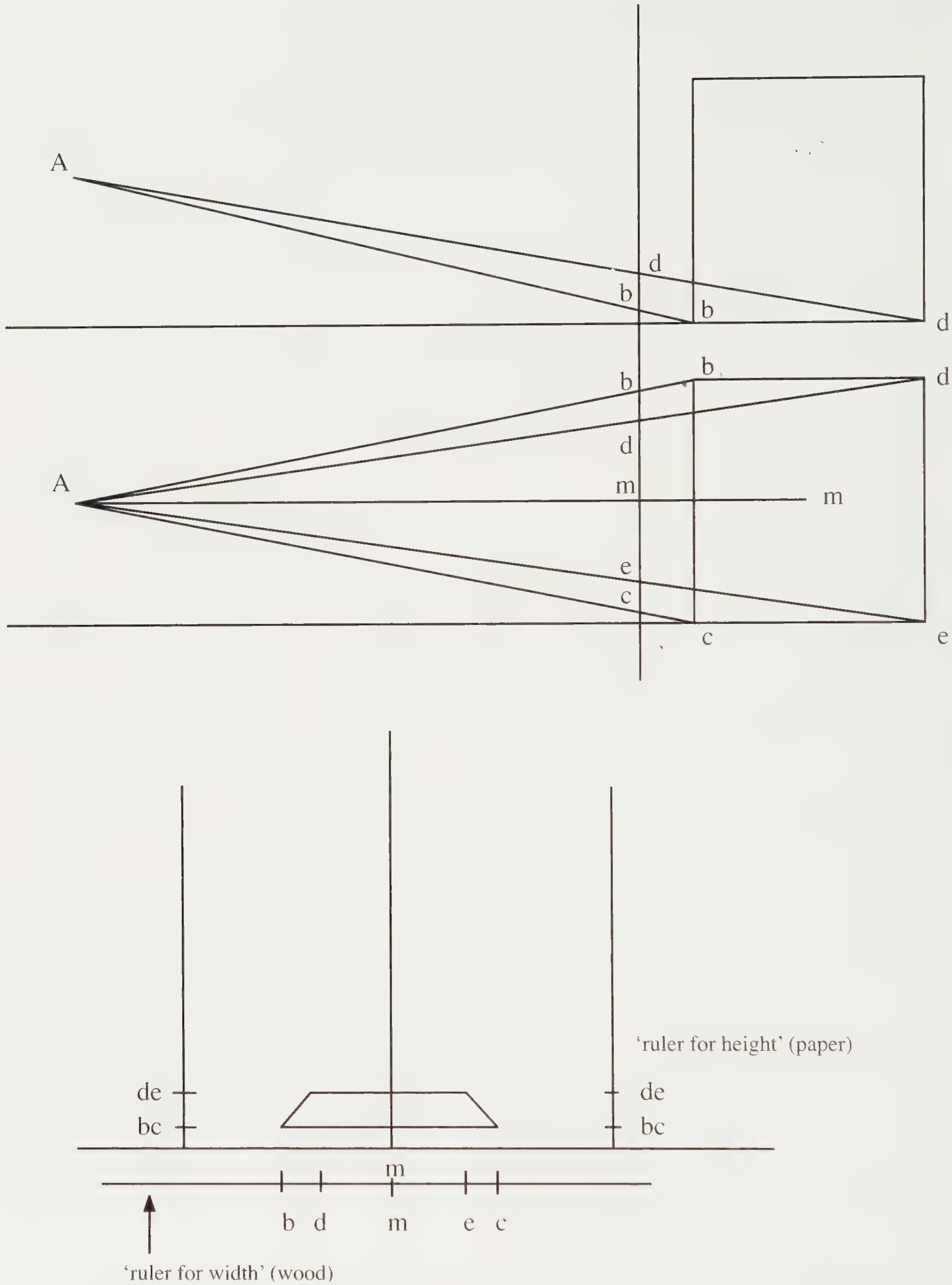
> Now you need to have wooden rulers [*righe/regule*], very thin and straight; then take one of these rulers, and put it touching .FG., and let it be securely fixed in position; then take hold of one end of the silk thread, and draw it taut over the point .B. of the surface, and where it hits against the ruler make the point .B.; then the thread is stretched over .C., and where it hits against the ruler, mark .C.; then the thread is positioned over the point .D., and where it hits against the ruler [is] .D.; the thread is drawn taut over .E., where it strikes against the ruler make .E.; the thread is stretched over .M., and where it strikes against the ruler mark .M. Now make an .A. on the ruler, which will be called ruler .A., and take it away, and it is put to one side, as the ruler for the width.[66]

We are then given correspondingly detailed instructions for making rulers for the vertical measurements:

64 In the prefatory letter to the *Libellus*, Piero claims not that his work is original but that it is related to the work of Euclid. See translation in Appendix 7.

65 Piero della Francesca, *De prospectiva pingendi*, Book 3, Section 1: Parma MS, p.32 verso; BL MS, p.37 verso; Piero ed. Nicco Fasola, p.130.

66 Parma MS, p.32 verso; BL MS, pp.37 verso–38 recto; Piero ed. Nicco Fasola, p.130.



5.23 The method of construction used for 'more difficult' bodies. Copy after Piero della Francesca, *De prospectiva pingendi*, Book 3, Proposition 1, Parma MS, p.33 recto. Some lettering has been omitted. Drawings by JVF.

> Now we want to see how far .DE. [on the picture plane] is above the surface .BCDE., which is [shown as] .BC.; so raise .A. above the line .CE. by as much as you want to raise it to see the said surface, neither shortening nor lengthening its distance from the line .FG. which is the limit [picture plane]. Putting the eye at .A., with the thread, as I said, a ruler is made from paper and put touching .FG., and .EC. is constructed dividing the paper ruler in the point .A., which will be ruler .A.;[67] then the thread is drawn taut over .E., and where it hits against the paper ruler are marked [the points] .E. and .D.; then the thread is stretched over .C., and where it hits against the ruler there is made the point .C. and [the point] .B. in one and the same place; then the ruler is taken away, and with it there is made another the same, with the same marks, and let it be marked .A. as the other [one was].[68]

The wooden ruler and the two paper rulers are then positioned against a ground line so as to allow us to construct the completed degraded square, using the rulers in the manner of the axes of a system of rectangular coordinates (see the bottom diagram in Fig. 5.23).

> After that construct the straight line where you want to make the degraded surface, which line will be .FG., and divide it into equal parts at the point .M., and up from .M. draw the perpendicular, which will be .MN., and draw up from .F., [a line to] .H. perpendicular [to .FG.], and above .G. draw [a line to] .I. perpendicular [to .FG.] which will be .FH. and .GI.; then take the two paper rulers marked .A.; one is placed touching .FH., and the other touches .GI., and .A, of each of them touches the line .FG. Then the wooden ruler marked .A. is taken, which is the ruler for the width, and it is placed over the two paper rulers, touching .E. and .D. of each of the two rulers, and .M. touches the line .MN., and where it coincides with the point .D. of the wooden ruler make the point .D., and where it coincides with .E. mark .E.; the ruler is moved to touch .B. and .C. of the two rulers, and .M. touches the line .MN., and where it coincides with .B. [is the] point .B., and where it coincides with .C. of the wooden ruler make .C.; and the surface is obtained.[69]

We have now found the four points that are the vertices of the degraded square. This is not the end of the matter, because Piero is determined to make sure the principle has been understood. Whether his explanations make matters clearer for a modern reader is open to doubt, but a fifteenth-century one was presumably less accustomed to coordinate systems than we are today. Piero apparently wishes to stress that there is nothing suspect involved in getting two points off the same ruler in one position (that is, marking the same horizontal or vertical coordinate). The lack of technical vocabulary makes the explanation as verbose as the detailed drawing instructions that preceded it.

> Take away the rulers and draw the lines .BC. .BD. .DE. .EC., which make the degraded square surface which we said to make. But if someone were to say: and if the thread is

67 This phrase seems to be out of context, but is found in the British Library Latin text as well as the Parma vernacular one.

68 Piero della Francesca, *De prospectiva pingendi*, Book 3, Section 1: Parma MS, p.32 verso; BL MS, p.38 recto to verso; Piero ed. Nicco Fasola, pp.130–31.

69 Piero della Francesca, *De prospectiva pingendi*, Book 3, Section 1: Parma MS, p.32 verso–33 recto; BL MS, p.38 verso; Piero ed. Nicco Fasola, p.131. Coordinate systems of this kind were well known in Piero's day. See Chapter 2, p.42.

> placed over .E. of the surface in its proper form .BCDE., and where the thread hits against the paper ruler, there is marked .E. and .D., and in the same way it is placed over .C. and there are marked .C. and .B., why is this done? I say that this happens on the surfaces that have [pairs of] corresponding points, that all such which are at the same distance from the line of the limit,[70] and neither is higher than the other, which points are placed on the paper ruler, which is [for] the height, in one and the same point, so that neither angle is higher than the other, as it is understood that .C. is on a level with .B. as to height, and .E. on a level with .D., and the line .FG. which is the limit is parallel to .BC. and .DE., and the paper ruler is always understood [as being] the ruler for the height.

This method is then applied to the problem of drawing a degraded octagon, lying a little back from the picture plane and with one side parallel to the ground line (Proposition 2). Like the first one, this second problem involves only two types of ruler: one for the width (in wood) and a pair of identical ones for the height (in paper). Proposition 3 presents us with a more elaborate problem: that of constructing the images of four concentric circles. We construct twelve points evenly spaced round each circle, and make a set of rulers for each circle so the complete tally of rulers is twelve: four wooden ones (width) and four pairs of paper ones (height). The next problem, Proposition 4, is that of a *torculo* (in Latin *torquis*, Piero's term for a *mazzocchio*, a light-weight ring designed to give form to a headdress). Piero proposes that the plan of the shape shall be twelve-sided and that the cross-section shall be octagonal. He deals with a simple case: that in which the *torculo* is lying flat on the ground plane. The solution begins:

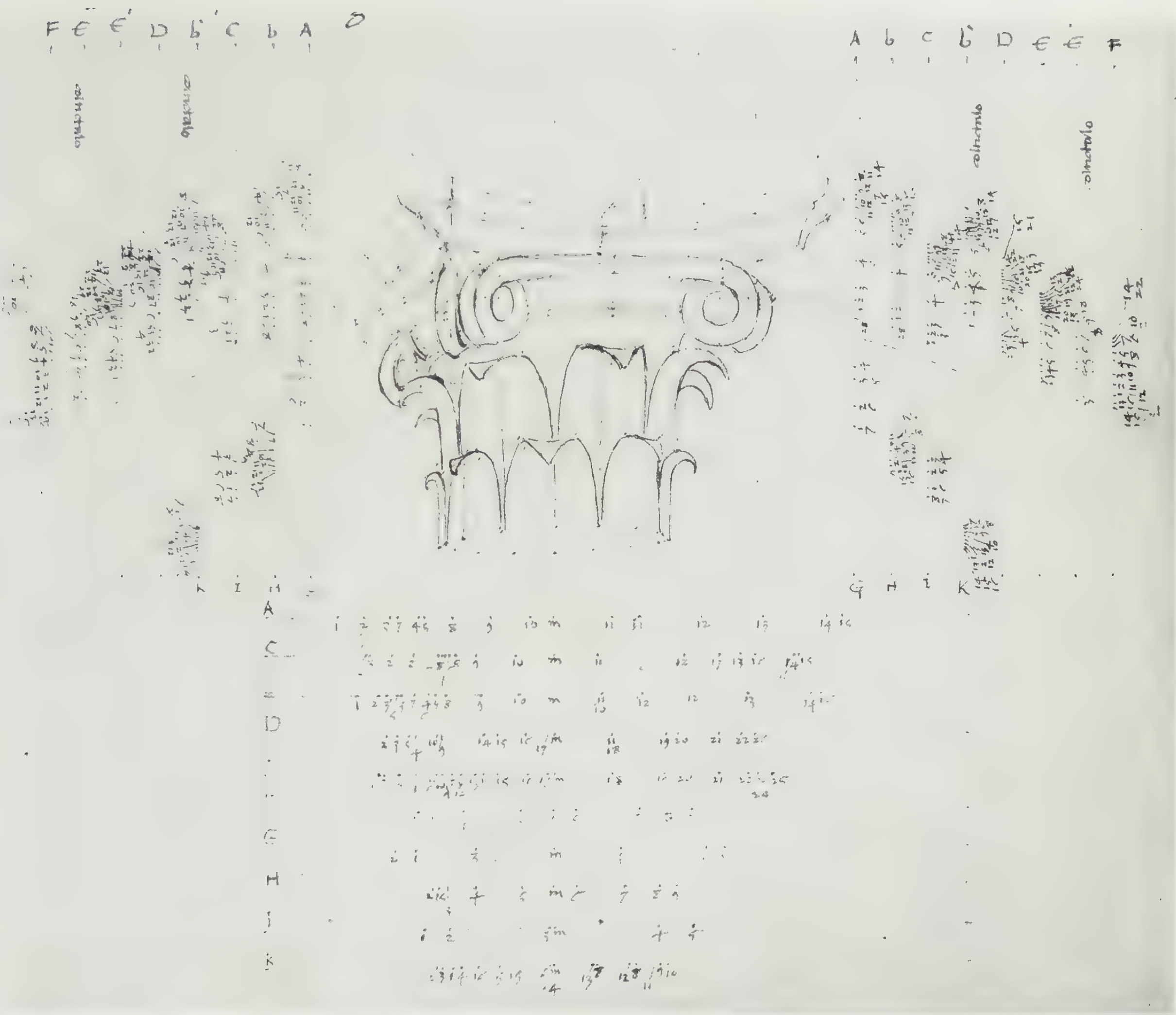
> This [proposition] is like the previous one for the circles, and the same method must be used in working on the width: although it is proposed [to use] eight circles, in this demonstration we shall do it with four, because we place the said *torculo* lying flat. But when it lies otherwise, it will be necessary for there to be as many rulers as there are circles by which the *torculo* is contained. But in this [case] I intend to make one of the octagons of the *torculo* perpendicular to the [ground] plane.[71]

Piero then tells 'tu' to construct a square, and gives instructions on how to cut its corners off to make a regular octagon, and so on. The reversion to elementary matters of plane construction is possibly another indication that he expects some readers may have begun with this third book, without having read the two earlier ones. About six folio sides later, in the Parma manuscript, the degraded *torculo* finally appears, with a retinue of eight rulers for the height on each side of it, and one for the width underneath.

Next, in Proposition 5, we have a much simpler body, namely a cube, but in a much more complicated pose, namely with one vertex in the ground plane and none of its edges parallel or perpendicular to the ground plane or the picture plane. What we need to do is to find the degraded positions of the eight vertices. Piero begins by finding the plan and, effectively, the elevation of the body. The procedures he uses are presumably considered standard, since there is no vestige of explanation, though the description of what should be done is, as usual, spelled out in detail. After the balletic cube, we have a column base, in

70 Since the 'limit' is the picture plane, this line must be what is now known as the ground line, that is the line in which the picture plane intersects the ground plane.

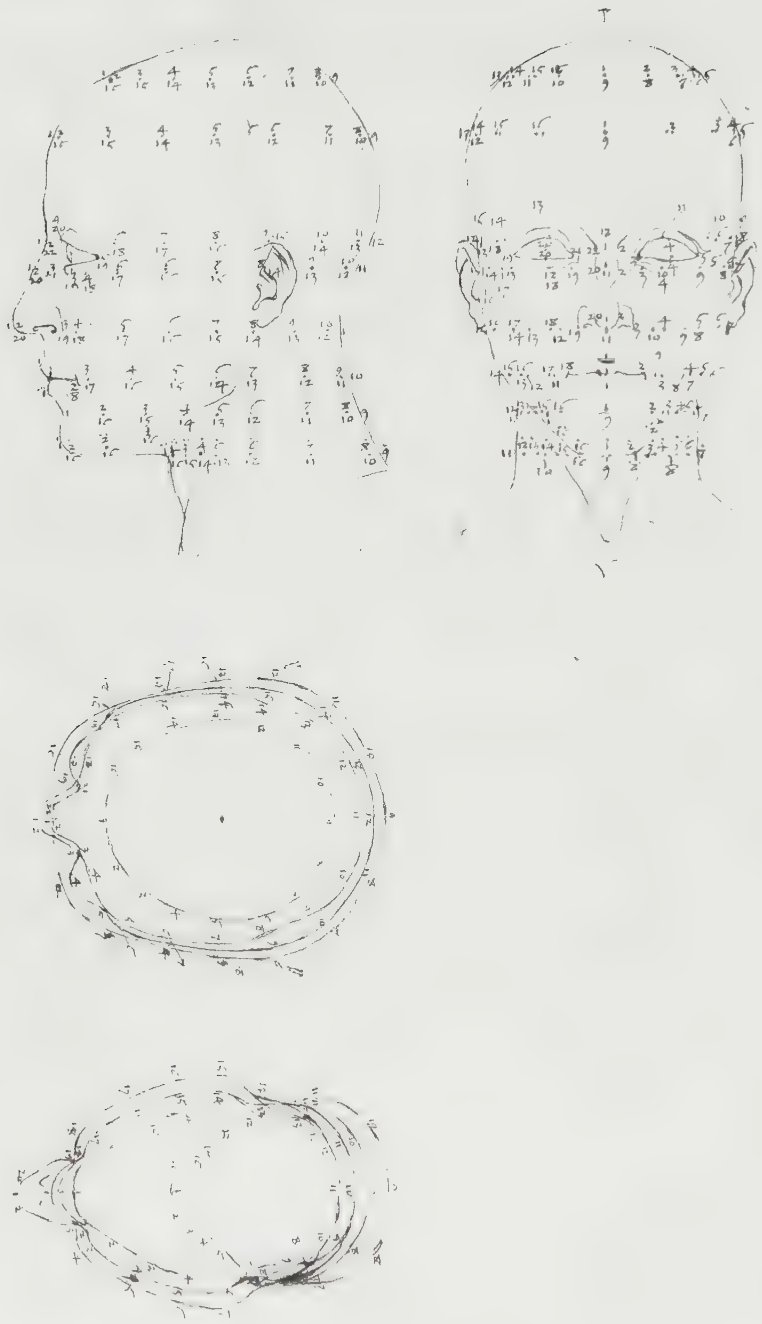
71 Piero della Francesca, *De prospectiva pingendi*, Book 3, Section 4: Parma MS, p.37 verso; BL MS, p.46 verso; Piero ed. Nicco Fasola, p.138.



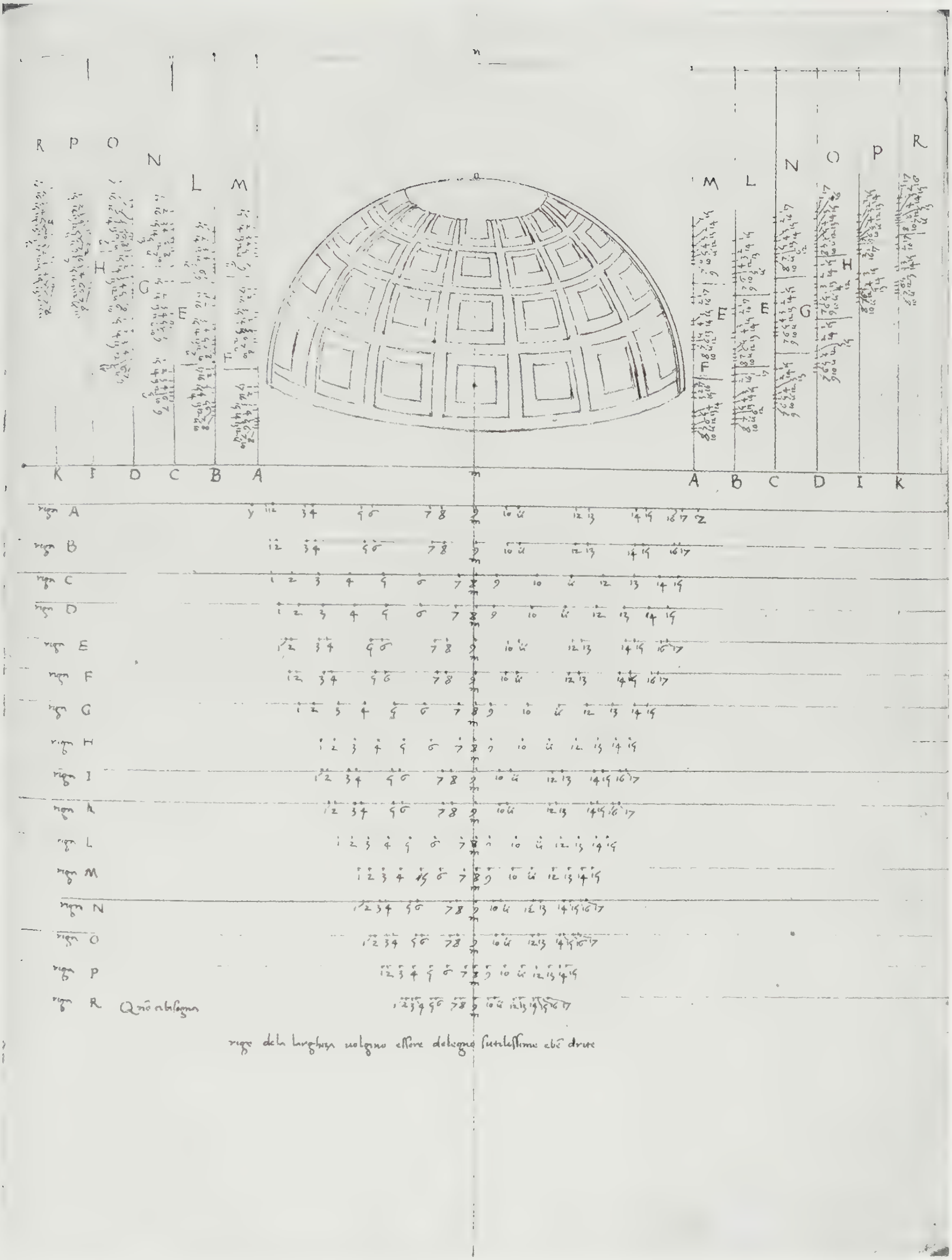
5.24 Drawing a column capital in perspective, third stage only, showing rulers for height and width and finished drawing. From Piero della Francesca, *De prospectiva pingendi*, Book 3, Proposition 7, Parma MS, p.57 recto.

classical style, with mouldings that are concentric circles in plan and variously curved in profile, standing on a plain square plinth (Proposition 6). Sixteen points are taken round each circle, and there are ten horizontal sections (eventually one more is added, for the ground level). Since this effectively produces a three-dimensional description of the shape, one is then free to draw it as it would be seen from any point one wishes, so the preliminaries need to be carried out only once. A new set of rulers will, of course, be needed for each view.

The same is true of the objects shown in the following propositions: column capitals of two designs, one approximating to Ionic and another that is a composite of Corinthian and Ionic (Proposition 7, Fig. 5.24); a human head (Proposition 8, Fig. 5.25); and a coffered half-dome (Proposition 9, Fig. 5.26). This half-dome is divided into seven coffers round its



5.25 Drawing a human head in perspective, first stage only, with points numbered. From Piero della Francesca, *De prospectiva pingendi*, Book 3, Proposition 8, Parma MS, p.61 recto.



5.26 Perspective drawing of a coffered half-dome, final stage. From Piero della Francesca, *De prospectiva pingendi*, Book 3, Proposition 9, BL MS, p.102 recto.

5.27 Piero della Francesca (*c.*1412–1492), *Madonna and Child Enthroned*, from *Sant'Antonio Altarpiece*, about 1467–8, tempera on panel, 141 × 65 cm, Perugia, Galleria Nazionale dell'Umbria. For the complete altarpiece, see Fig. 6.29.

base and four coffers up each vertical rib, leaving a small plain part above them. We have six pairs of rulers for the height and sixteen for the width. The design of the structure is like that of the half-dome behind the enthroned Madonna and Child in the central panel of the *Sant'Antonio Altarpiece* (Galleria Nazionale dell'Umbria, Perugia) (Fig. 5.27), but the painted version has only six coffers round (giving it a central rib) and three coffers up. The central rib has, of course, a significant part to play in what is otherwise a rather markedly asymmetrical composition.[72] The perspective treatise provides no hints on drawing the rosettes that are found in the painted coffering.

Book 3 ends with instructions for making drawings for three trick pictures: the first is of a ball that appears to be resting on the table on which it is painted, the second shows a goblet (*rinfrescatoio/refrigeratorium*) that appears to stand up from the dining table on which it is painted,[73] and the third is of a ring (the kind used to suspend lamps) that appears to hang down from the vault on which it is painted. As drawing problems, these last three are rela-

72 We shall return to this point in the next chapter.

73 Vasari tells us that Piero did actually make a painting of this kind.

tively simple in comparison with what has preceded them. It is clear that a successful hoax of the kind that appears to be envisaged would involve convincing rendering of colours and the effects of light, that is skills of painting in *trompe l'œil* that Piero has specifically excluded from his treatise. In fact, the last three problems seem somewhat out of place in much the same way as the last few problems were in Piero's *Trattato d'abaco* and *Libellus de quinque corporibus regularibus*. They appear to be either afterthoughts or deliberately proffered as an intellectual equivalent of after-dinner mints, a flourish to follow something more substantial.

The importance Piero ascribed to perspective

There can be no doubt that Piero did think of the main body of *De prospectiva pingendi* as being concerned with something intellectually substantial. All the same, much of the work proceeds at a lumbering pace since, as we have seen, a huge proportion of the text is taken up with drawing instructions. However, huge proportions of the texts of Piero's *Trattato* and his *Libellus* are similarly taken up with instructions for carrying out calculations. This is merely the style to be found in all the 'practical' works of the abacus tradition of mathematics. It is clear that *De prospectiva pingendi* does indeed belong to this practical tradition, and one might argue that Piero's elaborate examples are no more out of touch with genuine practicalities than his algebraic examples were out of touch with the everyday experience of calculation. All the same, the algebra, unrealistic though it no doubt was, had some history behind it: many abacus books contain algebra, though usually less than Piero chooses to include in his.

In writing about the mathematics of perspective, Piero is free to make up the syllabus as he goes along, so there is no obstacle to his designing the work as an idealized textbook rather than a totally practical one. Like Vitruvius' ideal architect, Piero's ideal painter has a serious interest in the theory behind his craft. Since he wrote the treatise in this way, one cannot but presume that Piero himself had a good grasp of *perspectiva* proper. The orderly succession of theorems with which his work begins suggests that we are by no means seeing all of Piero's learning. He is choosing from the standard optical tradition and adapting theorems to fit the use that will be made of them later in his work. There are, moreover, a number of theorems that seem to be original, for instance the one that is applied to the convergence of images of orthogonals (Book 1, Proposition 8). The weight of the mathematical preparation is explained by the references to perspective being a 'true science'.

Piero goes to some trouble to prove that his perspective for painting is a legitimate extension of the established science of *perspectiva* proper. In the context of fifteenth-century beliefs about the status of mathematics, this is rather more than a mere claim to intellectual respectability. Mathematics was believed to be true. Today we should say that Euclidean geometry is an axiomatic system, that is, one in which the theorems follow from the definitions and axioms, according to agreed rules of reasoning, but the axioms themselves are not subjected to any test in regard to acceptability other than that of being consistent one with another, even if there is generally an aesthetic preference for keeping their number to a minimum. (Definitions are unimportant in this philosophical context because they are merely a matter of establishing a terminology.[74]) To Piero's contemporaries, Euclid's axioms

74 See note 3.

(which he calls Common Notions and Postulates) were true. That is, Euclid's geometry is giving us true information about the actual world. We must, of course, hedge this about with provisos such as that geometry deals with abstract entities – and we have seen the standard form of this in Piero's comments on the nature of the point, line and area in the introduction to his first book. However, despite this concession to Aristotelian common sense, there is a lurking Platonism that gives particular authority to mathematics not only as a system of deductive proof but as a source of truth. The text of *De prospectiva pingendi* gives us every reason to believe that Piero saw perspective this way.

Since he was a practising painter as well as a writer, it is reasonable to look to Piero's paintings for traces of the craft practices described in his treatise on perspective. Some of the examples, such as that of a house (Fig. 5.21), are of things that are frequently shown in pictures, whoever the painter, so their appearance in Piero's works is not worthy of remark, though their inclusion in the treatise is, of course, evidence that Piero intends his writing to be useful in practice. On the other hand, it is interesting that the first column capital discussed in Book 3, Proposition 7, which has Ionic volutes, appears in a slightly more ornate version as the capital of the column to which Christ is attached in the *Flagellation* (Fig. 5.28). The second capital that Piero discusses, best described as 'composite', is a simpler version of the capitals of the four columns of the judgement hall in the same picture. This simpler form is found in the fresco cycle *The Story of the True Cross* (San Francesco, Arezzo), in the scenes of the Annunciation and of Solomon receiving the Queen of Sheba. The frescos will be discussed in the next chapter, but it seems appropriate to discuss the *Flagellation* in this one, since it is rightly well known as an example of correct perspective.

The Flagellation of Christ

It would perhaps be more accurate to describe Piero's *Flagellation of Christ* not as *an* example of correct perspective, but as *the* example. Despite long-standing disputes about the overall significance of the picture, and in particular the identification of the three figures in the right foreground, the *Flagellation* is almost invariably chosen as an illustration to elementary lectures on perspective and has had a long career as the subject of reconstructions.[75] The evidence is not only that the perspective is indeed correct, but also that the picture is, in this respect, extremely exceptional, at least among works painted in the fifteenth century. In any case, subsequent reconstructions have mainly had rather little to add to what was found in an investigation published in 1953 with illustrations in the form of drawings.[76] A physical model, built up from colour photographs of different sizes, and designed to give a front view the same size as the original painting, is shown in Figure 5.29.

In recent years, however, use has sometimes been made of computer-aided-design (CAD) programs, which essentially seize on real or supposed regularities in the architecture to reconstruct the scene. This technique makes far too many automatic assumptions for the

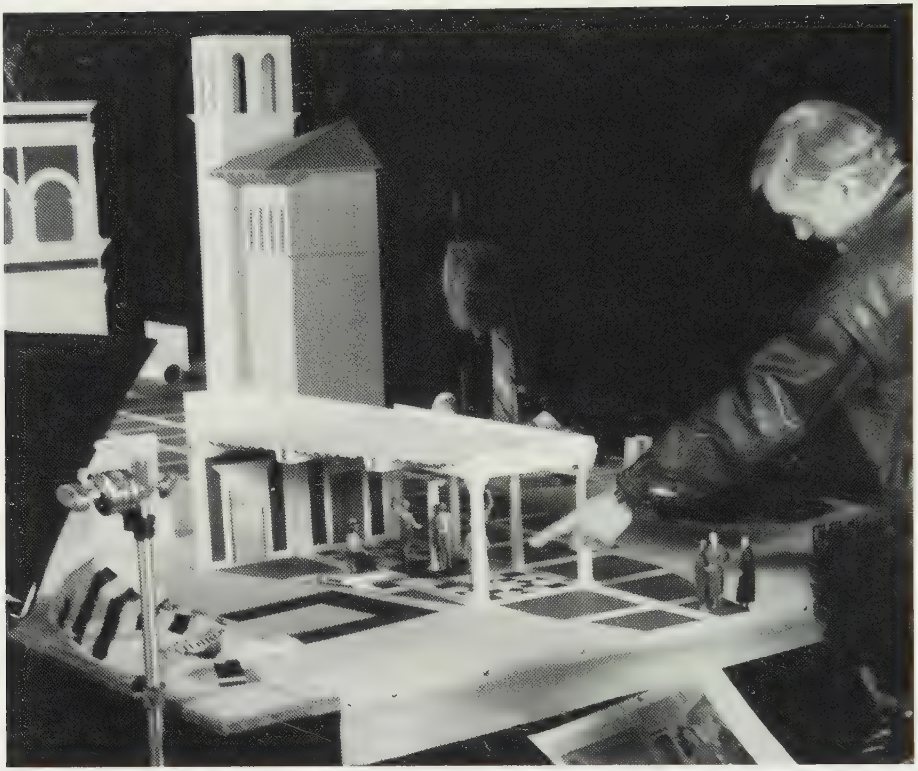
75 A full bibliography, to 1986, is provided in the second edition of Lavin, *Piero della Francesca: The Flagellation* (full ref. note 36). For an interpretation of the picture as 'The Dream of St Jerome', see J. Pope-Hennessy, 'Whose *Flagellation*?', *Apollo*, 124, 1986, pp.162–5, and more briefly in the same author's *The Piero della Francesca Trail*, London: Thames and Hudson, 1991.

76 R. Wittkower and B. A. R. Carter, 'The Perspective of Piero della Francesca's *Flagellation*', *Journal of the Warburg and Courtauld Institutes* 16, 1953, pp.292–302. The ground plan found by Wittkower and Carter is reproduced in Lavin, *Piero della Francesca: The Flagellation* (full ref. note 36), p.32, and in H. Pirenne, *Optics, Painting and Photography*, Cambridge: Cambridge University Press, 1970, p.75.

.28 Piero della Francesca (*c.*1412–1492), *The Flagellation of Christ*, tempera on panel, 59 × 81.5 cm, Galleria Nazionale elle Marche, Urbino.

results to be helpful in assessing paintings. Of much greater interest is the development of an artificial-vision program that analyses plane images (it is designed to read photographs) and makes no assumptions without asking for explicit instructions. A view of the reality constructed from such a reading of the painting is shown in Figure 5.30. This program really does test Piero's accuracy in rendering the perspective. The perspective of the *Flagellation* proves to be correct, with provisos that will be noted below. This comes as no surprise, but the method of analysis is likely to be able to extract useful information from pictures that cannot be tackled by conventional methods or by programs available on older, slower computers.[77] The *Flagellation* is a good test case for machine-assisted investigations

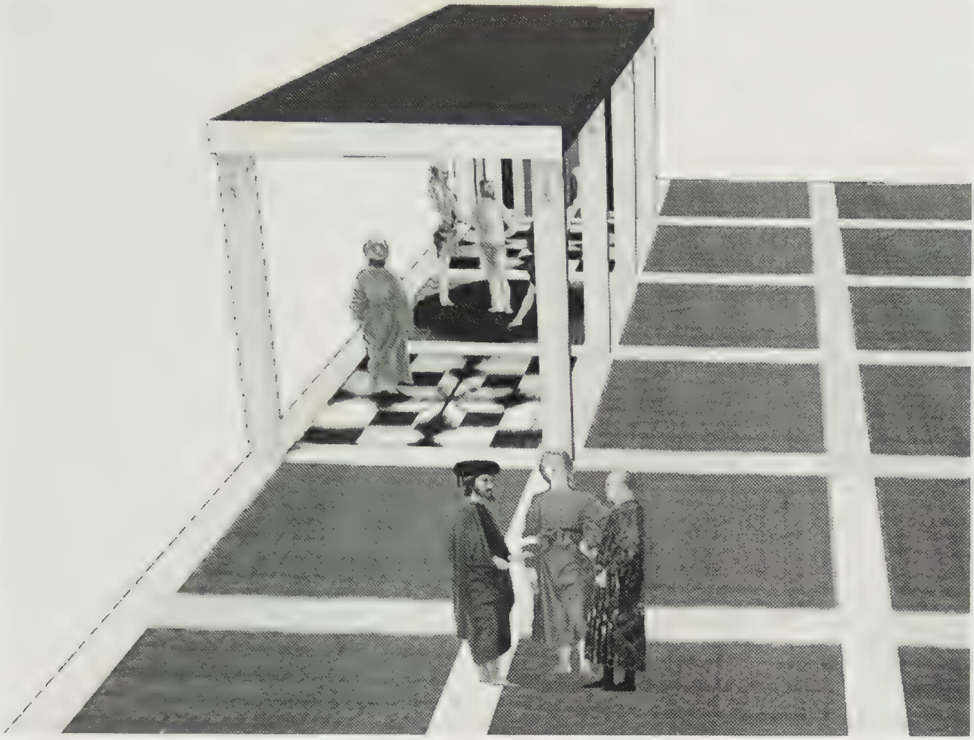
77 See Antonio Criminisi, *Accurate Visual Metrology from Single and Multiple Uncalibrated Images*, Distinguished Dissertation Series, London: Springer-Verlag, 2001. For a clear example of a mathematically unwarranted reconstruction, presumably made using a CAD program, see p.238.



5.29 Three-dimensional model of the scene shown in Piero della Francesca, *The Flagellation of Christ*, made from colour photographs and designed to give a front view the same size as the original painting. The model was made by Philip Steadman in 1992, for use in a television programme. Photograph courtesy of J. P. Steadman.

because it also yields to straightforward geometrical analysis. The lines one reads as orthogonals in the imagined scene – that is, images of orthogonals – all converge to a point that lies close to the inner edge of the moulding on the right (as we see it) of the back wall of the judgement hall, and at a height that is about at the level of the hip of the nearby man brandishing a whip. All images of orthogonals stop well short of this point, which seems to have no other compositional significance. The line taking us closest to it is the left edge of the white strip in the flooring, which drives almost straight into the imaginary space, and, together with the columns whose bases lie along it, serves to divide the part of the picture plane containing the background scene from that containing the foreground one. The eye is thus guided to the background scene, but to its edge rather than its centre. The background scene is, however, given prominence by the strong black-on-white contrasts in its architectural framing, and the brilliance of the lighting in the part of the ceiling directly above the figure of Christ.

There seem to be only two departures from mathematical correctness. The first is that the central white stripe in the floor, which leads the eye into the picture space, has been made slightly narrower than it should be, no doubt for straightforward pictorial reasons. The second is that the house on the far right, which defies exact and unambiguous recon-



5.30 The reality corresponding to part of Piero della Francesca, *The Flagellation of Christ* as reconstructed by a computer using an artificial vision program. See Antonio Criminisi, *Accurate Visual Metrology from Single and Multiple Uncalibrated Images*, Distinguished Dissertation Series, London: Springer-Verlag, 2001. Photograph courtesy of A. Criminisi.

struction because its base is not visible, turns out to defy reasonable mathematical interpretation at all. At best, it can be made to come out unconvincingly tall and thin.[78] In the picture its function is chiefly as a background framing element for the figure in blue and gold brocade; it is the sort of house he might have lived in – if we ignore the reconstruction problem, which a normal viewer is happy to do.

Apart from the perspective, there is another good reason for using the *Flagellation* as an illustration: its wonderful colour. Even by the standards of the fifteenth century, this is a very pretty picture. It is, moreover, painted with the utmost delicacy, for instance in the subtle rendering of the sheen of the grey robe of the man who is turning his back to us. It seems that one was meant to go close to the picture in order to admire such detail in its finish. Moreover, one was surely meant to look closely at the intricate folds of the turban worn by the man in grey. Its actual size is only 35 mm, but it was the subject of a preliminary drawing, transferred to the panel by *spolvero*.[79] However, the correct viewing distance

78 This came to light when Philip Steadman (then at the Open University) was building a three-dimensional model of what is shown in the picture for use in a programme about Piero shown on British television in 1992. His model is shown in Fig. 5.29.

79 See note 36.

for the perspective is about two and a half times the width of the panel. As we mentioned in Chapter 2, the same is true of Donatello's marble relief of the *Feast of Herod* (Musée des Beaux-Arts, Lille), which is also finished in great detail.[80]

While the quality of the finish Piero gave the *Flagellation* may seduce the eye away from the ideal viewing point built into the perspective construction, the colour also seems, in some passages, inclined to subvert one's reading of the spatial structure. For example, there is a very bright pattern of decorative floral motifs on the wall seen to our left of the central figure of the group of three in the foreground. Calculation, or optics-assisted reading of the spatial organization, shows that this wall is far back in the three-dimensional set-up. Thus its decoration must be realistically ancient Roman in its size and brashness, as well as being archaeologically correct in the in-or-out cubes pattern made of rhombi. However, the brightness of the colour pulls the wall forward and subverts our sense of the depth behind the picture plane at which the wall has been situated. It is, as it were, not only the Roman cube pattern that is playing games with depth. The nature of the surface composition and the way it asserts itself here against the composition in depth suggest that the wall and the tree that is behind it are in some way significant as framing for the figure of the young man in the middle of the foreground group. They would seem to be his in the same sense that the urban house belongs to the man in blue and gold brocade. It is accordingly tempting to suppose that the man on the left is to be read as characterized by the fact that his figure is seen against part of the official building in which the judgement hall is situated. Here our sense of the distance of the architecture is subverted by the strong colour contrasts it contains and our sense of the closeness of the human figure.

The identity of the three foreground figures, which might indeed provide some kind of key to the significance of the picture as a whole, has been the subject of much speculation, most of it based on sources external to the picture. From the internal evidence it seems to me that the figures are probably generic ones: a merchant on the right, associated with the town house; and a scholar of some kind, possibly a theologian or a jurist, on the left, associated with the judgement hall. The central figure does not, however, seem convincing as a mere peasant – a class that might be suggested by the vegetation seen behind him. Piero is utterly unsentimental about peasants in the Arezzo frescos. The youth shown here has the kind of dignity Piero usually gives to figures such as saints and angels. The deep rose colour of his clothes also suggests some title to spiritual nobility.

In the left side of the picture, constructed perspective and receding orthogonals are very much in evidence. Both floor and ceiling have patterns that divide them into squares and then into smaller squares. The spatial arrangement is as clear as mathematics can make it. The strength of the illusion depends on rectilinear elements: the plane floor and ceiling patterns, which are closely related to the examples Piero considered in the first book of his perspective treatise. For instance, the broad stripes of white marble in the flooring could have been drawn using the prescriptions for taking strips off the edges of squares, or adding strips to them, that are given in Book 1, Propositions 21 and 22. The double plinth under Pilate's chair would come from the second book, as would the columns, which seem to have sixteen flutes, though they are not exact prisms, since they taper slightly, as columns should. The capitals of the columns, both those of the building and the one supporting the small golden

80 See Chapter 2, p.53.

statue, are, as we have already noted, rather close in their design to those discussed in *De prospectiva pingendi*, Book 3, Proposition 7.

In view of the use of a preliminary drawing for the turban, the small size of the column capitals cannot be regarded as a strong argument for rejecting the suggestion that these capitals were also the subjects of preliminary drawings, using constructions like those described in the perspective treatise. If neither the treatise nor the panel were signed, there would be no problem in establishing a connection between the two. As it is, one may surely wonder whether the signature on the *Flagellation* was, possibly, intended to associate it with the treatise. The panel might, perhaps, even have been made as an exemplar of Piero's skill in practice, a counterpart to the treatise in which he displayed his skill by expounding the theory of perspective. In any case, Piero was in touch with craftsmen who made *intarsie*, and the main lines of the architectural setting do indeed appear in some inlaid work, confirming the reputation of the *Flagellation* in its own time as an example of correct perspective.[81] The house on the right was presumably taken on trust. However, the slightly ostentatious correct perspective of the scene in the left part of the picture may have a quite different explanation: not as an example of correctness for its own sake but as the careful establishment of an illusion that would be sufficiently robust to stand up to the use of very strange lighting in the scene that is shown.

Some oddity in the lighting became apparent when cleaning showed up the strength of the extra illumination in the compartment of the ceiling above the figure of Christ, and the shadows running up and to the left from the orthogonal beam on the left and the transverse beam in front of the compartment in question. In the foreground scene shown in the part of the picture on the right, the lighting is from upper left and is most naturally read as sunlight. We are presumably also seeing sunlight when we look through at the staircase on the extreme left. In natural terms, the extra lighting in the scene on the left is too bright to be anything but sunlight also. However, its source can be traced fairly accurately in three dimensions: its position would place it between two of the columns and allow Christ to be looking at it.[82] On the picture plane, the source of the extra light is almost directly above the point where the orthogonals meet, the point directly opposite the ideal eye. So if we are meant to notice the extra lighting, it will pull our attention upwards, and if we then follow the light itself we come to the group that is, presumably, central to the story. In purely pictorial terms, the additional lighting allows Piero to put further emphasis on the figure of Christ. However, in narrative terms, we cannot see what Christ is seeing, a situation that is no doubt highly defensible on theological grounds, but somewhat unsatisfactory for the historian.

It is perhaps worth remembering that in Piero's time the only way to obtain light comparable in brightness to sunlight was to use sunlight itself. So if a lighting effect like that in Piero's *Flagellation* were required in reality, say for a pageant, it would presumably have

81 In Ferrara, some time after 1449, Piero met Cristoforo and Lorenzo Canozzi da Lendinara, both of whom were craftsmen in wood. Piero was particularly friendly with the latter, who also wrote a (lost) treatise on perspective. See R. Lightbown, *Piero della Francesca*, London: Abbeville Press, 1992, pp.74–5. In his preamble to his discussion of architecture in *De divina proportione*, Luca Pacioli mentions that Piero was friendly with Lorenzo Canozi (*sic*) and says the latter was supreme in perspective; *De divina proportione*, Venice, 1509, Part 2, p.23 recto. On *intarsia* work in general in this period, see Luciano Cheles, *The Studiolo of Urbino: An Iconographical Study*, Wiesbaden: Dr Ludwig Reichert Verlag, 1986.

82 For details, see Lavin, *Piero della Francesca: The Flagellation* (full ref. note 36); summary in M. J. Kemp, *The Science of Art: Optical Themes in Western Art from Brunelleschi to Seurat*, New Haven and London: Yale University Press, 1990, p.32.

been attempted by the use of mirrors. High optical quality would not have been necessary, the purpose being merely to direct a suitable quantity of light onto the subject, so something quite crude would suffice, such as roughly flat sheet metal given a bit of a polish by an armourer. I do not, of course, wish to suggest that Piero's picture should be read as portraying the effect of a hidden mirror of this kind, but is it perhaps conceivable that, as in the case of the shiny-disc haloes sometimes shown on saints and angels (as in Piero's *Sant'Antonio Altarpiece*, see Fig. 6.29), we are seeing in idealized form something imported from the tradition of spectacle? The high visibility of the perspective construction in Piero's painting could be a consequence of a programme that included this non-natural lighting but, as the number of differing identifications of the foreground figures reminds us, a good grasp of the programme is one of the things we most conspicuously lack. So the *Flagellation* becomes an example of perspective. And we cannot even be sure that Piero did not simply intend it to be just that.

It is perhaps because Alberti's *De pictura/Della pittura* has been so much more widely read and written about than Piero della Francesca's *De prospectiva pingendi* that Piero's *Flagellation* has so often been described as 'Albertian'. The label does not seem to be particularly meaningful. There are indeed clearly defined spatial relationships between the objects shown in the picture, but there is no reason to suppose that this trait, which is found in many other pictures by Piero, is due to Alberti, except in the highly indirect sense that Alberti admires such pictures, and perhaps contributes to a fashion for admiring them. For the technical detail of the perspective construction, it is obvious that Piero's understanding of the matter went far beyond what we find in *De pictura*.

The part of the picture that can be proved to be in mathematically correct perspective is, of course, the architectural setting. As we have seen, in the right part of the picture, Piero allows other considerations to intervene in a manner that might interfere to some degree in our reading of the spatial arrangement. In this part, we have few clues as to the spatial relation of the figures to the architecture. On the left, however, the relationship is entirely clear and shows up a departure from naturalism that has been ignored by most scholars: the architecture is very small.[83] Piero must have known, as we do now, that Roman public buildings were large in scale. Indeed, the sources of the classicizing architectural elements shown in the *Flagellation* are all much larger than they appear in Piero's use of them.[84] He seems simply to have followed the convention that the setting, being of secondary importance, could be made smaller. A similarly undersized architectural setting is to be found in the Annunciation scene in *The Story of the True Cross*. It appears that in some contexts such scaling down of the setting could be employed without detriment to an overall impression of naturalism. Piero's compositional strategy in regard to the architecture in the *Flagellation* is like that in the predella panels of the *St Lucy Altarpiece* of Domenico Veneziano, and in other predella panels (see Chapter 3), but it is unlike, say, Mantegna's treatment of architecture in his *Circumcision* (Galleria degli Uffizi, Florence).

83 An honourable exception to this rule is provided by Lightbown (full ref. note 81).

84 For Piero's sources, see Christine Smith, 'Piero's Painted Architecture: Analysis of His Vocabulary', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no. 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art, 1995, pp.223–53. Though the style of Piero's architecture has frequently been described as Albertian, his sources now seem rather to have been antique (or what was at the time taken to be so).

It is possible that Piero's scaling down the architecture was not entirely a matter of his unconsciously following convention, since it has some formal consequences. By enabling him to show the ceiling as well as the floor, it allowed him to introduce a supplementary perspective grid in the upper part of the picture. If the floor alone had been responsible for defining spatial relationships, it would have had to be simple enough to be easily readable. As it is, the floor is rather complicated and Piero has avoided the slightly mechanical chequerboard effect one finds in the works of some of his contemporaries.

There seem to be two substantial clues to the date of the *Flagellation*. First, that Pilate is shown wearing a hat that appears to be copied from Byzantine models that Piero is most likely to have seen in Florence in 1439, when the Council of Orthodox and Catholic clergy, convened in Ferrara to discuss establishing union between the churches, had been moved to Florence and was holding its meetings in Santa Maria Novella.[85] Second, that the panel is painted in tempera, which suggests it is earlier in date than pictures in which Piero used oil. The second criterion tends to tilt the balance against any close association of the *Flagellation* with the Montefeltro portraits. That is, against any close association in date. There is, furthermore, no external evidence to connect the *Flagellation* with the 'small panels with figures' that Vasari tells us Piero painted for the Duke of Urbino. However, in context, the implication of what Vasari says is that these pictures were showpieces of Piero's art and served to establish his reputation. If so, it would surely be likely that they were signed. Few of Piero's pictures are signed, but the *Flagellation* is one of the exceptions: the plinth under Pilate's chair has the inscription 'OPUS PETRI DE BURGO SANCTI SEPULCRI'. So if Vasari is a reliable guide, which he may well not be, the *Flagellation* may be one of the works that served to establish Piero's reputation as a painter. And if it was painted for Federigo da Montefeltro, it would confirm that he took an interest in the practice of perspective. Such an interest in perspective is indicated by the *intarsia* panels in the decoration of Federigo's *studiolo*, and complements the interest in perspective theory that is suggested by the presence of a copy of *De prospectiva pingendi* in the Ducal library.[86]

Dating the perspective treatise

The *Flagellation of Christ* is exceptional in its prominent display of mathematical skill. The fact that *De prospectiva pingendi* in principle tells us how to go and do likewise, at least as far as the mathematical component of the drawing is concerned, scarcely detracts from the impression of virtuosity conveyed by the painting. To most readers, the long series of drawing instructions in the treatise, particularly in Book 3, would surely have been almost as intimidating as the painting. To follow such instructions successfully in practice would certainly require an unusual capacity for visualizing three-dimensional shapes. All the same, it is in principle possible to do so, and it seems perverse to doubt that Piero's text does indeed describe the methods Piero himself employed. This is, of course, not to say that we should suppose that constructions like those of Book 3 were used in each and every surviving picture. However, Piero's frescos show extensive indications of the transfer of

85 Much weight is put upon a possible connection with the Council in Carlo Ginzburg, *Indagini su Piero*, Turin: Einaudi, 1981 (English translation, *The Enigma of Piero*, London: Verso, 1985).

86 For the *studiolo* see Cheles, *The Studiolo of Urbino* (full ref. note 81). For the presence of the perspective treatise in the Ducal library, see Chapter 4 and Appendix 7.

5.31 Piero della Francesca (*c.*1412–1492), *Madonna del Parto*, fresco, 260 × 203 cm, Cemetery Chapel, Monterchi.

preliminary drawings, and the uniformity found among repeated elements such as heads, in panel paintings as well as in frescos, might well be explained by the reuse of a standard drawing. As we have seen, the method of Book 3 allows such reuse, since the viewpoint is decided only after the drawing of the 'proper' or 'perfect' form of the object is completed. The reuse of drawings is impossible to prove, but it is clear that Piero liked a rather high degree of uniformity among his figures, since there are several examples of figures that are more or less exactly a mirror image of another one in the same work.

In one famous case, however, in which it has often been asserted that the cartoon was simply reversed, that of the angels on either side in the *Madonna del Parto* (Cemetery Chapel, Monterchi) (Fig. 5.31), the mirror imaging is not as exact as it may at first seem.

Spolvero marks show where the drawing was transferred, but the wing of the angel on our left has been moved so that its tip is about 2 cm to the left of the *spolvero* version. This adjustment may have been made to take account of the asymmetry in the central figure, since the shapes of the spaces between the wings and each edge of the figure of the Madonna make a significant contribution to the formal balance of the picture as a whole. Piero may have had a taste for symmetry but, as we see it in his paintings, it is not a taste for exact symmetry. When we find a great deal of symmetry in the drawing, as in the *Madonna del Parto*, it is always accompanied by changes in intrinsic colour or in lighting. That is, the actual form is constant (or nearly so), but what we see is subject to one of the parts of painting that Piero explicitly excluded from his treatise: 'giving the colours as they are shown in the things, light and dark according as the light makes them vary'.[87]

The use of patterns, either as solid forms seen in various ways or as repeated or reversed plane elements, does not accord well with, say, Alberti's demand for variety in figures. It is nonetheless one of the characteristics that make Piero's style so easy to recognize. That the works are not individually dull or cumulatively monotonous is partly due to the elements that do vary, such as the lighting, which Piero handles with exquisite subtlety. The effect is also partly to be ascribed to relatively slight variations in the drawing which, no doubt being carefully designed to do just that, have relatively important effects upon the whole. One might perhaps sum the matter up by saying that Piero was clearly a very good draughtsman, but in paintings his other skills tend to direct our attention away from this one. All the same, the paintings suggest the existence of a series of preliminary drawings that were reused, and it is surely reasonable to suppose that at least something of this series of drawings is preserved in *De prospectiva pingendi*. It is much more likely that the more elaborate of the drawings in the treatise are copies after, or adaptations from, drawings that had already been made for use in pictures, rather than that the illustrations to the treatise were made specially for that purpose. There is no evidence Piero ever ran a large workshop. The lack of pictures in which only parts seem to be by his hand suggests he did not.[88] So it is not likely that he had a substantial stock of drawings that had come into existence merely for the purpose of training apprentices. Moreover, as we have seen, *De prospectiva pingendi* has a mathematical character that makes it seem in some ways impractical as a manual for apprentices.

This impracticality is like that of the *Trattato d'abaco*, also an ostensibly practical text, but known to have been written as an ideal example of a textbook at the request of a friend, rather than for use. It seems possible that *De prospectiva pingendi* has a similar kind of origin. Clearly this cannot be that it was the product of a specific request, since Piero does not mention one, but the work might perhaps have been written as a response to repeated requests for explanations of how he had drawn this or that in his pictures. It is not difficult to believe, for instance, that someone who had read Alberti's account of perspective

87 Piero della Francesca, *De prospectiva pingendi*, Book 1, Introduction: Parma MS, p.1 recto; BL MS, p.1 recto; Piero ed. Nicco Fasola, p.63. For the context, see above, at note 2.

88 The possible exception to this rule is the *Madonna and Child Enthroned with Angels* (Sterling and Francine Clark Art Institute, Williamstown), in which the overall design implies a degree of skill that is not found in most of the execution. The only one of Piero's pupils whose style is sufficiently close to Piero's own for him to have worked this piece up from drawings by his master would seem to be Lorentino d'Arezzo, of whom far too little is known for a rational comparison to be made. See Chapter 6 for reasons against attributing the picture to Piero himself or to his workshop under his supervision.

might recognize that there were things in Piero's pictures that it did not explain, and might be sufficiently emboldened by his reading to ask for further information. Alternatively, or by way of supplement, Piero himself may have wished to leave a record of his methods as a memorial to the use of his mathematical skill in the service of painting, or even, since he collaborated in the translation of the work into Latin, as a contribution to the further association of his style of painting with humanist learning.

All these possible explanations for how the treatise came to be written suggest the writing followed the painting, that is that the treatise was a consequence of Piero's being recognized as a competent painter rather than being, say, a young man's bid for fame (in which case he would have needed a prominent dedicatee for the work). It seems unlikely, however, that *De prospectiva pingendi* was written only in Piero's old age, since, in his prefatory letter to the *Libellus*, he contrasts that with the former in this respect.[89]

If Vasari's account of Piero's output is reasonably reliable, many of Piero's pictures have not survived. Only six of the eighteen works Vasari mentions are still extant – and this number involves the assumption that he has misidentified the *Misericordia Altarpiece* (Museo Civico, Sansepolcro) (*c.*1460–62) as a fresco. In particular, it seems we have lost frescos that Piero painted in churches and convents in and around Borgo San Sepolcro and Arezzo. Frescos are of particular interest in the present context since they are overwhelmingly likely to have occasioned the use of preliminary drawings. For example, the coffered half-dome in *De prospectiva pingendi*, Book 3, Proposition 9 (Fig. 5.26), might well have formed part of an architectural niche for one of the figures of saints that Vasari mentions, such as the two in Borgo parish church or the St Vincent in the convent of Monte Oliveto in Arezzo. On the other hand, it is possible that the problem in the treatise was developed as a more elaborate version of the half-dome shown in the *Sant'Antonio Altarpiece* (Fig. 5.27). It is also possible that the problem represents an earlier stage in the design we find in the finished picture.

Given the messy state of the available evidence, one cannot reasonably go much further than to say it seems that the writing of *De prospectiva pingendi* probably took place after Piero had painted the fresco cycle of the *Story of the True Cross* in Arezzo, and might have been occasioned by encounters with humanist intellectuals in Rome in 1458 to 1459. Not that there was a total lack of humanist intellectuals among Piero's countrymen, but just as it is apparently the thought of a picture being seen by strangers that prompts a signature (including the name of Borgo), so it seems likely that talking to strangers prompted the act of self-identification that is inherent in writing a treatise about the practice of one's craft.

89 The prefatory letter to the *Libellus* is translated in Appendix 7.

6

Optics and Illusionism

The *Flagellation of Christ* (Galleria Nazionale delle Marche, Urbino) provides an ideal case in which the three-dimensional organization of bodies in the painting is defined almost exclusively by means of geometrcal perspective. As we have seen, the lighting in fact works partly against the relations set up by the formal construction, since it is an inherently implausible combination of sunlight and light from a hidden source inside the judgement hall. However, the *Flagellation* is highly exceptional in both respects. Piero della Francesca's mathematical construction of rectilinear elements is usually far more discreet in relation to the overall composition, and his lighting tends to follow the direction of natural lighting, thereby implying a continuity of spatial relationships through the picture plane, that is from our space into that of the scene in the picture.

This implied continuity is not rigorous, which is to say that none of the surviving works contains a *trompe l'œil* trick of the kind described in the final three problems of *De prospectiva pingendi*. In fact, in the fresco cycle *The Story of the True Cross* (San Francesco, Arezzo), where the position of the eye of the observer is known, at least approximately, in regard to its height, Piero does not use a corresponding eye height for the ideal viewer of the picture. Nor does he generally adopt the eye height of a standing figure in the picture itself, as Masaccio had done in *The Tribute Money* (Brancacci Chapel, Santa Maria del Carmine, Florence) (Fig. 2.6), and as was recommended by Leon Battista Alberti in *De pictura/Della pittura*. Following this rule was common practice among Piero's contemporaries. Piero must surely have come across Alberti's work when he was in Florence in 1439, and it is even possible that he also met Alberti in person at that time.[1] All the same, Alberti may not be a direct source for any of Piero's usage: other elements in Piero's construction of depth suggest that a powerful exemplar was provided by Masaccio's fresco of *The Tribute Money*. In any case, like Masaccio's, Piero's figures are so strongly modelled that they are inevitably read as three-dimensional.

The clear sense of readable spatial relationships within the picture is universally acknowledged as one of the characteristics of Piero's style as a painter. As we have seen in Chapter 4, a corresponding characteristic can be found in Piero's mathematics. However, there is little that is unusually mathematical about the way the illusion of depth is set up in most of Piero's pictures. Apart from the occasional display of virtuosity in the use of perspective

1 Piero is documented as being in Florence in September 1439. Alberti had been there since January, as part of the papal entourage at the Council of Florence.

– to which we shall return – Piero conveys depth by the same pictorial means as most of his contemporaries. Formal perspective is useful only for simple geometrical objects, such as houses or colonnades, whose 'proper' form is well known and whose 'degraded' form can therefore be decoded, allowing the degradation to be read in optical terms as an indication of depth behind the picture plane. In the manner of the columns of a portico, the relative sizes of uniform objects, or human figures, can also act as an indication of depth, as they seem to do, for instance, in Piero's *Baptism of Christ* (National Gallery, London) (Fig. 4.2). On the crudest possible level, an effect of depth can also be achieved through the occultation of one object or figure by another. A more subtle variant is the use of shadows, both cast shadows falling on something else, which are relatively rare, and the shading produced by a body casting parts of itself into shadow. These effects, and those of aerial perspective, effectively belong to natural optics, *perspectiva* proper.

Aerial perspective is little used by most of Piero's Italian contemporaries, but with this exception Piero's usage is distinguished not by the means he employs but by his great skill in handling them. This too is an element in Piero's recognizable style. It accordingly seems possible that looking at the predominance of one or other means of constructing a sense of the third dimension in different pictures may not only provide some insights into Piero's art in a general way, but might also give us indications of the dates at which the pictures were painted. This is a matter of looking at longer-term changes, since the possibility of making neat groups is immediately excluded by the fact that the two dated pictures – the panel of St Jerome now in Berlin (Gemäldegalerie) (Fig. 3.2), dated 1450, and the fresco in Rimini (Tempio Malatestiano) showing Sigismondo Malatesta before St Sigismund (Fig. 6.1), dated 1451 – use very different means of constructing spatial relationships. However, with that warning in mind, we shall nevertheless examine the kind of knowledge of optics that Piero was bringing to bear upon the creation of illusion in various paintings.

Sigismondo Malatesta before St Sigismund

The discreet display of mathematical perspective in Piero's dignified profile portrait of Sigismondo Malatesta at prayer before St Sigismund should perhaps be seen as a salute to the antique, a complement to the use of classicizing architectural motifs in the framing of the work. Theatrical effects may be conceived in the best classical taste, so the line between the hieratic *tableau vivant* and the frankly theatrical is not an easy one to define. No doubt some of the Byzantine prelates entering Santa Maria Novella for the meetings of the Council of Florence in 1439 thought Masaccio's *Trinity* was a vulgar imitation of a waxwork. What we would now describe as being, for its time, an unusual level of naturalism in rendering effects of depth is held in check, to the modern eye, by Masaccio's firm control over the surface composition. Much the same is true of Piero's fresco of Sigismondo Malatesta and his patron saint.

The classicizing style of the architecture of the setting in which Piero has placed his figures may no doubt, for once, fairly be described as Albertian. Alberti himself was responsible for the overall design of the interior of the church in which the fresco was painted, and it is to be presumed that Sigismondo Malatesta would have taken Alberti's advice on his choice of painter – though it may also be relevant that the Malatesta family had ruled over Piero's native Borgo San Sepolcro from 1394 to 1430, and thus probably still had some contacts

6.1 Piero della Francesca (*c.*1412–1492), *Sigismondo Malatesta before St Sigismund*, 1451, fresco transferred to canvas, 257 × 345 cm, Cappella delle Reliquie, Tempio Malatestiano (San Francesco), Rimini.

there.[2] In any case, it is rather unlikely that architectural motifs of which Alberti disapproved would have been included in the picture, but that does not, of course, prove that Alberti's own architecture was a source for the setting.[3]

The classical style is not important in Piero's construction of depth. We have no vista, except for the tiny view of the castle through the fictive round window (*occhio*) on the right. Instead we are shown a shallow stage. Its back wall has panels of dark marble revetment, each panel enclosed by a white marble moulding, placed between fluted white marble

2 In 1449 Alberti wrote to Sigismondo concerning a possible painter, but the painter in question is not identifiable. See Battisti, vol.1, p.43. On the Malatesta rule of Borgo San Sepolcro, see James R. Banker, *The Culture of San Sepolcro during the Youth of Piero della Francesca*, Ann Arbor: University of Michigan Press, 2003, esp. pp.14–18.

3 See Christine Smith, 'Piero's Painted Architecture: Analysis of His Vocabulary', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no. 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art, 1995, pp.223–53.

pilasters with intricately carved capitals.[4] The floor of the stage has a simple pavement set out in squares. Extending the rather short images of orthogonals, we find that they meet close to the geometrical centre of the picture. Our eye height is thus deemed to be about 2 metres above a realistic level, and the point to which the eye is directed has no apparent significance in the organization of the composition in the plane. The eye height is, moreover, clearly not that of a standing figure in the picture, or that of either of the figures shown.

In this particular case 'artificial perspective' seems a thoroughly apt description of what has been used. It is not, however, clear how accurately any construction has been carried out. There have been suggestions that the images of the orthogonals of the pavement do not meet exactly at a single point as they should.[5] In view of the date of the picture, as well as the Albertian look of our being presented with figures set in their places on a horizontal grid, it seems highly unlikely that Piero did not know that the images of orthogonals should meet exactly. If they do not do so in the painting, the error may well be due to the transfer of the design from a drawing. Since the images of the orthogonals are short, a small error made in transferring the position of one end point will have a relatively severe effect on the direction of the line obtained. (This is the converse of the rule that measurements based on short images of orthogonals will be unreliable.)

Perhaps in its use of a method that can lead to such errors – rather than the method of direct construction on the 'degraded' plane that is implied by the corresponding propositions in *De prospectiva pingendi* – we may regard the Rimini fresco as an immature example of Piero's use of perspective. In fact, although the point opposite the eye (the meeting point of the images of the orthogonals) does not seem to have any compositional significance of its own, the horizontal through it – a line that plays some part in Alberti's discussion of perspective[6] – does run through the centre of the roundel showing the castle and through the centre of the circle outlining the sphere under St Sigismund's hand. Between these two, it grazes the edge of Sigismondo Malatesta's cuff. The horizontal line, which is the otherwise unseen horizon for the scene shown in the picture, thus has a certain importance in the surface design. All the same, the line is not visible as such. We are possibly meant to take notice of it by observing the congruence of shapes between the roundel and the sphere. In any case, we are clearly intended to read the pavement pattern in spatial terms, thereby putting Sigismondo Malatesta in the middle of the stage, in depth as well as width, with the dogs – echoing the colours of the pavement – nearer the front and St Sigismund nearer the back, though we may note that St Sigismund's position is less clearly established since he is not placed directly on the grid.

There is, however, a problem with this simple reading of the picture as showing a three-dimensional structure. In view of Piero's known later practice, the problem would seem to be of his own making and, to a certain extent, deliberate. At the top of the picture, he has shown us a strip of white marble, delicately carved in relief, which may at first appear to be a frieze placed over the architrave of the entablature on the back wall of the fictive space. Yet even in the present imperfect state of preservation of the fresco, it is clear that the struc-

4 My understanding of the spatial arrangement of the picture strongly endorses the reading of the architectural setting given by Smith, 'Piero's Painted Architecture' (full ref. note 3).

5 See, for example, the drawing given by Battisti, vol.2, p.425, fig.268. The drawing is ascribed to G. Joppolo and its significance is not discussed.

6 For instance, in *De pictura*, Book 2, §33.

ture under the architrave involves not columns but pilasters.[7] The slightly darker colour of the putative frieze suggests it may belong to a different structure. This is confirmed by its joining up at either end with similarly carved vertical members that clearly lie close to the picture plane and have their bases on the floor of the structure enclosing the figures. Piero has, as it were, provided a small proscenium arch.

The problem with reading the frame for what it is, rather than as an architrave, is due to its making only a weak colour contrast with the pilasters and mouldings on the back wall. Although the simulated frame serves to cut the scene off from the surrounding wall surface – which is, after all, only to say that it is indeed a frame – Piero seems to have been anxious to avoid too sharp a sense of recession at the level where the frame is seen against the back wall at the top of the picture. Similarly, the bright colour of the heavy swag hanging against the back wall seems designed to pull it forward. We have readable three-dimensional relationships, constructed by the pattern of the floor, but Piero is using colour – and specifically lack of colour contrast – to mitigate a sense of recession. He is apparently prepared to interfere with our sense of the spatial organization in order to assert the importance of relationships in the plane. We have seen such a use of bright colour in the background in *The Flagellation of Christ* (Fig. 5.28). The non-contrast effect is rather common in Piero's works, and whites of various textures or variously lit provide several examples.

One particularly clear example, no doubt made clearer because it has been wrenched out of its visual context as part of an altarpiece, is that of the panel showing St Michael, now in the National Gallery, London (Fig. 6.2). The altarpiece was painted for an Augustinian church in Borgo San Sepolcro some time between 1454 and 1469.[8] St Michael originally stood on the dexter side, immediately next to the now missing central panel, which, as is indicated by the fragments of dais and brocade visible at the bottom of this picture, may have shown the Coronation of the Virgin. The saint on the outside dexter edge, next to St Michael, was St Augustine himself (Museu Nacional de Arte Antiga, Lisbon) (Fig. 6.22). Like the remaining three accompanying saints,[9] St Michael stands a little in front of a balustrade, whose design is similar to that of the wall behind Sigismondo Malatesta. The lighting on all four figures is from above right. It is probable that a throne (or some similar structure) in the central panel established a depth for the mottled marble ledge on which the saints stand. This ledge is like that of the corresponding figures in the *Misericordia Altarpiece* (Museo Civico, Sansepolcro). It makes no contribution to establishing depth. It is the strong modelling of St Michael and the other saints that ensures the effect is three-dimensional. However, St Michael's wings are both partly seen against the white marble of the balustrade, and they have a tendency to merge or blend into it. The effect is particularly marked for the wing on our left, in which the striated pattern of white feathers is

7 As Smith has pointed out, if these uprights are to be read as columns, we must assume Piero made blatant errors in their perspective, showing diverging images of orthogonals. It is overwhelmingly more reasonable to read the diagonal lines as belonging to mouldings round the panels of revetment. See Smith, 'Piero's Painted Architecture' (full ref. note 3), esp. pp.231–8.

8 This dating is established by surviving documents: the contract is dated 4 October 1454 and the work is known to have been finished by 14 November 1469. See James R. Banker, 'Piero della Francesca's S. Agostino Altarpiece: Some New Documents', *Burlington Magazine* 129, 1987, pp.642–51. The documents are published in Battisti, vol.2, p.611, and vol.2, p.617.

9 St Augustine (Museu Nacional de Arte Antiga, Lisbon), St John the Evangelist (Frick Collection, New York) and San Niccolò da Tolentino (Museo Poldi-Pezzoli, Milan).

played off against the even neater geometry of the white mouldings below and behind it. We may notice also that the disposition of shadows allows parts of the outlines of St Michael's legs to form only weak contrasts with the darker marble against which they are seen. Piero seems to have been at some pains to ensure that the figure would not detach itself too strongly from the balustrade that unifies the design. On the other hand, the figures of the remaining saints are allowed to detach themselves, presumably because, since they are all heavily clad in voluminous garments of at least ankle length, rather little of the balustrade is to be seen behind them.

6.3 Piero della Francesca (*c.*1412–1492), *Sigismondo Malatesta before St Sigismund*, detail of St Sigismund's left hand, 1451, fresco transferred to canvas, Cappella delle Reliquie, Tempio Malatestiano (San Francesco), Rimini. The complete picture is shown in Figure 6.1 above.

In both the Malatesta fresco and the panel of St Michael, the unity of lighting is a significant element in our apprehension of three-dimensional relationships, not only through the shadows that give solidity to things, but also through the use of highlights. In the fresco, Piero seems to be happy to ignore Alberti's warnings against the use of pure white. White highlights define such things as the flutes of the marble pilasters and the shiny fur of the white dog. White is also used to indicate light on the view of the castle in the *occhio*, in which, as in the main part of the scene, the light comes from the left. The richest glow must surely have been that of gold, which has unfortunately now been lost, from St Sigismund's clothes and from those of Sigismondo Malatesta. All that remains of it is the colour of the sceptre held in the saint's right hand and the orb that rests under his left one. This latter detail, shown in Fig. 6.3, epitomizes Piero's mastery both in drawing and in rendering the fall of light. It is as accomplished as the glittering armour in *The Story of the True Cross*, which was to be admired by Vasari. Moreover, the simple geometrical form, whose pattern of lighting could in principle be found by means of geometrical optics, is held in place by a perfectly human hand. Piero seems to have found both equally worthy of careful attention.

In the panel of St Michael, the direction of the lighting is defined rather sharply by the shadow cast on the saint's left leg by the skirt of his armour and by the shadow of his chin on his neck. There is, however, also a considerable amount of what one might call reinforcing glitter: the sheen on the steel plates at the shoulders, the small highlight on each of the many gems, including the less brilliant ones on the cuffs of the fine-textured sleeves of the undertunic. The sleeves themselves are indicated with extremely thin strokes of white. When the painting was in position on the altar, they can hardly have been visible to anyone but the officiant. Even more gratuitous is the treatment of the lines of seed pearls that edge the top of the saint's bright red boots. As painted, each pearl is about the size of a pinhead, and each one is different, to take account of the different way the light is falling on it. Many other cases could be adduced to show features similar to those noted here. Any attempt to

6.2 (*facing page*) Piero della Francesca (*c.*1412–1492), *St Michael*, panel from an altarpiece for Sant' Agostino, Borgo San Sepolcro, 1454–69, tempera and oil on panel, 133 × 59.5 cm, National Gallery, London.

explain Piero's rendering of the fall of light as being indebted to a mathematical analysis of the problems presented by various different elements in the picture would accordingly necessitate taking some arbitrary decisions about where one stopped. Piero's powers of observation seem to have been far too acute to leave us any clues as to where he himself stopped. In his time, however, both observation and calculation would have been regarded as legitimate parts of *perspectiva*. Our niggling feeling that we ought to be able to separate the two is somewhat anachronistic.

Chance has removed the picture of St Michael to a situation far from its intended visual context. A positive aspect of this displacement is that it enables us to see clearly that our apprehension of depth in the picture is not dependent on the formal perspective elements that presumably linked up the panel and its surviving companions to the now missing central panel of the altarpiece. Piero could manage very well without visible reliance on mathematics. Effectively, the flow of light is doing the constructional work that might otherwise be expected of a tiled pavement. We find the same in many other pictures by Piero, including frescos, in which the decision to follow the natural lighting thus becomes important within the picture as well as in establishing its relation to the observer. For instance the emphatic shadows in the *Madonna del Parto* (Cemetery Chapel, Monterchi) (Fig. 5.31), which serve to establish spatial relations inside the picture, are too strong to be due to anything other than sunlight, if we rule out divine sources. The uneasy look of this picture in its now modified chapel is no doubt at least partly due to a change in the lighting. The rather subdued lighting, which comes from several sources, in the chapel at Rimini in which Piero painted his fresco of *Sigismondo Malatesta before St Sigismund* may well have been one factor in his decision to put in a pavement, which would ensure the correct reading of spatial relationships in the picture.

The fresco cycle *The Story of the True Cross*

Conveniently, when Piero painted the various episodes of *The Story of the True Cross* in San Francesco, Arezzo, highly directional lighting from nearby windows was available for the isolated figures placed in the lunettes on the altar wall, as it then was. The one on the right, possibly identifiable as Isaiah or Jeremiah, is shown in Figure 6.4.[10] Despite the damage to the blue robe, the figure still seems thoroughly solid. As in the Rimini fresco and the St Michael panel, Piero has made repeated use of what seems to be pure white, this time for an intricate set of highlights on the pale lining of the red cloak, as well as on the ankles and feet and the head, neck and hair of the figure itself. The light also somehow catches the eyes of the prophet, particularly his right one, and the white ruffling edge of the undergarment visible at the neckline of the robe. In a detail, as shown in Figure 6.4, the effect is theatrical. The picture is designed to be easily read from the floor of the church, about 10 metres below (see Fig. 6.5).

We do not know the date at which Piero began work in San Francesco. The commission for the fresco cycle had originally been given to Bicci di Lorenzo (1373–1452), so it is to

10 The direction of the prophet's glance connects him with the scene of the death of Adam and the planting of a seed from the tree of knowledge in his grave. Prophecies that were considered to be linked with these episodes are found in Isaiah 11: 10 and in Jeremiah 23: 5–6. R. Lightbown, *Piero della Francesca*, London: Abbeville Press, 1992, p.135, for Isaiah; and M. A. Lavin, *Piero della Francesca: San Francesco, Arezzo*, New York: George Braziller, 1994. p.95, for Jeremiah.

6.4 Piero della Francesca (*c.*1412–1492), *Prophet* (possibly Isaiah or Jeremiah), from *The Story of the True Cross*, 1452–66, fresco, width 190 cm, San Francesco, Arezzo.

be presumed that Piero took over after Bicci's death, in 1452, though it is possible that he had also been involved at some earlier stage. A document of 20 December 1466 mentions the cycle as completed.[11] So during the later 1450s and the early 1460s Piero was working on both the fresco cycle and the panel paintings for the Augustinian altarpiece. This gives legitimacy to the comparison of the panel of St Michael with the fresco of the prophet. The pictures are also comparable in their degree of finish. The vivid treatment of highlights in the fresco seems designed to take account of its being seen from a large distance, but Piero has nonetheless shown many details that would certainly not be visible to the normal viewer. Such detailed finish is in fact characteristic of the fresco cycle as whole.

The choice of scenes, and the narrative structure of *The Story of the True Cross* – that is, the positions chosen for the scenes within the chapel – were probably made by the patrons rather than by Piero or by Bicci di Lorenzo. The programme includes thematic symmetries such as that of the two battle scenes in the lowest tier (Figs 6.6 and 6.7), but details of the links between scenes were presumably left to Piero. Some of these links involve compositional pointers to the narrative sequence, such as the way the line of the wood in the picture of the middle tier on the right of the altar wall, *The Burial of the Wood*, points to the next scene in chronological sequence, *The Annunciation*, in the lowest tier on the left of the altar wall (Fig. 6.5). In some cases there are similarities of design in the figures that are shown.[12] However, it is also notable that Piero has imposed compositional unity on each wall as a whole by relating the scenes one to another, irrespective of any direct narrative connection.

For example, on the right wall, shown in Figure 6.7, there are links between the composition in the lowest tier, *The Battle of Ponte Milvio*, and that in the middle one, the scenes of the Queen of Sheba. The lance at the front of the group of riders at the left in the battle scene, whose line passes between the spectator and Constantine's head, is now colourless, but may once have been picked out in gold, as was almost certainly the case with the tiny cross held in Constantine's hand. This lance is prominent in the composition of the battle scene. It makes an acute angle with the contrasting dark lance to its left, producing an outline of an arrow that points at the wood in the centre of the scenes of the Queen of Sheba in the tier above. Indeed, the whole openwork structure of the lances tends to direct attention towards the centre of this composition, which is critical both in displaying the wood – which is the reason for the scenes in this tier being included – and in containing the point of concurrence of the images of the orthogonals shown in the architectural structure in the right-hand part of the picture (the scene in which King Solomon receives the Queen of Sheba). This point of concurrence is significant because it is the point to which the ideal observer of the picture is understood as directing his or her eye. As many scholars have noted, one of the Queen of Sheba's ladies is making a gesture that also tends to guide the eye towards this point. Since the wood is the spiritual centre of the scenes, Piero has also made it the centre of his perspective scheme, thus ensuring that there is only one ideal attention point for the mathematical reading of the three-dimensional structure of the picture and for appreciating its religious significance. Unfortunately the picture in the lunette in the highest tier on this wall, *The Death of Adam*, is badly damaged. However, the central

11 See Battisti, vol.2, p.615, document 96.

12 The narrative aspects of the cycle have received considerable attention. For a recent, fairly detailed discussion see Lavin, *Piero della Francesca: San Francesco, Arezzo* (full ref. note 10).

6.5 (*facing page*) Piero della Francesca (*c.*1412–1492), *The Story of the True Cross*, 1452–66, fresco, San Francesco, Arezzo. A view of the chancel (*cappella maggiore*) from the body of the church. On the far wall, at the left, working downwards: *Prophet*, *The Raising of Judas from the Well*, *The Annunciation*. On the right: *Prophet*, *The Burial of the Wood*, *The Dream of Constantine*.

6.6 Piero della Francesca (*c.*1412–1492), *The Story of the True Cross*, 1452–66, fresco, San Francesco, Arezzo. A view of the left wall of the chancel. In the lunette: *The Exaltation of the Cross*. Middle tier: *The Finding of the Three Crosses*, *The Proving of the Cross*. Lowest tier: *The Battle of Heraclius and Chosroes*, *The Beheading of Chosroes*.

6.7 Piero della Francesca (*c.*1412–1492), *The Story of the True Cross*, 1452–66, fresco, San Francesco, Arezzo. A view of the right wall of the chancel. In the lunette: *The Death of Adam* and *The Planting of a Seed of the Tree of Knowledge in Adam's Grave*. Middle tier: *The Queen of Sheba Recognizes the Wood, The Queen is Received by King Solomon*. Lowest tier: *The Battle of Ponte Milvio (The Victory of Constantine over Maxentius)*.

6.8 Piero della Francesca (*c.*1412–1492), *The Annunciation*, from *The Story of the True Cross*, 1452–66, fresco, 329 × 193 cm, San Francesco, Arezzo. See also Fig. 6.5.

6.9 Piero della Francesca (*c.*1412–1492), *The Dream of Constantine*, from *The Story of the True Cross*, 1452–66, fresco, 329 × 190 cm, San Francesco, Arezzo. See also Fig. 6.5.

tree clearly establishes a connection with the central accents in the picture below it, and since the tree was once green, its colour, together with the landscape, must have made the two pictures much more comparable than they are in their present state.

The directional lighting that plays such an important part in the two figures of prophets in the uppermost tier of the end wall of the chapel (Fig. 6.5) reappears, though in less emphatic form, in the two annunciation scenes in the lowest tier of the altar wall, the Annunciation to the Virgin on the left (Fig. 6.8) and, on the right, the angel's message in Constantine's dream (Fig. 6.9). In the latter picture, the main flow of light does indeed follow that from the nearby window, but it is not daylight. Its colour is golden, and its source is the small cross held in the hand of the angel who is flying down at a steep angle on the left side of the picture. This cross was presumably shown in gold. It can now only just be made out, in paler colour against the pink of the tent, almost directly above the line of the cuff of the angel's robe.[13] In narrative terms, the near invisibility of the cross is an important loss, since in the story told in the *Golden Legend* of Jacob of Voragine (*c.*1230–*c.*1298), which was almost certainly Piero's source, Constantine dreamed that an angel appeared and directed his attention to a vision of a cross made of bright light and carrying in golden letters the words 'in this sign you will conquer' ('in hoc signo vinces'). The matching cross in the neighbouring scene of *The Battle of Ponte Milvio* (Fig. 6.7) is the cause of the rout of Maxentius, which is seen to be miraculous because, since the lances are unbroken, battle has not yet been joined.

In the scene of Constantine's dream, the main lighting, on the foreground, originates from the cross, thus making the cross a determining element in our viewing of the scene. This is true in an entirely literal sense, since the light belongs to the dream and is clearly not present to the guards and the attendant sitting beside the bed. Piero presumably knew that he had a talent for portraying impassivity and made good use of his skill in this seated figure, visible to us in a light he himself does not see. The dream light provides shadows and modelling for the human figures and for the shape of Constantine's tent and its furnishings. Behind the tent, we see the natural lighting, pale moonlight in which colours become bluer. Piero may have thought of this as an example of 'giving the colours as they are shown in the things, light and dark according as the light makes them vary' (as he says in the introduction to his perspective treatise), but in this case he has gone rather further than that, since he has shown a change in colour not merely a change in intensity.[14] The diminishing heights of the tents indicate their distance, but essentially our sense of the spatial organization of the scene is determined by the lighting.

The pendant to this scene, on the same level, to the left of the central window (see Fig. 6.5), is *The Annunciation* (Fig. 6.8). The minor iconographic variant by which the angel is holding a palm frond instead of the more usual lily is fairly common in this region at this time and was no doubt the patron's choice.[15] A more interesting iconographic variant is Piero's avoidance of a deep perspective vista, blocked off in a symbolic representation of

13 For clear photographs of this passage see Lavin, *Piero della Francesca: San Francesco, Arezzo* (full ref. note 10), p.63, and M. A. Lavin, *Piero della Francesca*, New York: Harry N. Abrams Inc., 1992, p.87.

14 This picture is usually regarded as the first to show such change of colour, which is now known as the 'Purkinje effect'. See, however, M. J. Kemp, *The Science of Art: Optical Themes in Western Art from Brunelleschi to Seurat*, New Haven and London: Yale University Press, 1990, p.312.

15 On the inclusion of an Annunciation scene in the cycle and the detail of the palm frond, see Avraham Ronen, 'L'*Annunciazione* nel ciclo della Croce di Piero della Francesca e la tradizione aretina', in *Città e Corte nell' Italia di Piero della Francesca. Atti del Convegno Internazionale di Studi Urbino, 4–7 ottobre 1992*, ed. Claudia Cieri Via, Venice: Marsilio, 1996, pp.205–18.

virginity.[16] Such vistas, which are extremely common in Annunciation scenes, are included in the *Annunciation* (Fitzwilliam Museum, Cambridge) in Domenico Veneziano's *St Lucy Altarpiece* (Figs 3.11 and 3.12) and, in an extreme form, in Piero's *Annunciation* in the *Sant' Antonio Altarpiece* (Galleria Nazionale dell'Umbria, Perugia) (see below and Fig. 6.29). Instead of a vista, in the *Annunciation* scene in *The Story of the True Cross* Piero has provided a shallow loggia, which is grandly classical in style but unrealistically small in height. Thus, although the figure of the Virgin is about the same size as that of the standing soldier to the left of Constantine's tent, she appears to be larger. This effect must presumably be deliberate, since it is exaggerated by the monumental folds of the red robe. The solid golden halo, a disc seen in perspective, blocks the head off from a too close encounter with the architecture behind but not much above it.

The perspective of the loggia may have been worked out in detail, but we have not been left with enough clues to analyse it. There are only two visible images of orthogonals: one along the edge between the white marble moulding and the dark red panel in the ceiling,[17] and one in the base of the column. Both lines are short. However, it is clear that they cannot be made to meet within the picture. Unfortunately, there has been damage to the picture in the area of the column base, so it is possible that its line is now incorrect. If there were any measurable perspective in the picture of Constantine's dream, one might wonder whether Piero had designed the two pictures to have a common centre of perspective, on the central axis of the window (and the chapel). As things are, it is merely clear that we are looking into the *Annunciation* scene from the right, and that our centre of attention has everything to do with the Virgin and nothing to do with the perspective construction.

The flow of light is also from the right, illuminating the inside of the window opening in the wall above the loggia, and casting a shadow of the loggia onto the wall behind the angel. In a contrary direction, approximately parallel to that of the arm of the angel in *The Dream of Constantine*, we have a sheaf of gold rays running out from the hands of God the Father towards the Virgin. The position of His hands is rather as if He were releasing the dove that is often shown as a symbol for the Holy Ghost. The gold rays have now almost disappeared but must originally have been a prominent element in the picture. The direction of these rays is not clear in spatial terms, since we have no definite indications of the position of God the Father – which is no doubt perfectly correct in theology, but is not helpful for a naturalistic reading of Piero's composition.

This picture is one of those that stands up least well to being examined in isolation. It works much more satisfactorily in its correct visual context. As can be seen in Figure 6.5, the strong visual accent provided by the column, continued by the edge of the wall running vertically above it, is taken up by the vertical in the high wooden framework that is being used to raise the Jew, Judas, from the well in the scene above (which is the next scene in chronological sequence). This part of the end wall of the chapel has a heavy vertical that, as it were, absorbs the thrust of the downward-pointing diagonals on the other part of the wall. The line of the wooden beam in the picture in the middle tier on the right thus takes us as far as the *Annunciation*, but not beyond it.

16 The virginity symbolism has apparently been transferred to the window and the wooden bar across it, whose shadow passes through a ring. See Lavin, *Piero della Francesca: San Francesco, Arezzo* (full ref. note 10), pp.56–7.

17 There are, of course, several lines in the moulding associated with this one, but they can hardly be counted as separate from it in a practical sense as opposed to a strictly mathematical one.

6.10 Piero della Francesca (*c.*1412–1492), *The Raising of Judas from the Well*, from *The Story of the True Cross*, 1452–66, fresco, 356 × 193 cm, San Francesco, Arezzo. See also Fig. 6.5.

6.11 Piero della Francesca (*c.*1412–1492), *The Burial of the Wood*, from *The Story of the True Cross*, 1452–66, fresco, 356 × 190 cm, San Francesco, Arezzo. See also Fig. 6.5.

In the remaining two scenes on this wall, *The Burial of the Wood* (Fig. 6.11) and *The Raising of Judas from the Well* (Fig. 6.10), the quality of the paint handling is not as subtle as in the two lower pictures, and it is generally agreed that their execution is probably the work of assistants. The overall design and the cartoons must, however, be by Piero himself. *The Burial of the Wood* is designed as a frieze, with a landscape background that provides no definite clues as to its distance. The figure on the left hints at a scene of Christ carrying the cross, but the workmen are shown as undignified and they wear simple clothes. Their costume and their sloppy stockings are like those of the workmen shown in *The Finding of the Three Crosses* (Fig. 6.18). They do not resemble the central figure in the group in the foreground in *The Flagellation of Christ* (Fig. 5.28), a figure that has sometimes been identified as representing a peasant. The costume of that figure in fact looks like the clothing Piero normally gives to angels. Thus, despite his apparent association with vegetation, it nonetheless seems difficult to identify the figure in the *Flagellation* as merely a peasant. *The Burial of the Wood* is lit from the left, in accordance with natural lighting, but there is little sense of depth in the picture. Its composition chiefly serves to emphasize the picture plane and, as already noted, to point to the next scene in chronological order, the *Annunciation*, in the lowest tier on the left part of the wall.

The Raising of Judas from the Well (Fig. 6.10), directly above the *Annunciation* (see Fig. 6.5), has a fairly well-defined spatial structure, partly because the figures are clearly to be read as grouped round the opening of the well, and partly because we have the receding edge of a building on the left. The two edges of the roof of the building and the line of the wooden pole across its windows provide what seem to be three orthogonals. In principle, a fourth is provided by the lower edge of the nearer window, but it is extremely short. The images of the three longer orthogonals meet close to the right edge of the picture, about on the level of the hem of the blue garment of the man who has Judas by the hair. If these lines really are images of orthogonals, then the centre of the perspective scheme is within the picture, as Piero prescribes in *De prospectiva pingendi.*[18] It accordingly seems possible that the centre of the perspective scheme of *The Annunciation* also lies within that picture, perhaps again on the very edge – which, using the more reliable orthogonal, namely the one from the ceiling, would put the height of the eye of an ideal observer about on the level of the palm of the Virgin's raised right hand. This hypothesis does not, of course, enable us to reconstruct a ground plan. It is merely more in accord with what we know of Piero's understanding of perspective than is the otherwise rather attractive hypothesis that the two lowest pictures had a common centre. Also, if the ideal eye height is connected with the Virgin's hand, this would fit in rather well with a sense of the picture's being readable in terms of the interaction of the hand gestures of the three figures. All three pairs of hands seem to have been designed to catch at our attention.

The scene of *The Raising of Judas from the Well* shows a curious departure from Piero's normal practice: the natural light comes from the right, but the light in the painting is from the left. As the lighting from the left is consistent, it seems exceedingly unlikely that its unexpected direction is, as some scholars have suggested, the result of an error by the assistant who actually painted the picture. Since frescos were made a *giornata* at a time, any such

18 Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 13, final paragraph: Parma MS, p.6 verso; BL MS, p.7 recto; Piero ed. Nicco Fasola pp.76–7; see Chapter 5 and Appendix 8.

6.12 Piero della Francesca (*c*.1412–1492), *The Queen of Sheba Recognizes the Wood* and *The Queen is Received by King Solomon*, from *The Story of the True Cross*, 1452–66, fresco, 356 × 747 cm, right wall of chancel, middle tier, San Francesco, Arezzo. See also Fig. 6.7.

error would have been noticed quickly. The painter, about to start on his second *giornata* by applying a layer of fresh plaster to the allotted area, must surely have looked at the first *giornata*, which would almost certainly have been in the top left corner of the picture, showing the walls of the house. This passage is slightly problematic in any case, since it is not clear how we come to have light inside the window openings, but the overall direction of the lighting is obviously from the left. Since the present lighting has the face of the official on the right fully lit, it is obvious that if the assistant was relying upon Piero's drawings, which presumably included some indication of the boundaries separating dark from light, the putative original design, from which the assistant would be departing, must have shown the face completely in the shade, which hardly seems likely. Moreover, most of Judas's face would also be in the shade. It seems more reasonable to suppose that Piero did indeed intend the lighting that we now see, and that its direction is meant to link this picture with the next picture in the chronological sequence, which lies on the left wall, at the same level as this picture. There, St Helena, having extracted from Judas the information about where the cross was hidden, sets about having the cross dug up.

The scenes of St Helena, in the middle tier on the left wall (Figs 6.6 and 6.18), balance those of the Queen of Sheba in the middle tier on the right one (Figs 6.7 and 6.12). In both cases we have two separate scenes, divided one from another by being given separate settings. In each case one setting is architectural and visibly 'in perspective', whereas the other is landscape, in which depth is defined in a less rigorous manner. This scheme is similar to that adopted by Masaccio in his fresco of *The Tribute Money* (Fig. 2.6), which Piero must

6.13 Piero della Francesca (*c*.1412–1492), *The Battle of Ponte Milvio*, from *The Story of the True Cross*, 1452–66, fresco, 329 × 764 cm, right wall of chancel, lowest tier, San Francesco, Arezzo. See also Fig. 6.7.

surely have studied when he visited Florence in 1439. Moreover, on both walls in San Francesco, Piero has also followed Masaccio's example in having the landscape part on the left side – the part one reads first – and the architecture on the right. In the two scenes of the Queen of Sheba, the palace in which the queen is received by King Solomon provides enough reasonably long images of orthogonals to establish that their point of concurrence lies in the wood that the queen has recognized as that which will become the cross. Thus, as we have already noted, the point to which our attention must be directed to read the perspective is a significant one for understanding the meaning of the picture. In this also, Piero has done as Masaccio did in *The Tribute Money*, where the meeting point of the extended images of the orthogonals was in Christ's face.

This picture in the middle tier is the only one of the three on the right wall in which we see orthogonals and can thus reconstruct at least part of a perspective scheme. This does not, of course, mean that Piero did not have definite perspective schemes for the pictures in the other tiers. However, we have no way of finding out much about them, except in rather vague terms. One could wish for more definite terms, since the scene in the lowest tier, *The Battle of Ponte Milvio* (Fig. 6.13) includes, near its left edge, a spectacular figure of a rearing horse, which was admired by Vasari and was no doubt one of the figures he drew when he made studies from these frescos in his youth. This may, of course, be no more than a good piece of freehand drawing by Piero. The sense of solidity is greatly enhanced by the natural lighting in the church, which makes the horse leap out in the metaphorical sense as well as the more literal one. On the other hand, anyone who could propose the

6.14 Piero della Francesca (*c.*1412–1492), *The Battle of Heraclius and Chosroes* and *The Beheading of Chosroes*, from *The Story of the True Cross*, 1452–66, fresco, 329 × 747 cm, left wall of chancel, lowest tier, San Francesco, Arezzo. See also Fig. 6.6.

kind of analytical engineering methods that Piero applies to drawing the human head in *De prospectiva pingendi* (Book 3, Proposition 8, see Fig. 5.25) is clearly not entirely committed to vague concepts of organic wholeness when it comes to drawing living things. The rearing horse may be an exercise in geometry. However, it does not appear in the perspective treatise, and when one examines the picture from various different positions on the floor of the church there is no sense that any particular (reasonable) viewpoint is more correct than another. Perhaps we have here an indication that somewhere between a human head and a rearing horse was where Piero drew the line when it came to doing a full perspective construction.

On the left wall (Fig. 6.6), it is again in the picture in the middle tier that we have the most clearly defined perspective scheme. However, the openwork cylindrical structure over the throne dais in the small scene of the beheading of Chosroes to the right of the battle scene in the picture in the lowest tier (Fig. 6.14) provides a number of images of orthogonals, which meet at a point close to the central vertical axis of the picture and rather close to its lower border, about at the level of the meeting of the hooves of the black and white horses. If this perspective scheme is meant to apply to the whole picture in this tier of the decoration of the left wall, it nevertheless makes rather little difference to our reading of the closely packed mass of figures, which gives an unpleasantly realistic impression that all one needs to know is who is trying to kill whom. The mass of figures is almost a caricature of an overpopulated panel from a Roman sarcophagus. The effect is simultaneously static and violent. A tiny piece of clear stage is made only at the far right of the picture,

6.15 Piero della Francesca (*c.*1412–1492), *The Exaltation of the Cross*, from *The Story of the True Cross*, 1452–66, fresco, 390 × 747 cm, left wall of chancel, highest tier, San Francesco, Arezzo. See also Fig. 6.6.

and that for a formal act of violence, the beheading of Chosroes. Apart from the repeated 'Roman' imperial soldiers and their horses, the comparison with *The Battle of Ponte Milvio* on the opposite wall is one of contrast.

The perspective of the scene in the lunette, *The Exaltation of the Cross* (Fig. 6.15), is less well defined. It seems likely that the cross on the left is intended to be presented straight on to the line of kneeling citizens of Jerusalem on the right, so we might hope to use the edges of its transverse member as orthogonals. Unfortunately, the lines concerned seem to be nearly parallel to one another. The centre line of the transverse piece points more or less towards the bottom right corner of the picture. There is thus no question of its meeting the apparently well-defined horizon (visible in the central part of the picture) in any meaningful point. One must therefore assume that the cross has been turned to make it more visible to the true spectators as opposed to the painted ones. The part of the city walls shown on the right is clearly not straight, and it is accordingly not certain that the lines of battlements, which seem to be aligned with one another, are orthogonal to the picture. We have better luck elsewhere: the two upper edges of the battlements really are aligned, and their line meets the putative horizon close to the central vertical of the lunette. The line of the upper edge of the right-hand part of the city wall seems also to pass through this point. So there is evidence for the existence of a perspective scheme.

The rather low eye height implied by the line we have assumed to be the horizon is confirmed by the fact that we seem to be looking up at the standing figures on the left. The light falling on these figures is from the direction of the window, that is from the right. It falls directly on the face of the cross (which could be another reason for Piero's having turned it). The light also casts noticeable shadows, particularly on the figure of the man dressed in white, second from the left, who is turning away from us. In the natural lighting of the church, this figure is very prominent. The sinuous line of the bottom of his robe is marked in *spolvero*, but it is not clear whether this should be interpreted as an indication that Piero took special pains over this figure; it may well be that the *spolvero* merely shows up here because of the pale colour. However, close inspection from the restorers' scaffolding in 1992 did not reveal any other uses of *spolvero* in drapery, though it was clearly visible in all faces and hands in all the pictures of the fresco cycle, and there were also indications of the transfer of drawings for architectural elements.

Infrared reflectograms, made in the course of the restoration work, have revealed some of Piero's drawings on the *intonaco*, which not only show his extensive use of *spolvero* – even in some passages of drapery – but also include his freehand drawings, sometimes in charcoal and sometimes done with a brush. For example, in the head of a male bystander in the scene of King Solomon receiving the Queen of Sheba (Fig. 6.16, compare Fig. 6.12), the profile and main features have been transferred by *spolvero* but the ear has been added freehand in charcoal, and shows shading. Again, the juxtaposed heads of two of the Queen of Sheba's ladies (Fig. 6.17), from the scene where the queen recognizes the wood (Figs 6.7 and 6.12), show *spolvero* for the profile and freehand drawing with a brush for the head of the lady facing us. Piero seems to have been most unusual in making such extensive use of drawings, and we may note that they were used everywhere, that is not only in the areas where the painting was to be done by assistants, but even in the areas that Piero proposed to paint himself. Nevertheless, the guidance the drawings give is not always fully detailed: in the scene of *The Death of Adam* (Fig. 6.7) the same drawing was found under the head of Adam and under that of Eve.[19] Unfortunately, the history of repair work on this wall has been such that there seems to be no chance of anyone ever being able to have a look at Piero's *arriccio* in the hope of recovering his larger scale preliminary drawings (*sinopie*).[20]

19 Most of these drawings have not been published, but some of them are discussed in Roberto Bellucci and Cecilia Frosinini, 'Ipotesi sul metodo di restituzione dei disegni preparatori di Piero della Francesca: il caso dei ritratti di Federigo da Montefltro', in *La pala di San Bernardino di Piero della Francesca. Nuovi Studi oltre il restauro* (*Quaderni di Brera* 9), ed. Emanuela Daffra and Filippo Trevisani, Florence: Centro Di, 1997, pp.167–87. I am grateful to Dr Frosinini for allowing me access to the relevant computer files.

20 One nineteenth-century restoration involved pouring cement into a cavity behind the paintings on the left wall. It is thus most unlikely that a way could be found to separate the *intonaco* from the underlying *arriccio*. Various other heroic measures had earlier been adopted to prevent collapse following earthquake damage.

6.16 Piero della Francesca (*c.*1412–1492), underdrawing of head of a bystander in *The Queen is Received by King Solomon*, from cycle *The Story of the True Cross*, 1452–66, fresco, San Francesco, Arezzo. Infrared reflectogram courtesy of the Opificio delle Pietre Dure, Florence. See also Figs 6.7 and 6.12.

6.17 Piero della Francesca (*c*.1412–1492), underdrawing of heads of attendant ladies in *The Queen of Sheba Recognizes the Wood*, from *The Story of the True Cross*, 1452–66, fresco, San Francesco, Arezzo. Infrared reflectogram courtesy of the Opificio delle Pietre Dure, Florence. See also Figs 6.7 and 6.12.

6.18 Piero della Francesca (*c.*1412–1492), *The Finding of the Three Crosses* and *The Proving of the Cross*, from *The Story of the True Cross*, 1452–66, fresco, 356 × 747 cm, left wall of chancel, middle tier, San Francesco, Arezzo. See also Fig. 6.6.

The picture in the middle tier of the left wall (Fig. 6.18) shows, in the left part, Helena supervising the discovery of the crosses and, on the right, Helena kneeling with her attendants, when the true cross declares its nature by bringing a dead body back to life. This second scene is set in a townscape, which provides a number of lines that can be read as orthogonals and can thus be used to find the position of the foot of the perpendicular from the eye of the ideal observer to the picture. This turns out to be disarmingly easy to do. The left edge of the marble-inlaid building behind the kneeling women is clearly being seen absolutely edge on: we see the orthogonal lines of the white moulding at the lower edge of the roof as vertical. Thus the point opposite the ideal eye must lie on the line of the edge, which, convincingly, lies on the centre line of the complete picture. The point we are looking for turns out to be the point of intersection of this line with the line in the furrows of the ploughed field behind that runs into it, a little below the level of the shoulder of the kneeling lady with a white cloak. That is to say, the lines in the roof, the wooden bar and the moulding under the windows of the grey house on the far right can all be extended to meet at this point. The same may be true of the tiny top and bottom edges of the end of the transverse member of the cross.

Piero seems to have made a vertical alignment of the three centres of the perspective schemes in the three tiers on this wall. In the left part of the picture in the middle tier, the crosses are obviously not aligned in any simple way, but this is not to say that Piero had no definite spatial arrangement in mind. Since we know their shape, the crosses do at least give us some sense of reading the pattern of the figures surrounding them, and Piero may well have carried out exact constructions for their 'degraded' forms – presumably by dissecting them into prisms in the manner of some of the problems in *De prospectiva pingendi*, Book 2. The perspective effects in the right-hand part of the picture obviously contribute to our reading of depth in the left part, but the left does contain indications of depth in the hilly landscape and the portrait of Arezzo as Jerusalem in the background. The view of the city is fairly like that one sees by looking back from the road that leads out through the Porta San Domenico. The pink bell tower seen above the gate is probably that of the present Badia, and the large church at the top of the hill is clearly the cathedral, with the fortress (now demolished) behind it. San Francesco is down the hill on the right. Piero seems to have moved the buildings about a bit, but the look of incorrect perspective is entirely naturalistic.

In the scenes of the Queen of Sheba on the right wall, our one determinable point of concurrence of lines representing orthogonals is an important point not only for the perspective but also for the religious significance of the picture. This is not so in the picture that shows scenes of St Helena. Both parts include the cross, and the simplest possible reason for putting the point to which the eye should be directed in the middle of the picture is very probably the true one: attention is intended to be equally divided between the scenes. Piero has, in fact, done his best to give us nothing to see at the point towards which we are, in principle, meant to look (see detail in Fig. 6.19). The point itself, being on the edge of an orthogonally receding face of a building, has no precise location in depth. Moreover, it has been given the white-on-white treatment that we have seen in the non-contrast of St Michael's wing against the balustrade in the panel from the Augustinian altarpiece (Fig. 6.2). The white column or pilaster at the edge of the building is partly obscured by the white sleeve of the kneeling lady in the white cloak, but the line of her sleeve follows the edges of the marble so closely that the two forms run into one another. Similarly, the pale colour of the ploughed field is such as almost to allow the line of the lady's shoulder to be read as a continuation of a furrow. The first reading problem would be liable to take one's attention to *The Proving of the Cross* on the right, whereas the second problem would send it off towards the scene of *The Finding of the Three Crosses* on the left.[21]

There is some more white-on-white at the other side of the same marble-faced building: at first glance, we may seem to have a wider strip of white at the right than at the left, but a closer look shows that the extra white is part of a pale tower that stands behind the building and whose upper part can be seen above the right side of the pediment. The pale building a little further to the right is also further back. As in the St Michael panel, Piero is avoiding detaching forms too sharply one from another. In the fresco, however, he goes further. He seems to be prepared to risk some of the sense of depth that his use of perspective on the far right has created for this townscape.[22] This should perhaps not surprise us too much; we have seen something rather similar in the right-hand part of the *Flagellationof Christ* (Fig. 5.28).

There is, in any case, no danger of our finding the foreground forms difficult to read, since they are strongly modelled as lit from the right, the illumination as usual following

21 This is the only case I know in which the centre of a perspective scheme apparently justifies both its usual name in sixteenth-century Italian treatises, namely 'flight point' ('punto di fuga'), and the 1715 English name 'vanishing point' – but unhappily in the sense that there is flight away from the point and the point itself vanishes.

22 If we assume the grey house on the far right is square, and that Piero's construction of this image followed the correct rules laid down in *De prospectiva pingendi*, Book 2, Proposition 9, then we can make a beginning on constructing a ground plan of the scene of *The Proving of the Cross*. However, I do not understand the mathematical basis for the detailed reconstruction proposed in Thomas Martone, 'Piero della Francesca e la prospettiva dell'intelletto', in *Piero teorico d'Arte*, ed. Omar Calabrese, Rome: Gangemi, 1985, pp.173–86.

6.19 Piero della Francesca (*c.*1412–1492), *The Finding of the three Crosses* and *The Proving of the Cross*, from *The Story of the True Cross*, 1452–66, fresco, 356 × 747 cm, left wall of chancel, middle tier, detail of Figure 6.18, showing the part where the scenes overlap, San Francesco, Arezzo. See also Fig. 6.6.

the natural lighting in the church. In this natural light, the figure that catches one's attention is the man on the right wearing the sculptural pink–purple cloak. Unlike some of the other draperies, this part does not seem to have been the work of assistants. The man's head then takes one straight into the image of the cross and forward to the brightly lit back of the young man rising from the dead. All the lines in the back show *spolvero*. The verticals of the buildings, and the outlines of the various surfaces of the cross show scribed lines that suggest they too were transferred from preliminary drawings. In fact, if one looks closely at the pictures – as was possible from the restorers' scaffolding – the overwhelming impression is that Piero has taken so much care over everything that it is impossible to decide, on these grounds, what was most important.

This is, indeed, also what one sees on looking closely at Piero's panel paintings. The obvious difference is that, while the owner of, say, the *Madonna di Senigallia* (Galleria Nazionale delle Marche, Urbino) (Fig. 4.15), would be able to peer closely at it and marvel happily over Piero's display of skill, only scaffolding can put anyone in a position to see the beautiful quality of the detail in the frescos in San Francesco. For instance, from such close quarters, the 'incorrect perspective' look of Arezzo as Jerusalem is replaced by a sense of real spatial relationships, so that one can find one's way through the streets of this almost familiar city, and can, moreover, see the pairs of metal pincers that are laid out on folding counters in front of the shops just inside the gate. Most of the painting is not quite as fine-textured as it is in, say, the panel of St Michael, but the size of many of the details is of the order of 2 or 3 mm. The quantity of detail is not even – there are, for instance, large visible brushstrokes in draperies – and the quality of parts of the detail suggests assistants were merely having some fun, but a great deal of the finish is so delicate that one is compelled to take it seriously. The effect is, of course, visible for a normal viewer, on the floor of the church, in the sense that the pictures convey an impression of a complexity that, like that of the real world, cannot all be taken in by the eye. One does not reach for an adjective such as 'economical' to describe Piero's method of arranging to convey this impression. However, one must assume he did indeed intend to convey it.

The two most obvious explanations for his choice of method may in fact be equivalent, from his point of view. They are that he believed getting things right in detail was the best way (for him) of getting them to look right, or that he painted the detail to the greater glory of God. The first suggestion is, of course, the one that is more amenable to being unpacked in a historically interesting way. The detail of the finish would then work with the careful rendering of light to compensate for the inevitable departure from naturalism caused by the fact that the centre of attention in the picture cannot in general be identified with the point to which the ideal observer of its perspective scheme is deemed to direct his eye, and that the eye is also inevitably at an incorrect height, so that the sight line to this ideal point cannot, as it should, be perpendicular to the picture plane. Many painters may simply have accepted without further thought the convention that the eye height in a perspective picture was generally not in accord with real viewing conditions, but it is very difficult to believe Piero took such an attitude. As we have seen, there is a great deal of ordinary natural optics in his perspective treatise. He must certainly have understood the mathematical implications of his own constructions. Thus every picture becomes a compromise, even in regard to its mathematics.

In making this compromise Piero must obviously also have recognized that a picture in correct perspective could, in practice, be viewed from a position some distance away from the correct viewpoint and still retain much of its power to suggest depth. He presumably found this unexpected, maybe even counter-intuitive, since it seems to extend the 'power of lines' further than it should go. It would be extremely interesting to know how Piero accounted for the phenomenon, if indeed he regarded it as having the status of an optical phenomenon in its own right – after all, one cannot be sure what other people see. Piero's perspective treatise of course gives us no hint, being concerned with getting the picture right in the first place. Today we are inclined to regard the phenomenon as real (that is, true for essentially all observers) and explanations involve visual psychology. Nevertheless, we sometimes externalize it by referring to the image itself, rather than our sense of it, as being robust. In fact, the history of the discussion of the phenomenon tells one immediately that we are not dealing with a matter of physiological optics, since the working of the human eye (as far as the retina) was understood more or less in the modern manner from 1604 (see Chapter 2), and 'robustness' surfaces as a matter for investigation only in the mid-twentieth century.

It is in some ways convenient that the phenomenon of which Piero seems to be aware is, in some degree, explicable by today's theories about the working of the human mind. And we may notice, also, that the standard visual theory of Piero's time, with its emphasis upon central vision, is in good agreement, as far as perception is concerned, with today's explanation in terms of the structure of the retina. However, as we have seen in examining Piero's mathematics, we are liable to be misled if we allow today's theories to guide us in investigations of work that depended on different theories. Investigations of our own response to Renaissance art – that is, a response assessed in terms of today's understanding of perceptual psychology – seem in principle to belong to the realm of art criticism, not that of art history. In this respect, historians of art perhaps have something to learn from the newer discipline of history of science. We should not allow ourselves to be led too far by the apparent similarities between Ibn al-Haytham's concept of the 'central ray' and our own concept of 'macular vision'.[23] For our present purposes, we fortunately merely need to observe that Piero seems to have noticed the phenomenon we call the robustness of the perspective illusion. And it is also fairly clear that he distrusted it – after all, it does run flat counter to the certainty that everyone believed was given by mathematics – since he is at pains to devise compromises when designing his fresco cycle.

What we see in San Francesco is that within their context the pictures are self-consistent, each a world of its own, but carefully related to the others around it. Piero's treatment of light is one of the ways in which he achieves this relationship, but he has also designed each wall as a whole, by organizing pictures so that their compositions interlock, and possibly also by aligning the centres of their perspective schemes. It should probably be seen as an indication of a wish for greater naturalism in the perspective that ideal eye heights seem on the whole to be lower than the level of the eyes of the people in the pictures, even when, as in *The Proving of the Cross*, some of them are kneeling. A similar explanation might be offered for the fact that the ideal eye height in the Rimini fresco is below that of either St Sigismund or Sigismondo Malatesta.

23 Here I have consequently preferred to rely on Henri Pirenne, *Optics, Painting and Photography*, Cambridge: Cambridge University Press, 1970, which makes straightforward use of geometrical optics, rather than appealing to the more complicated descriptions found in such works as Michael Kubovy, *The Psychology of Perspective and Renaissance Art*, Cambridge: Cambridge University Press, 1986, which shows an inadequate grasp of the differences between fifteenth-century visual theories and those accepted today.

The Story of the True Cross presents us with paintings that are in something fairly close to their complete original context, and thus gives us an important insight into the kinds of relation Piero wished to establish between the world of the observer and the fictive world of the picture. Each of Piero's pictures is in principle self-contained, as can be seen from the fact that they look satisfactory when isolated as illustrations in a book. This internal coherence tends to ensure that little attention is directed to the framing elements that divide them one from another on the wall. The framing elements are highly conventional, though the simulated horizontal mouldings may well be intended to look properly classical. However, at the corners Piero has not provided the conventional vertical framing members employed in the fresco cycle this one is clearly designed to resemble most, namely Masaccio and Masolino's unfinished decoration of the Brancacci Chapel. There, the vertical framing takes the form of simulated classical pilasters with pretty pink capitals.[24] Unlike Masaccio, Piero has allowed his pictures to run into the neighbouring wall. Indeed, as has been pointed out by several scholars, there is even a degree of wraparound by the repetition of forms in neighbouring pictures on either side of the corners of the altar wall.[25] For instance, the gold cross held by the angel in *The Dream of Constantine* (Figs 6.5 and 6.9) reappears in Constantine's own hand in *The Battle of Ponte Milvio* (Figs 6.7 and 6.13), and the features of Chosroes (Figs 6.6 and 6.14), who had claimed to be God, are the same as those of God the Father in the nearby *Annunciation* (Figs 6.5 and 6.8), while the house on the left in *The Raising of Judas from the Well* (Figs 6.5 and 6.10) associates this scene with the St Helena scenes to its left (Figs 6.6 and 6.18), with which it belongs since they come next in chronological sequence.

These visual echoes seem to be designed to reinforce the meaning of scenes as well as their narrative order. The programme of the cycle does not follow a transparently simple chronological sequence but instead presents a more reflective treatment in which the scenes of a well-known story are arranged in a pattern that emphasizes spiritual and historical analogies. The fact that Piero did not cut scenes off sharply one from another could be interpreted as expressing something of the same spirit. It is lucky for a modern viewer that Piero's talent for compositional clarity compensates for the lack of instant recognition of the episodes that are shown. However, we may note that even in this respect Piero exercises a variant of the non-contrast technique. For example, in the scenes with St Helena, the long white cloak of the lady furthest to the left in *The Proving of the Cross* flows into the immediately preceding scene of *The Finding of the Three Crosses* to the left. Just as Piero is apparently willing to jeopardize our reading of spatial relations to keep the surface coherence of his composition, so he is willing to risk the separation of the episodes in favour of ensuring the coherence of the picture across its whole width.

There is one more point that should be noted in regard to the overall organization of the fresco cycle: Piero has made the height of the upper registers, and the heights of standing figures in the pictures they contain, progressively a little larger than that of the lower ones. (The height of the pictures in the lowest registers is 3.29 metres, in the middle registers 3.58 metres, and in the lunetttes 3.90 metres.) That is, he has made allowance for the pictures

24 These pilasters could be described as echoing the architecture in Masaccio's *Trinity* (Fig. 2.7), but I am inclined to think that the latter was painted later. The perspective of the *Trinity*, in which the chief figures are seen from below, is more daring than that of any of the scenes in the Brancacci Chapel.

25 For juxtapositions, see Lavin, *Piero della Francesca: San Francesco, Arezzo* (full ref. note 10), pp.23 and 32–3.

higher on the wall being at a greater distance from the eye of the viewer.[26] It is clear that Piero has organized *The Story of the True Cross* as a complete system, rather than treating it as a series of quasi-independent pictures. In consequence, some of the pictures, such as *The Burial of the Wood* and the *Annunciation* work much better in context than they do in isolation. We should perhaps bear this in mind when looking at other planned assemblages of pictures, such as the panels of the *Sant'Antonio Altarpiece*. However, before turning to this altarpiece, we shall examine some more of Piero's frescos, since it is among them that we may most reasonably hope to find relationships with *The Story of the True Cross* – if only because it is rather likely that at least some of them were commissioned by people who had seen Piero's work in San Francesco.

Frescos in frames

Piero painted a number of frescos in Arezzo, Borgo San Sepolcro and places nearby. Almost all of those that survive incorporate a painted frame. The most notable exception is the *Madonna del Parto* (Fig. 5.31), which seems to have entirely filled its wall and therefore did not need to be explicitly cut off from its surroundings. The commission is possibly explained by the fact that the family of Piero's mother, Romana, came from Monterchi, but there is no specific reason to associate its date with that of Romana's death, 6 November 1469, since when the picture was painted the church in question, Santa Maria a Momentana (or della Selva), was not yet the cemetery chapel. So, from the point of view of dating, this picture has no special status within the group of works Piero painted in the region of his birthplace. Suggestions for dating it have consequently been based upon stylistic similarities to some of the frescos in San Francesco.[27] Such comparison necessarily refers only to specific parts of the fresco cycle and relies upon an examination of the actual execution. There is general agreement among scholars that broader treatment is more characteristic of the later phase of Piero's work than of the earlier one – which leads to general agreement that the *Madonna del Parto* belongs with the earlier part of Piero's work in San Francesco, before his visit to Rome in 1458–9.

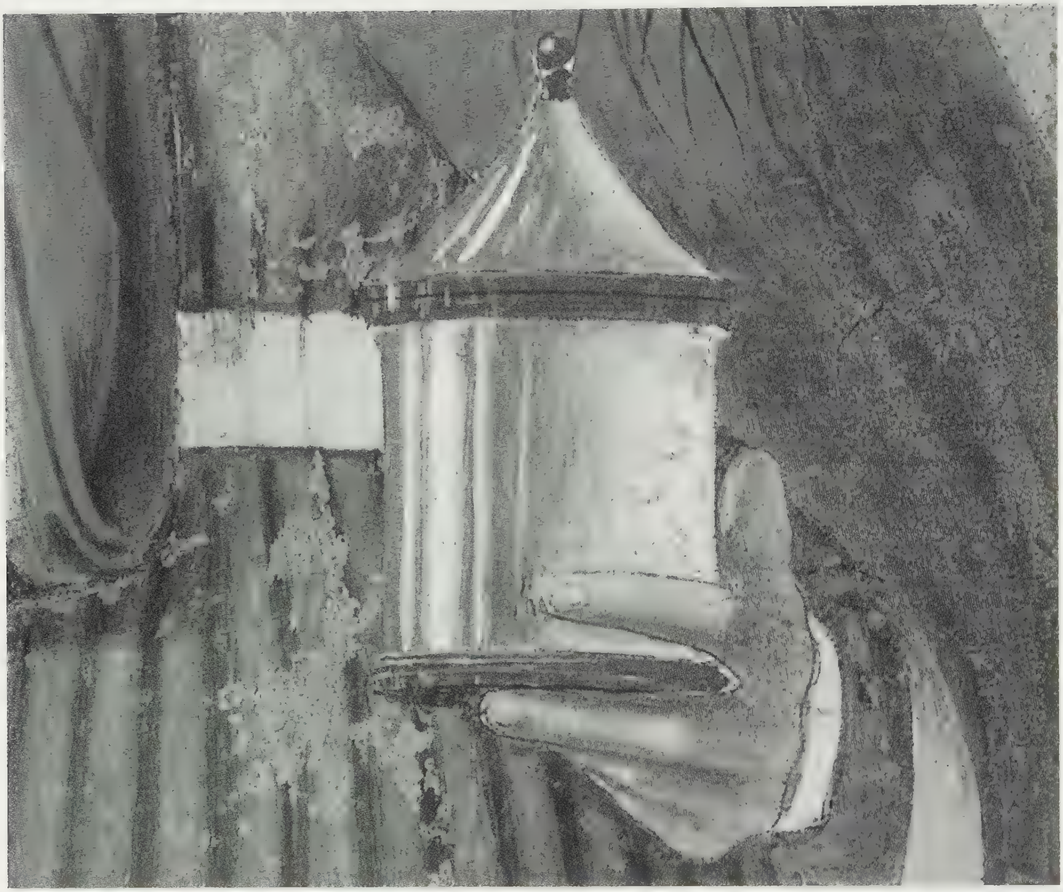
The fresco of *St Mary Magdalene* in Arezzo cathedral, shown in Figure 6.20, is generally dated to some time after Piero's return from Rome. It shows the saint standing behind a classical archway, some of which has now been lost.[28] The archway opens onto an outside world, since above the low marble wall, set some way behind the figure, we can see the sky. However, the figure is not backlit. The lighting, which is fairly strong, comes from our side of the arch and from above left. The symmetry of our view of the underside of the arch indicates that the ideal viewpoint is on the central axis of the arch, which was presumably originally also the central axis of the fresco. The tiny orthogonal lines supplied in the capitals of the pilasters can apparently be extended to meet this central line at a point just

26 Such allowance is prescribed by Albrecht Dürer (1471–1528) in his *Underweysung der Messung mit dem Zirkel und Richtscheyt* (Nuremberg, 1525), Book 3, sig. Ki verso; see also J. V. Field, *The Invention of Infinity: Mathematics and Art in the Renaissance*, Oxford: Oxford University Press, 1997, p.117–18.

27 See, for example, Antonio Paolucci, *Piero della Francesca. Catalogo completo dei dipinti*, Florence: Cantini, 1990, p.94.

28 The elaborate marble tomb of Bishop Guido Tarlati (fourteenth century) was moved to a new position, cutting into the left part of the fresco, in 1783. Further loss at the lower right was caused by the installation of a basin for holy water in the 1820s; see Lightbown, *Piero della Francesca* (full ref. note 10), p.183.

6.20 (*facing page*) Piero della Francesca (*c.*1412–1492), *St Mary Magdalene*, fresco, 190 × 80 cm, cathedral, Arezzo.



6.21 Piero della Francesca (*c.*1412–1492), *St Mary Magdalene*, detail of ointment jar, fresco, cathedral, Arezzo.

below the saint's right hand, which holds the wide and rather angular folds of her cloak. Thus the ideal viewpoint, while still too high for an observer standing on the floor of the church, is nevertheless low in the picture and accordingly gives the impression that we are looking up at the figure.

Given the prominence of the folds in the cloak, one could argue that this point of concurrence has a certain significance in regard to the overall composition in the plane as well as for reading the perspective of the architectural setting, but it has no clear connection with the religious content of the picture. The saint's identifying attribute is the jar of unguent held in her left hand (Fig. 6.21). This is shown as having a simple geometrical shape: a cylinder with a pointed, almost conical, lid. The material of the jar seems to be translucent, since we see a small amount of green, which may be the colour of the robe showing through. In any case, the jar is highly polished. The bands of highlights parallel to its vertical edge are pure white, neat but vivid rather than subtle when seen in detail. They are much cruder than the fine white lines that constitute Piero's Northern minimalist treatment of the crystal stem of the crozier in the figure of St Augustine from the *Sant'Agostino Altarpiece* (Museu Nacional de Arte Antiga, Lisbon) (Figs 6.22 and 6.23).

6.22 (*facing page*) Piero della Francesca (*c.*1412–1492), *St Augustine*, panel from an altarpiece for Sant'Agostino, Borgo San Sepolcro, 1454–69, tempera and oil on panel, painted surface 132 × 56.5 cm, Museu Nacional de Arte Antiga, Lisbon.

Unlike much of Piero's work, including some of the frescos in San Francesco, *St Mary Magdalene* shows a certain theatricality. That is, the picture seems to have been painted in full awareness that it can be seen only at a certain distance, or perhaps that it is unlikely to be seen in a bright light. In any case, the balance of real and fictive lighting across the fresco has certainly been disturbed by the loss of gold. The halo is now visible as a pale oval.[29] This suggests it was once shown as solid gold, like the halo of the Virgin in the *Annunciation* in San Francesco, rather than as resembling the more discreet discs of rays in, for instance, the *Baptism* (Fig. 4.2). There was certainly an *a secco* addition, and almost certainly some gold, on Mary Magdalene's belt, which is now colourless. On the upper left, some of the green of the tunic has been allowed to stray into the area left blank for the belt (Fig. 6.21). Such untidiness is clearly inadmissible and the stray patch of green pigment must originally have been covered in some way. It is thus probable that the fictive gold of the mounts of the jar was, as it were, in direct competition with real gold in the belt. The vivid colour of the real gold would no doubt have lessened the prominence that the bright highlights on its cylindrical body now give to the jar. It seems possible that there was also some gold in the saint's hair, counterbalancing the shine of the pale lining of her cloak.

29 The shape may be an ellipse, as mathematics would decree for a circle seen at an angle, but on this scale it would be impossible to measure the differences in shape between an ellipse and a four-centred oval of the kind that is described in Sebastiano Serlio (1475–1554), *Libro di Geometria e di Prospettiva*, Paris, 1545. Serlio's recipe is undoubtedly a traditional one, and may well go back to Roman times; see Vladimiro Valerio, 'Sul disegno e sulla forma degli anfiteatri', *Disegnare Idee Immagini* (Rivista semestrale del Dipartimento di Rappresentazione e Rilievo, Università degli Studi di Roma 'La Sapienza'), Anno 4, no. 6, 1993, pp.25–34. As an exercise in mathematics, drawing the oval would probably be easier than drawing the ellipse. In any case, Piero was no doubt capable of drawing a suitable curve freehand.

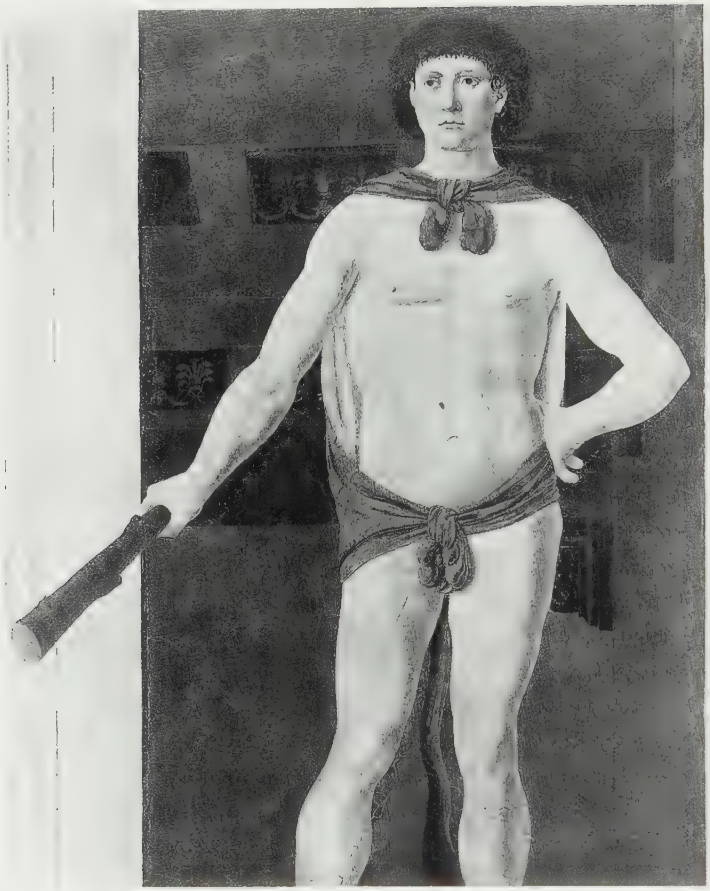
6.23 Piero della Francesca (*c*.1412–1492), *St Augustine*, detail of the stem of the crozier, from a panel from an altarpiece for Sant'Agostino, Borgo San Sepolcro, 1454–69, Museu Nacional de Arte Antiga, Lisbon. For the complete panel see Fig. 6.22.

6.24 Piero della Francesca (*c.*1412–1492), *St Julian(?)*, fresco, 135 × 105 cm, from the Presbytery of the church of Sant' Agostino, Borgo San Sepolcro, now in the Museo Civico, Sansepolcro.

6.25 Piero della Francesca (*c.*1412–1492), *St Louis of Toulouse*, fresco, 123 × 90 cm, from the Palazzo del Capitano, Borgo San Sepolcro, now in the Museo Civico, Sansepolcro.

Among the isolated figures that survive from this group of frescos (a grouping that is, of course, entirely the historian's creation) *St Mary Magdalene* is unique in being shown in a setting that is essentially open air. The fragment of *St Julian(?)* (Museo Civico, Sansepolcro) (Fig. 6.24), shows him against what seems to be a panel of black marble, resembling that of the revetment in the Rimini fresco; the now badly damaged *St Louis of Toulouse* (Museo Civico, Sansepolcro) (Fig. 6.25), which was originally in a niche, has a similarly classicizing background.[30] So too does the *Hercules* (Isabella Stewart Gardner Museum, Boston, Massachusetts) (Fig. 6.26), which is the sole surviving example of Piero's work in what one may regard as a completely secular scheme of decoration, since the *Resurrection* fresco (Museo Civico, Sansepolcro), although it was painted to be seen in a secular context, can hardly be considered secular in itself.

30 On the condition, ascription and dating of this work, see Lightbown, *Piero della Francesca* (full ref. note 10), pp.203–5.



6.26 Piero della Francesca (*c*.1412–1492), *Hercules*, fresco, 151 × 126 cm, from a private house, Borgo San Sepolcro, now in the Isabella Stewart Gardner Museum, Boston (Massachusetts).

Hercules appears to be standing in an arch opening onto an interior, a setting probably, like the figure itself, related to those of the series of famous men and women painted by Andrea del Castagno around 1447–51 in the Villa Carducci, Legnaia (now in the Galleria degli Uffizi, Florence). Like some of Castagno's figures, Piero's *Hercules* is brandishing a weapon, in this case a club, which comes out beyond the fictive frame. The gesture of the right arm is rather close to that of the guard on the right in front of the tent in *The Dream of Constantine* (Fig. 6.9). The hand-on-hip gesture of the left hand, which seems excessively relaxed in comparison, is, moreover, disconcertingly close to the gesture of the foreground lady in green in the scene of the Queen of Sheba being received by Solomon (Fig. 6.12), though the figures are seen from different angles and there is, accordingly, no question of the simple reuse of a drawing. Overall, Hercules' slightly self-conscious, if heroic, stance suggests the use of a study from a live model, though the musculature is no doubt indebted, directly or indirectly, to classical sculpture. Indeed, the rather coarse features may perhaps owe something to a late Roman portrait of a pugilist. There is, however, a profoundly non-classical look to the prissy way the knotted back paws of the lion skin do the office usually assigned to a fig leaf. This draped lion skin of course makes for a neater outline in what is an essentially static figure.

Hercules also provides us with yet another example of non-contrast: the lion-skin cloak that identifies the figure as Hercules has somehow acquired what seems to be a white silk lining, whose colour blends into that of the exposed torso, giving a smooth edge to the figure, rather than a sinuous one. This adds to the static, posed, effect of the picture – which invites comparison with the *St Sebastian* from the *Misericordia Altarpiece* (Fig. 3.9). Perhaps Piero reused some studies he had made earlier for the sacred figure, but strove to interpret them in a way that fitted his view of a classical subject in the light of things he had seen when visiting Rome? Since it seems he did not necessarily react quickly in registering in his own works what he had seen in those of others, it is impossible to suggest a date at which he might have seen Andrea del Castagno's frescos in Legnaia, near Florence, but it does seem highly likely that he had seen them, or at least knew them indirectly.

Piero's *Resurrection* (Museo Civico, Sansepolcro) (Fig. 6.27) was considered in his lifetime to be his best work, and this opinion was shared by Vasari. Piero was no doubt put on his mettle by receiving the commission, which came from the city fathers of his native town and was for a picture in the room in which they held their meetings, a place of great civic importance. There is, however, nothing specifically civic about the painting. We have no view of Borgo, but simply the religious scene associated with the city's name. It seems clear, not only from the character of the picture itself but also from Piero's provision of a weighty classical frame, that the fresco was not intended as part of a decorative scheme, but was designed to be seen in isolation. There is no secure external evidence for a precise dating of the picture, but we may probably assume the applicability of the rule that no man is a hero to his own countrymen and deduce that the commission would have come only after Piero had acquired extra prestige by having worked for Pope Pius II in Rome. A reference in 1480 to repairs to the wall 'on which Piero has painted' establishes that the picture was then complete.[31]

The scene is lit from the upper left, presumably in accordance with the natural light on the wall, about which there is a modicum of doubt because we know that the picture has been moved. The doubt must also extend to the exact height of the picture's position on the wall, though it is surely safe to assume that the fresco was painted above the chief officer's seat, in the centre, and at a height roughly similar to that at which it is now seen. The view of the picture is undoubtedly intended to be from below, so, if we are to judge by the Arezzo fresco cycle, we might expect that the ideal viewpoint would be low in the picture. The sprawling soldiers do indeed give the impression that we are looking up at them. This qualitative assessment is confirmed by a tiny piece of geometrical evidence. The bases of the columns that frame the scene not only look as if we were looking up at them but also include two very short orthogonals, namely the lines of their upper edges. These short representations of orthogonals can plausibly, but inevitably also with some uncertainty, be extended to meet at a point on the central vertical of the picture and about halfway up the now almost vanished inscription that ran along the frieze below the frame. This is indeed an appropriate point on which to fix our attention. Unlike that of the ideal observer, our glance towards this point will not be at right angles, but in this respect also we have a situation like that for viewing the scenes of *The Story of the True Cross* in San Francesco.

31 See Lightbown, *Piero della Francesca* (full ref. note 10), p.198. The document is not included in Battisti.

6.27 Piero della Francesca (*c.*1412–1492), *The Resurrection*, tempera and fresco, 225 × 200 cm, Museo Civico, Sansepolcro. See also frontispiece.

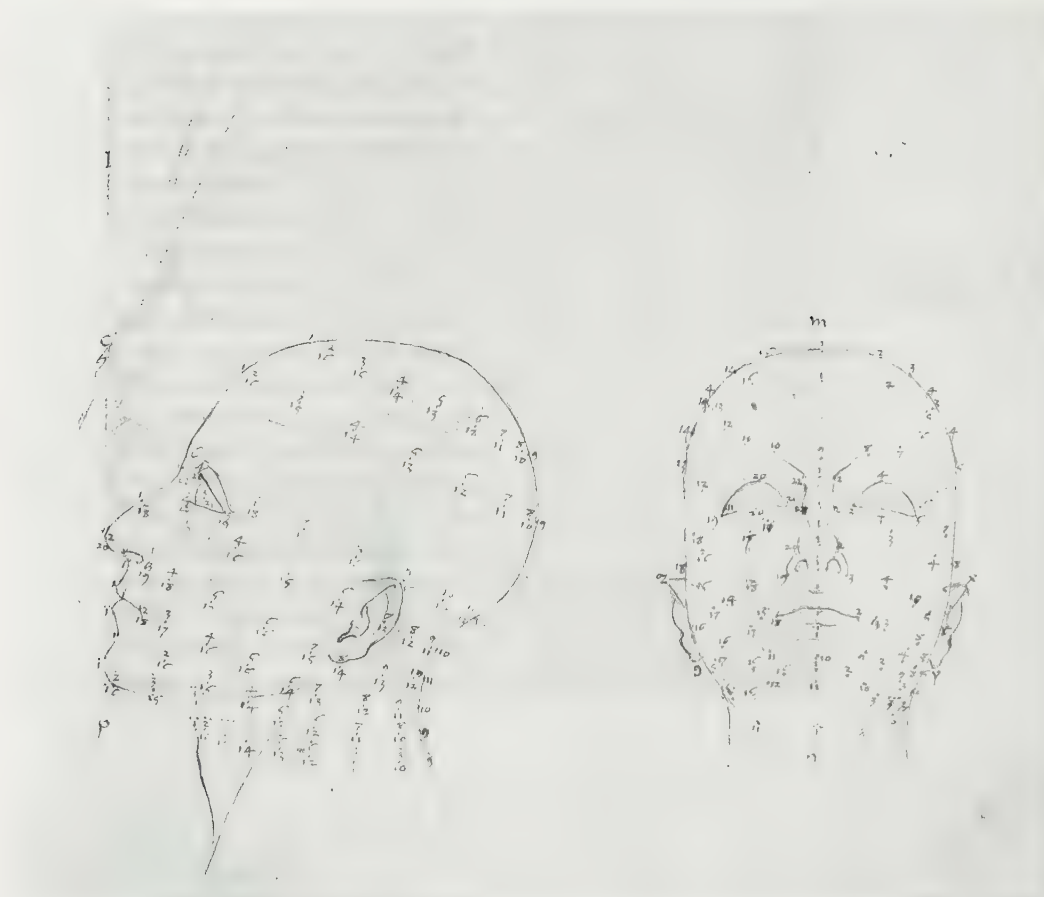
Since the orthogonals are so short, they surely cannot play an important part in our reading of what we see. It seems highly likely that our sense of depth is largely mediated by the figures of the soldiers. It also seems likely – and further arguments for this will be presented below – that Piero has made mathematically correct perspective images of the four figures, or at the least of their heads, so that the figures of the soldiers correspond to this low viewpoint at the level of the frieze. However, this cannot be the height of the ideal viewpoint for the whole picture, because we see the upper surface of Christ's foot. It rests on the upper edge of the tomb, and we seem to be looking at it from about the eye height of the sleeping soldier whose head is seen partly against the drapery falling from Christ's left arm. It is conceivable that this new level could be the ideal eye level for the rest of the picture: we can, for instance, see some of what seems to be the underside of Christ's right arm.

The landscape behind Christ is hilly. We cannot tell quite how it slopes, so it gives no precise indications of how we are looking at it. The effect is simply that we are looking uphill, and that the scene is wintry on the left but coming to life again on the right, the direction towards which Christ's triumphant flag is blowing. It has been suggested that the trees on the left have lost their leaves simply because the leaves were added *a secco*, in which case the leaves on the trees on the right were presumably painted in a different pigment.[32] As the restorers found, Piero used a wide variety of pigments in *The Story of the True Cross*, so there would be nothing incongruous in his having used two different pigments here. On the other hand, the symbolism seems too apt to be due to chance. The flag is, however, much less prominent than in some other versions of the Resurrection, curving into a narrow shape that seems to be designed not to distract attention from the figure of Christ. This landscape, and the dawn sky above it, are naturalistic, but they also carry a huge freight of symbolic meaning. There is, however, no sign that Piero let such meanings interfere with his normal methods of conveying a sense of the spatial arrangement of hills and trees. For instance, on the left, the height of the trunk of the nearer tree, measured to the fork, is about twice the corresponding measure on the tree further up the hill behind it. Nevertheless, the sleeping soldiers, the tomb and Christ himself have no precise connection with the background landscape. As in the portraits of Battista Sforza and Federigo da Montefeltro (Figs 4.11 and 4.12) and the *Madonna di Senigallia* (Fig. 4.15), foreground and background are disjunct.

In the *Resurrection* there is also some disjunction between the foreground figures themselves. The viewpoint that is apparently appropriate for the soldiers is apparently not appropriate for Christ. In a narrative or theological sense that is no doubt entirely reasonable, and Piero has made it look reasonable for the viewer. All the same, there is undoubtedly a departure from the naturalistic construction of spatial relationships that we expect in Piero's paintings. A further glance at the figure of Christ suggests we are really looking at Him straight on, in some vague sense that I am not tempted to unpack. Surely if our eye were at the level of that of the sleeping soldier we should see Christ's knee rather higher up than we do? The beard hides the lines of His chin, but the drawing of the features indicates that the face is not seen at a steep angle. This ambiguity, or at least absence of precision, in our viewpoint for Christ gives the figure an unreal quality that is entirely at variance with its otherwise impeccable solidity, conveyed by the modelling and the cast shadows of the nose and the forearm. Piero seems to have played one part of *perspectiva* off against another part to give us a sense that we are seeing a miracle.

This is not a matter of Christ's being regarded as not subject to the laws of optics, for it is clear that His body is indeed subject to them: it shows the effects of lighting and it casts shadows. In addition, the left foot is definitely seen in what one can most reasonably describe as naturalistic perspective. The overall effect is, to a degree, naturalistic. On the technical level, however, Piero is playing tricks on our visual system. On the theological level one can no doubt make out a good case for taking Christ, risen from the dead, as belonging to an order of reality different from that of the everyday world inhabited by the soldiers and by

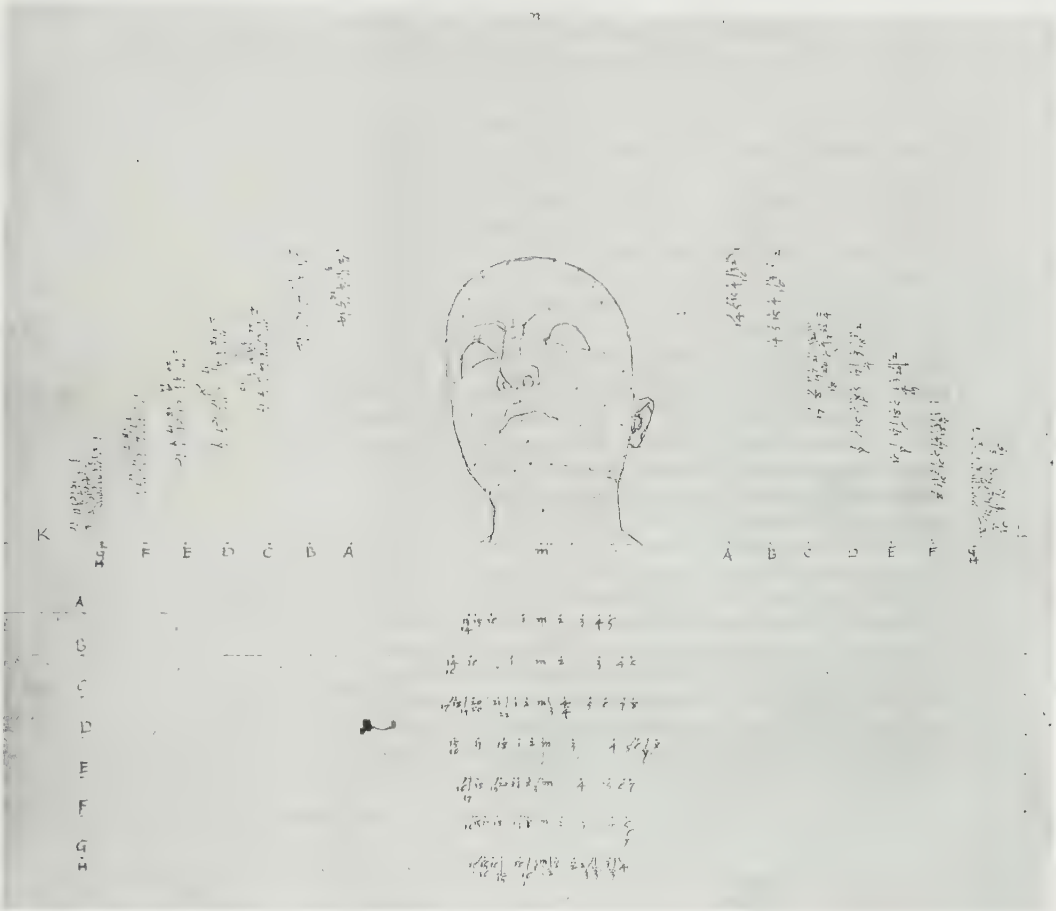
32 On the composition generally, see M. Baxandall, 'Piero della Francesca's *The Resurrection of Christ*', in *Words for Pictures: Seven Papers on Renaissance Art and Criticism*, New Haven and London: Yale University Press, 2003, pp.117–64.



6.28a Drawing a human head in perspective, from Piero della Francesca, *De prospectiva pingendi*, Book 3, Proposition 8, Parma MS: first stage, for the head thrown back, p.69 recto.

the spectator. It seems that in Piero's painting this theological truth has found a rather literal-minded interpretation, or merely a literal-minded counterpart, which – perhaps contrary to the expectations of those accustomed to non-naturalistic art – contrives to exercise great power. The partial unnaturalness of Christ is noticeable and impressive because we see it in juxtaposition to the ordinary world of the soldiers. It is thus crucial to our understanding of the picture that we take the soldiers as part of a world that, if not exactly our own, is at least a recognizable extension of it.

The fresco shows extensive evidence of the transfer of preliminary drawings, and reference to the illustrations provided for *De prospectiva pingendi*, Book 3, Proposition 8 (Fig. 6.28), shows that these particular examples of human heads bear rather more than a mere family resemblance to the heads that appear in the fresco. It is probable that computer analysis of the images in the painting could eventually tell us whether the heads of the soldiers in the *Resurrection*, and indeed those of the figures in the *Story of the True Cross*, might all be derived from a small set of drawings of heads like those shown in *De prospectiva pingendi*



6.28b Drawing a human head in perspective, from Piero della Francesca, *De prospectiva pingendi*, Book 3, Proposition 8, Parma MS: final stage, for the head thrown back and to one side, p.76 verso.

(Fig. 5.25). In any case, since there can be no doubt that Piero did use preliminary drawings, and no reasonable doubt that he recognized it was important that the soldiers should be convincingly naturalistic, it seems that we have here some very strong evidence that Piero did, at least on occasion, really carry out the long construction procedures that he describes in detail in his treatise. It is extremely difficult to believe that he deliberately forbore to use a technique that would give optically reliable results for these important components of his picture. Moreover, it was his own choice of the overall design (or his acceptance of the specification supplied by others) that had given such importance to the soldiers, and it seems natural to assume that such a decision was made in the knowledge that he would be able to solve the problems they presented.

It is chiefly the soldiers' heads that attract our attention in the figures themselves. The three-dimensional relationships of the limbs are readable, but there are few strong colour contrasts, and there is a complete non-contrast of brown armour against brown armour where the leg of the soldier whose head is at the far right is seen against the arm of the man whose head is second from the left. The colour of the armour is, moreover, close to that of the front of the tomb behind it. These non-contrasts ensure that we have rather little to notice in the area directly below the figure of Christ, except the shadow cast on the tomb by the head of one of the sleeping soldiers. There is, of course, nothing unusual in the fact that the three-dimensional structure of the *Resurrection* is established by means that, while they may have been mathematical, do not provide the viewer with substantial clues for a precise mathematical reconstruction. The lack of the sort of rectilinear elements that give a clear indication of the correct viewpoint is entirely normal in Piero's output. In the case of the *Resurrection* there may, however, have been a specific reason for avoiding such elements. The picture was liable to be viewed by councillors seated in positions that offered a far from ideal view of it. Accordingly, Piero may have deliberately preferred to rely upon methods that gave the least obvious clues that one was not in an ideal position. That is, he designed the picture so that reading its composition in three dimensions would depend on things for which the illusion of solidity would be robust.

The *Sant'Antonio Altarpiece*

The altarpiece that Piero painted for the convent of St Anthony of Padua in Perugia (Fig. 6.29) is mentioned by Vasari and remained in the possession of the convent until 1810. It is thus, in a sense, a well-documented piece. However, it was only in the early 1990s that documentation emerged that established the date at which the altarpiece was painted, by providing evidence that the work was complete by 21 June 1468. The commission probably dates from 1467.[33] The lower part of the present frame is a reconstruction, and the upper part has almost certainly been modified. In particular, it is likely that the *Annunciation* originally had a more elaborate edging, perhaps like that of the triptych to which Piero's *Baptism* once belonged (Fig. 4.9).[34] The present framing of the lunette, which would no doubt have seemed unacceptably austere in the 1460s, tends to exaggerate the already considerable difference in style between the central panels and the uppermost one.

The three panels in the central row form a single picture, since the corners of the red dais that supports the Madonna's throne protrude into the side panels. This dais, together with the architectural throne, establishes that the white marble slab is horizontal and gives it a depth. The lines that represent receding orthogonal edges of the dais can be extended to meet one another at a point close to the central vertical of the panel, about where the Child's wrist rests against His leg. The eye height of the ideal viewer is thus well below that of any of the figures in this set of panels, though somewhat above a realistic level for an actual viewer in the church, and possibly correct for the officiant. That is, Piero is adopting his usual solution for the problem of the eye height in constructing perspective pictures that

33 For a discussion of various scholars' various datings of the work, and an account of the documentary evidence establishing a *terminus ante quem* of 21 June 1468, see Francesco Federico Mancini, '"Depingi ac fabricari fecerunt quamdam tabulam". Un punto fermo per la chronologia del polittico di Perugia', in *Piero della Francesca. Il Polittico di Sant'Antonio*, ed. Vittoria Garibaldi, Perugia: Electa Editori, 1993, pp.65–72.

34 See Christa Gardner von Teuffel, 'La collocazione originale e la struttura del polittico', in *Piero della Francesca. Il Polittico di Sant'Antonio*, ed. Vittoria Garibaldi, Perugia: Electa Editori, 1993, pp.89–92.

6.29 (*facing page*) Piero della Francesca (*c.*1412–1492), the *Sant'Antonio Altarpiece*, *c.*1467–8, tempera and oil on panel, 338 × 230 cm, Galleria Nazionale dell'Umbria, Perugia. Central part: *Madonna and Child Enthroned with* [from the left] *St Anthony of Padua and St John the Baptist, St Francis and St Elizabeth of Hungary*. In the lunette: *The Annunciation*. In roundels in the minor predella: *Sts Clare and Agatha*. In the narrative predella: (from the left) *St Anthony Brings a Baby Back to Life, St Francis Receives the Stigmata, St Elizabeth Rescues a Boy from a Well*. For the *Madonna and Child Enthroned*, see Fig. 5.27 above.

6.30 Piero della Francesca (*c.*1412–1492), the *Sant'Antonio Altarpiece*, *c.*1467–8, tempera and oil on panel, 124 × 62 cm, Galleria Nazionale dell'Umbria, Perugia, detail showing the figures of St Anthony of Padua and St John the Baptist. See also Fig. 6.29.

6.31 Piero della Francesca (*c.*1412–1492), *St Anthony Brings a Baby Back to Life*, from the narrative predella of the *Sant'Antonio Altarpiece*, *c.*1467–8, tempera and oil on panel, 36.5 × 49 cm, Galleria Nazionale dell'Umbria, Perugia. See also Fig. 6.29.

are to be seen from below. However, the slight depth of the marble floor is of no significance in the scene itself, except in allowing us to feel comfortable with the real volumes that Piero has given to the saints. Great care has been taken over highlights and shadows in these figures, and, in a bizarre touch, the haloes have not only been rendered as if they were solid discs but each is also shown as reflecting the top part of the head of the saint beneath it (Fig. 6.30). The haloes of the saints in the minor predella are also reflective, as is that of the Virgin in the central panel. Christ's halo, however, merely shows a cross, red and in perspective (Fig. 6.29). The golden background, seen above the white marble floor, is shown as if it were brocade, making it a kind of extended cloth of honour. Thus the panels of the *sacra conversazione*, while highly conventional in their overall effect, can nevertheless be regarded as a little more naturalistic than the corresponding parts of the *Misericordia Altarpiece* (Fig. 3.7).

The *sacra conversazione* seems to be taking place in a pageant version of Heaven. The scenes of the narrative predella and of the lunette take place on Earth. This difference allows Piero to adopt a style rather like that of Domenico Veneziano's predella for the *St Lucy Altarpiece* (Figs 3.11–3.13).[35] In *St Anthony Brings a Baby Back to Life* (Fig. 6.31), we

35 Piero may have seen this altarpiece in Florence on the same journey that took him to see Andrea del Castagno's frescos at nearby Legnaia. Unfortunately we have no documentary evidence for any such journey. The road from Borgo San Sepolcro or Arezzo to Rome may take one via Perugia, but hardly via Florence.

6.32 Piero della Francesca (*c.*1412–1492), *St Francis Receives the Stigmata*, from the narrative predella of the *Sant'Antonio Altarpiece*, *c.*1467–8, tempera and oil on panel, 36.5 × 51.5 cm, Galleria Nazionale dell'Umbria, Perugia. See also Fig. 6.29.

have two fairly short lines and one longer one that look as if they are orthogonals: the line of the edge of the chimney breast on the far left, the line in which the wall on the far left meets the floor, and the line in which the right wall meets the floor. When extended, the images of these three lines meet at a point that lies on the lower edge of the architrave of the alcove, directly above the head of the friar standing behind the kneeling saint. This high position for the point that is directly opposite the eye of the ideal viewer is presumably chosen to take account of the fact that Piero expects the observer (perhaps the officiant?) to be looking down at the picture. The fact that the point is near the right edge presumably indicates that the observer is expected to be to the right of the picture, which is entirely reasonable. However, the fact that the point is so near the edge of the picture may perhaps be regarded as evidence for a reconstruction of the predella in which vertical framing elements are narrower, and this panel was directly under the figure of St Anthony in the *sacra conversazione*, at the extreme left, leaving room on its right to fit in a predella panel, now missing, for John the Baptist.[36]

36 See Piero Bianconi, *Tutta la pittura di Piero della Francesca*, Milan: Rizzoli, 1957, pp.60–2. Bianconi's diagram of the arrangement he proposes is reproduced in Vittoria Garibaldi, 'Introduzione', in *Piero della Francesca. Il Polittico di Sant'Antonio*, ed. Vittoria Garibaldi, Perugia: Electa Editori, 1993, pp.19–44, see p.22 and fig. 5 (p.26).

6.33 Piero della Francesca (*c.*1412–1492), *St Elizabeth Rescues a Boy from a Well*, from the narrative predella of the *Sant'Antonio Altarpiece*, *c.*1467–8, tempera and oil on panel, 36.5 × 49 cm, Galleria Nazionale dell'Umbria, Perugia. See also Fig. 6.29.

Neither of the two other preserved predella panels, *St Francis Receives the Stigmata* and *Saint Elizabeth Rescues a Boy from a Well* (Figs 6.32 and 6.33) provides evidence for finding the precise position of the foot of the perpendicular from the eye of the ideal viewer. The fact that the vista of landscape in the St Francis picture stretches away to the right can be read as merely descriptive of the landscape, and in the plane of the picture has the effect of aligning the direction of the vista with the most important axis – also not precisely defined in three-dimensional terms – that connecting St Francis with the glowing figure of Christ on the Cross. In the foreground of the scene of St Francis, spatial relationships are defined by the lighting, which comes from Christ. The direction of this light is contrary to that of the illumination shown in all the other panels, which presumably corresponds to the natural lighting in the church. The background landscape is much more faintly lit. A sense of its depth is conveyed by Piero's accustomed use of hills and trees. The darkness makes the distance relatively unobtrusive but the depth is certainly greater than in either of the other predella panels and corresponds to the greater sense of depth in the middle panel in the main register above and, more forcefully, in the *Annunciation* in the lunette. This is not necessarily an argument for the St Francis scene truly belonging in the centre of the predella. The hypothesized missing panel of St John the Baptist could well have included a landscape of suitably complementary design.

In the scene of St Elizabeth (Fig. 6.33), small visual clues as to the ideal position of the eye are provided by the right part of the door in the archway at the left. As the panelling is presumably rectangular, and we see the panels as rectangular as far as one can judge, we are presumably looking straight at the door. However, the top edge of the door, which seems to be parallel to the top edge of the arch, is not exactly parallel to the horizontal edges of the panel, so we cannot be looking exactly straight on at the door. That appears, however, to be the only clue we have. What it suggests is made a little more plausible by the fact that the ideal eye would then be directed to a point aligned vertically with the eye of the kneeling woman. The symmetrical shape of the well-head – which should in principle be an ellipse[37] – is no indication of its relationship to the ideal viewpoint. Except under weird or extreme conditions, perspective views of circles are symmetrical, however off-axis the position of the circle.[38] The shape of the small piazza in which the scene takes place is not precisely defined, and matters are further confused by the small size of the figure of St Elizabeth, appearing at the top right. It is, of course, possible that some further definition was provided by whatever was shown near the lower rim of the picture, where there has been fairly extensive loss of paint. However, it seems unlikely that this scene ever had such a clear three-dimensional structure as that of *St Anthony Brings a Baby Back to Life*. Like the fresco showing Judas being taken out of the well in *The Story of the True Cross* (Fig. 6.10), the three-dimensional layout of the scene of St Elizabeth is established largely by our reading the figures as grouped round the simple geometrical shape of the well.

In contrast, the use of perspective in *The Annunciation* in the lunette (Fig. 6.34) is explicit to the point of being blatant. The foreground part of the architectural setting presents us with a pavement that seems to be divided into squares in the style of the foreground of the *Flagellation* (Fig. 5.28). The restoration carried out in the 1980s has led to the discovery of a number of incised lines that must certainly represent orthogonals, for instance along the top of the wall that supports the row of columns on the left, and in the corresponding position on the right.[39] These lines, which are all reasonably long, can be extended to meet at a point that lies on the vertical axis of the lunette and is at about the level of the angel's hand. Thus the height of the eye of the ideal viewer is lower than that of either of the figures in the picture, though it is, of course, unattainably high for any actual viewer. The eye height is about a third of the total height of the lunette, and is thus, in relative terms, slightly lower in the picture than the eye height was in the middle register.

In *The Annunciation*, the foreground squares on the right are edged with incised lines, and the ones on the left have been drawn in ink.[40] However, none of these squares is shown complete in the painting, and Piero's guide lines do not extend to invisible parts. So a little reconstruction has to be done to make some complete squares whose diagonals will meet the horizon (that is a horizontal at the ideal eye height) at points whose distance from the point of concurrence of the images of orthogonals will be the ideal viewing distance of the picture

37 However, see note 29.

38 This is not obvious, but the corresponding theorems are proved in the *Conica* of Apollonius of Perga (active *c.*230 B.C.), and were known to some professional or university mathematicians of Piero's time, since they are relevant to the construction of lines on plates for a planispheric astrolabe (then the commonest astronomical observing instrument).

39 See Sergio Fusetti and Paola Virilli, 'Il restauro', in *Piero della Francesca. Il Polittico di Sant'Antonio*, ed. Vittoria Garibaldi, Perugia: Electa Editori, 1993, pp.137–51, and especially the diagram (fig. 98) on p.141.

40 See Fusetti and Virilli, 'Il restauro' (full ref. note 39).

6.34 Piero della Francesca (*c*.1412–1492), *The Annunciation*, from the lunette of the *Sant'Antonio Altarpiece*, *c*.1467–1468, tempera and oil on panel, 122 × 194 cm, Galleria Nazionale dell'Umbria, Perugia. See also Fig. 6.29.

– if the picture is in correct perspective.[41] This time, the lines we are using are short, making squares that are small compared with the length that will be given to the extended versions of their diagonals. That is to say that the process is not going to yield very reliable results.

For what it is worth, my estimate is that the ideal viewing distance is about twice the width of the base of the lunette. This result enables us to draw a ground plan of the nearer part of the scene, but as we have no definite information about the spacing of the columns

41 Using this inverse version of the distance point construction does not assume or imply anything about how the perspective of the picture was constructed. The image concerned simply has to be in correct perspective, which is to say that one can find distance points for a photograph.

in the receding colonnade there is no way of reconstructing that part of the architectural setting, unless we make some subsidiary assumptions.[42] Piero has drawn himself a vertical (as an incised line) only for the first column on the angel's side. The painting and the lighting of the receding colonnade are wonderfully detailed, but the perspective construction seems not to have been. All the same, the sense of recession is powerful. This may be a deliberate attempt to compensate for the flattening effect that is introduced when one is forced to look up at the picture at a steep angle. (A similar consideration might explain the rather deep vista provided in *The Exaltation of the Cross* in San Francesco, see Figs 6.6 and 6.15.) This effect is much more noticeable in reality than in photographs in a book, but some sense of it can be obtained by tilting the page.[43]

In its present state, *The Annunciation*, with its deep perspective, has an unsettling effect on the appearance of the altarpiece as a whole. This is not in any way a criticism of the restoration work of the 1980s, which, following the widely accepted precepts of minimal intervention, clearly could not have made adjustments to remove the effects of such things as local changes in colour and texture due to the differential ageing of different areas of paint. Nor would it have been reasonable to make drastic alterations to remove the disturbing effect of the severity of the uppermost part of the frame. The *Sant'Antonio Altarpiece* in fact presents a suitable case for the use of computer simulations that could show the results of such reversals of the effects of time and chance. What remains is sufficient to show that Piero has imposed a certain degree of coordination upon the designs of the panels that make up the altarpiece. In doing this he has used the same means he employed in establishing relationships between the various scenes on each wall in the frescos of *The Story of the True Cross*. In the *Sant'Antonio Altarpiece* both the row of panels in the central register and the lunette have what Alberti would have called the 'centric point' of their perspective schemes on the central vertical axis of the complete altarpiece. This fact is perhaps trivial but it serves to emphasize the choice of making the middle register a single scene rather than treating it as three separate panels.

It may well have been the potential distraction of introducing additional vertical axes for individual pictures that Piero wished to avoid by his treatment of the scenes in the narrative predella. In many altarpieces, it is the predella panels that show the most adventurous use of perspective. In the *Sant'Antonio Altarpiece* we have the central predella panel – or one of the central predella panels, if we assume there never was a panel for St John the Baptist – namely *St Francis Receives the Stigmata* (Fig. 6.32), which has no sign of mathematical construction and thus no precise pointers to an ideal eye position, though there is a strong sense of landscape receding into the distance. In the two lateral panels, only the interior scene of St Anthony (Fig. 6.31) allows one to establish the exact position of the foot of the perpendicular from the ideal eye to the picture plane. That position turns out to be rather close to the right of the scene, implying that we are glancing in from that direction, and thus reading the scene

42 A complete reconstruction of the ground plan, plus angled views of the whole structure, is given in Martone, 'Piero della Francesca e la prospettiva dell'intelletto' (full ref. note 22). It is not clear to me what the computer was told. See comments on computer-assisted reconstructions in Chapter 5, pp.174–5.

43 If one stands at the 'correct' horizontal distance from the picture in the lunette of the *Sant'Antonio Altarpiece* – that is, twice the width of its base – one is indeed looking up at a steep angle, but this is probably irrelevant since Piero must have known by observation that in practice the 'correct' viewing distance acted more or less as a minimum. See Chapter 2.

from right to left, with the strong black–white contrasts in the covers of the baby in the cradle establishing that area as a centre for attention. The perspective of the outdoor town scene, showing the miraculous intervention of St Elizabeth (Fig. 6.33), is much vaguer, and serves mainly to establish that this time we are reading the picture in the more usual direction, namely from left to right. In each case the perspective has been designed to allow the observer to feel reasonably comfortable while viewing the picture from a position on the central vertical of the altarpiece, which is obviously the correct position for the main register and the lunette.

The ideal heights of the eye in various pictures have been chosen so as to establish that one is looking down at the narrative predella but up at the other parts. This is purely qualitative: establishing a single eye height would have meant arranging for images of orthogonals to converge towards points that lay outside the area of the picture – a scheme that Piero explicitly excludes in his perspective treatise.[44] The solution adopted in the *Sant' Antonio Altarpiece* is more or less the same as that adopted in *The Story of the True Cross* and other frescos, namely that when we are looking up the ideal eye height is lower than that of people in the picture and, conversely – a problem not encountered in the frescos – when we are looking down the eye height is above that of people in the picture.

The only picture in the *Sant'Antonio Altarpiece* that provides us with enough information to find the horizontal distance of the ideal eye from the picture is the *Annunciation*. Its correct viewing distance of about twice the width of the altarpiece does not seem plausible for the dais and architectural throne of the central *Madonna and Child*, or for the small interior in the *St Anthony* panel. As it is applied to the positions of the point directly opposite the eye of the ideal observer for each picture, Piero's coordination process seems to be designed, first, to emphasize the central vertical axis of the altarpiece as a whole and, second, to take account, in a qualitative way, of the different heights of the pictures in different registers. There does not appear to be any attempt to coordinate the ideal distance of the eye from the picture, perhaps because Piero has noticed that the eye is not very sensitive to the effects of being at an incorrect distance, provided only that it is not less than the ideal one. However, this ideal distance plays a part in the construction of the perspective within the picture. So by apparently ignoring the distance of the eye, Piero has applied his coordination process not to the full construction of the perspective of the picture, but only to the position of the point opposite the eye in the plane of the picture. That is, in planning the altarpiece as a whole he has been concerned with relationships in the plane, not in three dimensions. Unity is imposed upon the various scenes by their using the natural lighting of the church – though the night scene of St Francis, mainly illuminated by the vision of the crucified Christ, is inevitably more or less an exception to this rule. This method of imposing unity not only brings together the various scenes in the pictures but also links them with the rest of what is seen by the viewer in the church.

The *Montefeltro Altarpiece*

The *Montefeltro Altarpiece* (Galleria di Brera, Milan) (Fig. 6.35) is often associated with the church of San Bernardino, just outside Urbino, which was designed by Francesco di Giorgio

44 Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 13, final paragraph: Parma MS, p.6 verso; BL MS, p.7 recto; Piero ed. Nicco Fasola, pp.76–7; see Chapter 5 and Appendix 8.

6.35 Piero della Francesca (*c.*1412–1492), *The Virgin and Child Enthroned with Saints and Angels* (*The Montefeltro Altarpiece*), tempera and oil on panel, 251 × 172.5 (±0.5) cm, Galleria di Brera, Milan.

6.36 Ascribed to Federico Barocci (1532–1612), *Interior of San Bernardino, Urbino*, sixteenth century, pen and wash, 28.7 × 21.6 cm, Galleria degli Uffizi, Florence, 245 Ar.

Martini (1439–1501/2) as a mausoleum for Federigo da Montefeltro.[45] The presence of Federigo as sole donor is explained by the fact that his wife, Battista Sforza, who died in 1472, had chosen to be buried in the convent of Santa Chiara. The tall rectangular shape of Piero della Francesca's picture, which was unusual at the time, is sometimes attributed to the demands of the architectural setting. There is a sixteenth-century drawing that shows the painting in position on the high altar of the church, before this part was modified by the removal of the apse and the substitution of a rectangular choir (Fig. 6.36). This drawing does indeed show that the proportions of the picture are appropriate to the setting. However, it also shows the direction of the natural lighting, which, as can be seen from the shadow of the classicizing frame – perhaps designed by Francesco di Giorgio – is from the right. Piero's picture shows the main lighting, falling on the apse behind the figures, as coming from above left. The lighting in this setting in San Bernardino thus seems inappropriate.

The main altar of San Bernardino faces south-east, so in this position Piero's picture shows the light on the apse as coming from somewhere between the north and the north-west.[46]

45 On the possible connections of the picture with San Bernardino, see, for instance, Lightbown (full ref. note 10), pp.245–52.

46 For a plan of the church, with compass directions, see Howard Burns, 'San Bernardino a Urbino', in *Francesco di Giorgio Architect*, ed. Francesco Paolo Fiore and Manfredo Tafuri, Milan: Electa, 1993, pp.230–8 (p.230 for the plan).

As the source of the light, presumably the Sun, is high up, this is entirely absurd for a location in the northern hemisphere. Nor are matters much improved if we assume that Piero, not having advance knowledge of the orientation of Francesco di Giorgio's church, designed the picture for an altar facing east. In this situation, the light in the painting would be coming from between west and north-west. For the direction of the light in the picture to correspond to natural lighting coming from a reasonable position, roughly south, the spectator would have to be looking in a direction between north and north-east.[47] It is, of course, possible that Piero did not intend any such correspondence with natural lighting. Moreover, we may note that in the *Flagellation of Christ* (Fig. 5.28) and also in the *Madonna di Senigallia* (Fig. 4.15) sunlight is falling from the upper left. After all, not all pictures that, to the modern eye, seem to be religious in their import, were in reality intended to stand on an altar.

The *Montefeltro Altarpiece* is known as an altarpiece because it was indeed placed on the altar of San Bernardino, apparently when the church was consecrated – which was probably in 1491, but certainly before 1496.[48] However, the most weighty reason for connecting the picture with San Bernardino is that the church was intended as a mausoleum for Federigo da Montefeltro. The funerary character of the painting has been accepted by most scholars, but, as is usual with Piero's works, there is considerable disagreement about the date of execution, though the picture is generally believed to have been at least nearly finished before Federigo's death in 1484. It is also generally agreed that at some time after Piero had finished his work, modifications were made to Federigo's hands, to which an extra ring has been added, and to the Virgin's headdress, from which a jewel has been removed.[49] The recognition of these modifications very much weakens the argument for a relatively early date for the picture on the grounds that it does not show the insignia of various orders of chivalry (among them the Golden Fleece and the Garter), awarded to Federigo in August 1474. If the insignia had seemed important, or even appropriate for inclusion in the picture, they could surely have been added, and if they did not seem so, then their absence is not good evidence for the date at which the picture was painted.

It has been suggested that a structural modification was made to the work, removing the lowest section, in which case we would most likely be missing a piece of the foreground. This suggestion appears to be based upon a misunderstanding of the results of restoration work carried out in the 1980s.[50] There have been several suggestions that the picture was originally intended to be placed not over an altar but over Federigo's tomb.[51] The close inspection of the structure of the panel in the course of restoration procedures showed that

47 A fuller account of the lighting in this picture is given in Appendix 9.

48 See Burns, 'San Bernardino a Urbino' (full ref. note 46).

49 Both these modifications are discussed in greatest detail in Carlo Bertelli, *Piero della Francesca*, trans. E. Farelly, New Haven and London: Yale University Press, 1992.

50 For the suggestion that a piece of Piero's painting is now missing, see Bertelli, *Piero della Francesca* (full ref. note 49), especially p.212; and Carlo Bertelli, 'La pala di San Bernardino e il suo restauro', *Notizie da Palazzo Albani* 11, 1–2, 1982, pp.13–20. For detailed comments on the structure of the panel, which is made up of nine planks (Bertelli mistakenly supposed that only eight of these were present), see Filippo Trevisani, 'Struttura e pittura: i maestri legnaiuoli grossi e Piero della Francesca per la carpenteria della pala di San Bernardino', in *La pala di San Bernardino di Piero della Francesca. Nuovi Studi oltre il restauro* (Quaderni di Brera 9), ed. Emanuela Daffra and Filippo Trevisani, Florence: Centro Di, 1997, pp.31–83.

51 Some of these are mentioned by Burns in connection with plans for the tomb; see below and Burns, 'San Bernardino a Urbino' (full ref. note 46). See also Guido Ugolini, *La pala dei Montefeltro: una porta per il mausoleo dinastico di Federico*, Pesaro: Nobili, 1985.

the panel was almost certainly made as an independent structure, the use of tongue-and-groove joins, and metal inserts between the nine planks, being similar to what is found in other fifteenth-century pictures from Urbino, such as the portraits of famous men from Federigo da Montefeltro's *studiolo* (now in the Louvre) and the altarpiece of the *Communion of the Apostles* painted by Justus van Ghent (Joos van Wasserhove, active *c*.1460–80) (Galleria Nazionale delle Marche, Urbino). For Piero's panel, as for these others, the metal inserts provide rings that would allow the picture to be fixed securely to some kind of mounting.[52] It is thus a practical possibility that the panel was intended to form part of a stone tomb structure. And some real stonework surrounding the picture might have balanced the elaborate simulated stonework of the architecture that dominates the upper part of Piero's painting. At present the composition looks top-heavy – something it has in common with a *sacra conversazione* of rather similar format that we may regard as one of its descendants (see below), the San Giobbe altarpiece of Giovanni Bellini (1431–1516). Bellini's painting was certainly designed to have a stone frame and to stand on an altar. It is now displayed in the Accademia Gallery, Venice, without the frame, but with an accompanying photograph to show how much difference the frame makes. To return to the *Montefeltro Altarpiece*, it seems likely that the loss of the Virgin's jewel has seriously interfered with the balance of Piero's design, since it has removed a piece of attention-catching glitter from the middle of the composition.

The most reasonable explanation for the removal of the Virgin's jewel is that the Observant Franciscans of San Bernardino did not consider it fitting that the Queen of Heaven should wear such a thing. The Virgin of the *Montefeltro Altarpiece* is the most richly clothed of all Piero's figures of the Virgin, which is perhaps another indication that the picture was not originally intended for this church. As we can see from his other works, Piero did not need to show sumptuous clothing in order to give dignity to his figures. It is not known exactly when Federigo decided that he would be buried in San Bernardino, and there is evidence that his earlier plans included a funerary chapel inside the Ducal palace. This is mentioned by the court painter Giovanni Santi (1435/40–1494) in his poem about Federigo.[53] Santi, writing between 1482 and 1487, speaks of the construction of San Bernardino in the present tense, and it is not certain that plans for the church were in fact completed before Federigo's death.[54] This is, of course, not to say that Piero could not have known something of what Francesco di Giorgio was planning, but it does open up the possibility that Piero designed his funeral painting for Federigo for a location such as a chapel in the palace, rather than the Observant church, well designed though that proved to be as an art gallery, with its uncluttered lines, pale walls and bright lighting. Unfortunately, no designs for Federigo's tomb are known to survive and we do not even know where it was to be located within the church of San Bernardino. It may be that he was interred beneath the altar and that this accounts for the decision to place Piero's picture above it.

The spatial organization of the *Montefeltro Altarpiece* has been a subject of sufficient disagreement to lend weight to the claim that the arrangement is not easily legible. Colour changes due to ageing of the pigments may have contributed to the difficulty. We shall return

52 See Trevisani 'Struttura e pittura' (full ref. note 50).

53 See Burns, 'San Bernardino a Urbino' (full ref. note 46), p.233.

54 See Burns, 'San Bernardino a Urbino' (full ref. note 46), p.234.

to this problem of legibility later, since it is rather curious to encounter it in a picture that has what seems to be a well-defined perspective scheme. The perspective scheme is, however, visible almost exclusively in the architectural elements of the upper part of the picture. The only orthogonals in the lower part of the painting are provided by the pattern in the carpet over the dais under the Virgin's chair. These lines are all unhelpfully short. In comparison, the orthogonals in the architecture are relatively substantial, particularly since the horizontal mouldings at the base of the vaulting on either side of the arches opening out to left and right are clearly meant to be aligned. Extending all these images of orthogonals establishes convincingly that they meet at a point a little to our right of the Virgin's mouth. This point lies on the vertical axis of the architecture, which is marked by the chain from which the egg is hanging.[55]

The meeting point of the images of orthogonals – that is, the point directly opposite the eye of the ideal observer – thus lies more or less on the eye level of the four attendant angels and a little below that of the saints. This picture is the one in which Piero comes closest to adopting the convention that the ideal eye height should be that of a standing figure in the picture. However, since this eye height is so unusual for Piero, it seems much more likely that it was not adopted as a convention but reflects the eye height of an actual viewer – as is the case in all those of Piero's pictures for which we have enough relevant information to allow us to investigate the matter. Rough measurements from photographs show that the ideal eye height lies about two-fifths of the way up the picture, at a height of about 115 cm from its lower edge. A reasonable eye height for a fifteenth-century European man is about 156 cm.[56] This leaves roughly 41 cm for a frame and a tomb resting on the floor of the mortuary chapel, a set-up that would bring the ideal eye height to that of a standing spectator and make the world behind the picture plane a continuation of our own.[57] If the design really was for something like this, one must presume the part of Federigo's tomb above floor level was to be no more than a low table structure, but such an arrangement would be in keeping with the emphasis on humility and penitence in the choice of the saints included in the *sacra conversazione*.[58]

St Peter Martyr, the middle saint in the group of three on our right – no doubt included because of his suffering – was a Dominican, and, in view of the rivalry between the two orders, is a rather unlikely figure to find in an altarpiece for a Franciscan church. His inclusion could, of course, be regarded as an additional argument against Piero's picture having been originally intended for the church of San Bernardino. It is nonetheless a moot point whether one would wish to extend the notion of tolerance to allow the possibility that, as some historians have suggested, the figure shown as the Dominican St Peter Martyr is a portrait of Luca Pacioli, who was himself a Franciscan. The features do resemble those in

55 A photograph of the picture, with superimposed white orthogonals from the architecture is given in Bertelli, *Piero della Francesca* (full ref. note 49), p.133, fig.125. These lines were obtained by running strings over the actual picture (Bertelli, p.212), so they are presumably reliable. The figure also includes a number of additional lines whose significance eludes me.

56 This is to assume that the average height of a fifteenth-century European man was about the same as that of a twentieth-century European woman, namely 165 cm, and that eye height is about 9 cm less in each case.

57 If we assume the eye height considered was that of a man of ideal height that is, 3 *braccia* (about 175 cm), then we shall have room for a correspondingly deeper frame of 51 cm.

58 On the saints, see, for example, Lightbown (full ref. note 10), pp.246ff. A detailed proposal for the association of Piero's painting with Federigo's tomb, taking account of the eye level, is put forward in Guido Ugolini, *La pala dei Montefeltro* (full ref. note 51).

the only known portrait of Pacioli.[59] However, Vasari, who, at the very beginning of his *Life* of Piero, comments in a moralizing way on what he considers to be Pacioli's appropriation of Piero's mathematical work, does not mention that Piero had ever painted Pacioli's portrait – a fact he surely might have remarked upon, since it could be seen as an aggravating circumstance in regard to conduct of which he is expressing disapproval. Vasari's silence may, however, be explained by his not knowing the *Montefeltro Altarpiece*, which he does not list among Piero's works. Pacioli became a well-known figure – no doubt in a limited circle – only after the publication of his *Summa de arithmetica* . . . in 1494. The book is shown in the known portrait of Pacioli. So a portrait by Piero would be simply a personal tribute to a friend, and, given that Pacioli was born in 1445, argues for a later rather than an earlier date for the picture. The man shown as St Peter Martyr can hardly be aged twenty-seven, as Pacioli was in 1472 (which is usually taken as the earliest plausible date for the painting, and even adding about ten years hardly helps). It would be interesting to know more about Piero's relationship with Pacioli, but the *Montefeltro Altarpiece* does not seem to be a good source of evidence on the matter.

The architectural setting not only provides us with a usefully large number of orthogonals, but also supplies two horizontal squares: the plan of the area under the barrel vault[60] and the plan of the area enclosed by the nearer arch of the barrel vault and the two arches at extreme left and right. Both plans are visible through the intermediary of the mouldings running round the upper edges of the walls. Piero has not shown all the sides of either of these squares sketched by the mouldings, but we have perfectly clear indications of the corners of the nearer one, since we can see the corners of the mouldings concerned. We have slightly less clear indications for the further square, since the meeting of the mouldings at the back of the barrel vault is obviously not quite comparable with the corner defined by the meeting of mouldings at the front. However, extending the diagonals of both squares shows that they meet the horizontal through the point directly opposite the ideal eye – that is, the horizontal through the point of convergence of the images of orthogonals – at points whose distance from the point opposite the ideal eye is twice the width of the panel. Which is to say that the ideal viewing distance of this part of the picture is twice the width of the panel.[61] The simple ratio between panel width and viewing distance is of no particular technical significance since it would not have helped Piero with the task of putting in the transverse ribs in his barrel vault, which are a rather prominent feature. The vault is probably intended as a recognizable homage to Masaccio's *Trinity* fresco – which we may note is presented as standing over a tomb.

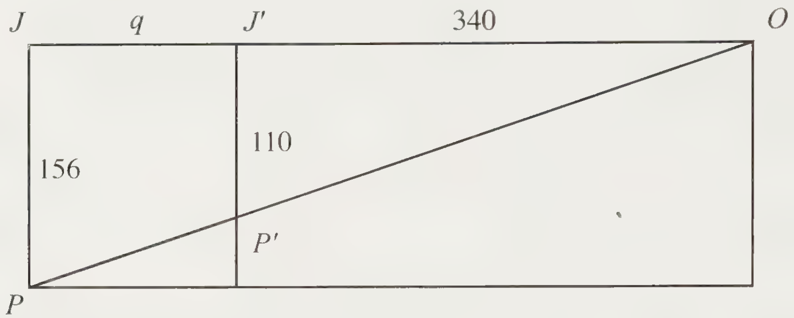
The figures are all placed in front of the architecture that we see in the upper part of the picture. At the far left there is a tiny indication of where the left wall meets the dark floor on which the angels are standing. This proves – if one notices it – that the square defined by the arches to left and right lies behind the figures. The indication is, however, far too slight to afford a basis for calculation, even if there were a suitably patterned floor. Since the restoration in the 1980s showed no evidence for the picture having been cut at the left or right edge, we must assume that this visually elusive clue was all Piero intended to

59 Reproduced in Lightbown, *Piero della Francesca* (full ref. note 10), p.255.

60 For its shape, see Appendix 9.

61 This use of the inverse of the distance point method does not imply that the distance point method was used to construct the image. See note 41 and Appendix 3.

provide. If we assume that the human and divine figures belong to the same perspective scheme as the architecture, as we are presumably meant to do, we are now in a position to make some rough but mathematics-aided guesses (guesstimates) of the distance of the figures from the ideal viewer. The diagram for this procedure is shown in Figure 6.37.



6.37 A sketch to show how an eye at the ideal viewing position for the *Montefeltro Altarpiece* will see the figure of St John the Baptist. The point O represents the ideal viewing position, J is at the level of St John's eyes, and P at the level of his feet, so that JP represents his position in the fictive space, his height being assumed to be the same as that of the ideal viewer. $J'P'$ represents the figure of St John on the picture plane. What we want to find is the length q, St John's distance behind the picture plane. All lengths are in centimetres, and are rough estimates. Drawing by JVF.

We shall consider the figure of St John the Baptist, on the viewer's far left. In Figure 6.37, let St John's position in the fictive space be given as JP, J being at his eye height, and P his foot. Let the image of St John on the picture plane be $J'P'$, as shown. The height of the image, from the front foot up to the saint's eye level, is about 110 cm. The distance of the viewer's eye from the picture is 340 cm (that is, 2×170 cm), and his eye height is 156 cm, which we shall assume will also be the true eye height of St John the Baptist. The fact that St John's eye is not on the same level as our own does not affect the mathematics: the method involves only similar triangles, and putting St John's eye on a level with that of the observer, as in Figure 6.37, merely has the effect of making the triangles right-angled, which is irrelevant to the calculation. However, it is important to remember that our measurements and our estimate of St John's true height are all approximations, so the answer we get will accordingly also be approximate.

Let the distance of St John behind the picture plane be q, as shown in Figure 6.37. We have $OJ = JJ' + J'O = q + 340$, $J'O = 340$, $JP = 156$ and $J'P' = 110$.

The triangles OJP, $OJ'P'$ are similar. Therefore

$$\frac{OJ}{OJ'} = \frac{JP}{JP'}.$$

That is, substituting known values, we have

$$\frac{q + 340}{340} = \frac{156}{110},$$

giving

$$q = 340\left(\frac{156}{110}\right) - 340,$$
$$= 142 \text{ (to 3 sig. fig.)}.$$

That is, the distance of St John behind the picture plane in the fictive space is 142 cm to the nearest centimetre. A similar method could be used to find the distances of the other standing saints.[62]

This computation is mathematically valid, but, as noted, the results are unreliable because we made an assumption about the true height of St John, and used unreliable measurements. One could take measurements directly from the painting, but the element of guesswork for the true size of St John is irremovable.

If we want to go further in exploring possible spatial relationships in the fictive scene, we need more guesswork. There is no direct way of finding the size of the architecture, since it nowhere meets the picture plane, which would have allowed us to measure some part of it, as we can measure part of the paving in *The Flagellation* (Fig. 5.28). Moreover, there is no way of relating the figures to the architecture, which is simply seen as somewhere behind them. As in the *Madonna di Senigallia* (Fig. 4.15), the figures and the architecture of the *Montefeltro Altarpiece* are disjunct. In the former picture, this is not disturbing, since the viewer is not likely to be in any serious doubt about the approximate size of the architecture, whose style is essentially domestic. Since we know, or believe we know, the size of what we see, we imagine it as being at a certain distance. In the *Montefeltro Altarpiece*, however, the architecture is not familiar. Finding the correct viewing distance for the picture does not tell us how far away the architecture is but merely establishes the different distances that would correspond to its being of different sizes. Standing at the correct viewing distance is, in my own experience, similarly ineffective, there being no 'obvious' hypotheses that the visual system can supply to supplement the inadequate data.

What we can do by way of mathematical calculation naturally exposes the need for this kind of supplementary information. The only clue that seems to be provided is the egg. Rough measurement shows that the width of the vaulted area is about 16.6 times the length of the egg – a proportion that makes no allowance for the fact that the egg is under the vault and thus a little nearer to us than is the back wall of the vaulted area. Given this relation to the architecture, the egg will certainly need to be an ostrich egg if the architecture is to be large enough to be behind the figures.[63] The curators of the Galleria di Brera in Milan have kindly hung the picture at a reasonable height and placed it with windows to its left. It would be excessively literal-minded for the visitor to ask for a real ostrich egg as an additional aid. If we ignore the possibility that Piero may have expected there would be a real ostrich egg nearby for comparison purposes, we are driven back on the dubious intuition that the egg, being a sort of visual echo of the head of the Virgin, may be supposed to be the same size in actuality. Unfortunately, we need additional guesswork to find the size of the Virgin's head.

62 This is explained in more detail in Appendix 9, where there is also a slightly different version of the calculation relating to St John, using a different measurement and arriving at a slightly different answer.

63 For the identification of the egg as that of an ostrich, together with documentation regarding the display of such eggs, see Millard Meiss, '*Ovum struthionis*: Symbol and Allusion in Piero della Francesca's *Montefeltro Altarpiece*', in *Studies in Art and Literature for Belle da Costa Green*, ed. D. E. Miner, Princeton: Princeton University Press, 1954, pp.92–101; and Millard Meiss, 'Addendum ovologicum', *Art Bulletin* 36, 1954, pp.221–4.

The choice is between direct guesswork and guessing the distance of the figure from the picture plane and using a measurement of the image to find the size of the head. Direct guesswork might suggest that the figure, which looks a little larger than those of the saints, had been made the 'perfect' height of 3 *braccia* (175 cm) and that the head might be a tenth of the body, giving us a size of 17.5 cm. However, the height of the image of the head is about 14.5 cm, and calculation shows that this indicates that if the true size is 17.5 cm the Virgin is much closer to the picture plane than St John is.

The indirect method is more successful. Putting the Virgin at a distance of 180 cm behind the picture plane, a position that is reasonable in relation to the saints, gives us the size of her head as 22 cm. The figure has clearly been made decidedly larger than anything one might call 'life size'. Taking the size of the egg as 22 cm gives us the width of the barrel vault as 365 cm to the nearest centimetre. All these calculated values are necessarily approximate, partly because of the guesswork, but also, in the present case, because they use measurements of small elements in the picture – measurements that have been taken from small photographs. In view of the necessary use of guesswork, it is not obvious that more exact measurements would give more accurate answers.

With this proviso about exactness in mind, we may proceed to some further calculations. For example, measurement shows that, in height, the image of the egg is about one-third as large as the Virgin's head, which enables us to find the distance of the back of the barrel vault behind the picture plane as 1220 cm, which is about 3.3 times the width of the vault. Such proportions duly appear in various scholars' reconstructions.[64] Neatness is, however, no guarantee of precision. All reconstructions necessarily depend upon assumptions of the kind that have been detailed here. Piero himself did not, of course, need to guess. He could make – indeed presumably did make – all the relevant decisions, and then carry out calculations accordingly, either numerically or by means of drawings. At least, there is evidence for extensive use of preliminary drawings that have been transferred to the panel by means of *spolvero*,[65] so it is extremely likely that some mathematical preparation went into the drawings. Juggling with similar triangles was a thoroughly familiar process to any mathematician of the time.

There is no particular mystery about how the preliminary drawings could have been made. What is much less easy to understand is why Piero should have constructed a picture whose three-dimensional structure is so full of uncertainties for the viewer. As we have already mentioned, the *Montefeltro Altarpiece* is not unique among Piero's works in having a background that is disjunct from its foreground. The same structure is found in the Montefeltro portraits (Figs 4.11–4.14), in the *Madonna di Senigallia* (Fig. 4.15), and indeed in the *Nativity* (National Gallery, London) (Fig. 6.38). Part of the much greater problem we encounter in reading the three-dimensional structure of the scene in the *Montefeltro Altarpiece*, compared with those of the other works, may be attributable to the state of preservation of the picture. First, the removal of the jewel from the Virgin's headdress has taken away an accent that would surely have held our attention in the group of figures – if only by relating the Virgin more closely to the angels on either side – rather than allowing our glance to plunge too readily backwards to the distant egg.[66] Second, the ageing of the

64 More detailed accounts of the above calculations are given in Appendix 9.

65 Infrared reflectograms have revealed some transfer of drawings; see Bellucci and Frosinini, 'Ipotesi sul metodo di restituzione' (full ref. note 19).

66 The visual implications of the loss of the jewel are remarked upon in Bertelli, *Piero della Francesca* (full ref. note 49), p.138.

6.38 Piero della Francesca (*c.*1412–1492), *The Nativity of Christ*, oil on panel, 124.5 × 123 cm, National Gallery, London.

pigments used for the flesh tones has almost certainly given the figures a browner colour than Piero intended, and by making them noticeably darker than the background tends to force them backwards while the background is pulled forwards. This comment is not intended in any way as an adverse criticism of the restoration carried out in the 1980s, which naturally could not arrange to reverse the effect it inevitably exposed more fully. On the positive side, it also exposed the delicacy of Piero's paint-handling in such passages as the jewels around the hem of the Virgin's blue cloak and the white highlights used to show the crystal cross held by St Francis.

A computer simulation might perhaps be used to produce false-colour images showing possible reconstructions of Piero's original intentions. A simulation might also be used to add a conjectural reconstruction of a missing frame, so as to bring architectural elements up to the picture plane – or slightly proud of it – as do the frames of Giovanni Bellini's San Giobbe and San Zaccaria altarpieces. Experience with those pictures, and indeed with Domenico Veneziano's *St Lucy Altarpiece* (Fig. 3.11), suggests that a frame can make a considerable difference to one's sense of a picture's being readable. There remains, however, the irreducible fact that – unless a now lost frame did indeed make a crucial contribution – the picture presents itself as perspectival in a mathematical sense, but in reality defies exact reconstruction. Indeed, although the correct viewing distance for the background architecture is twice the width of the panel, the picture is extremely insensitive to the actual viewing distance.

This may be a characteristic of barrel vaults with reasonably long viewing distances: the vault in Masaccio's *Trinity* fresco (Fig. 2.7) similarly continues to look right (that is, to seem three-dimensional) when viewed from a distance much greater than the correct viewing distance, which is the width of the aisle, rather more than twice the present width of the fresco.[67] Like Piero, Masaccio has disengaged his figures from the architecture, but there are two important differences. First, in the case of the *Trinity* fresco, the architecture surrounds the figures, so that at the very worst our sense of its size would be mediated by our sense of the size of the figures. Second, the architecture of the *Trinity* comes right up to what seems to be the picture plane, so we can measure its size in absolute terms.

Piero's picture invites technical comparison with Masaccio's not only on stylistic grounds, and perhaps because the latter contains a representation of a tomb, but also because they both have attainable naturalistic ideal viewpoints. At least, the evidence we obtain by comparing the *Montefeltro Altarpiece* with Piero's other pictures suggests the panel was meant to be seen with an eye at more or less the natural height. This would make the perspective of the picture unlike that in any of the scenes in the fresco cycle *The Story of the True Cross*. Perhaps that is a clue to Piero's decisions. He may have consciously preferred to avoid the *trompe l'œil* possibilities that seemed to offer themselves. It is conceivable that in context such illusionism would have seemed indecorous. In any case, in the *Montefeltro Altarpiece* Piero did not provide a rigorous mathematical framework that would connect the positions of everything in the picture in an unambiguous way with the positions of objects in the real world of the viewer. Among his surviving works, the only one in which he did impose such rigour is the *Flagellation of Christ*, in which the figures are small, so that the relationships in the picture cannot possibly be read as literally continuous with those in the full-scale world. We may note, too, that the ideal eye height in the *Flagellation* is low, well below the eye height of a standing figure, which, from what we know of Piero's practice, suggests we were intended to be looking up at the picture.

67 See Chapter 2 and J. V. Field, R. Lunardi and T. B. Settle, 'The Perspective Scheme of Masaccio's *Trinity* Fresco', *Nuncius* 4.2, 1988, pp.31–118.

Some of the more obvious descendants of Piero's *Montefeltro Altarpiece* are to be found among Venetian works of the later fifteenth and early sixteenth centuries, such as Giovanni Bellini's *sacra conversazione* altarpieces for San Giobbe (1480–85) and San Zaccaria (1505). It is tempting to suppose that Piero also supplied an antecedent for such altarpieces presenting their scenes as taking place in an architectural structure that is a direct continuation of the architecture of the church or, at least, that of the frame round the altarpiece. However, in the Venetian pictures the eye height for an ideal viewer is never, in my experience, entirely correct. For instance, the *San Zaccaria Altarpiece* adopts Piero's convention, seen in the frescos of *The Story of the True Cross* and in the upper parts of the *Sant'Antonio Altarpiece*, that since the actual viewer is looking upwards at the picture the ideal eye height is made lower than the eye height of figures shown in the painting.[68] This is to say that although Bellini does present the scene in his picture as taking place in an extension of the church, he does so in a less rigorously naturalistic manner than that of Masaccio in the *Trinity* fresco.

Since the distance of the architecture in the *Montefeltro Altarpiece* is in effect a free variable, it would be possible to frame the picture so that the frame supplies the near arch of the 'crossing', in the manner of the framing of Bellini's San Giobbe and San Zaccaria altarpieces. If such framing were envisaged or actually put in place in the period before the painting was moved to the altar in San Bernardino, Piero's picture would provide a highly plausible antecedent for Bellini's practice.[69] It seems rather likely that Bellini did see Piero's *Montefeltro Altarpiece*, and this possible adaptation is surely conceivable as an exercise of the painter's predatory eye, on the lookout for what can be used, not necessarily in search of another painter's intentions. However, in the absence of precise evidence concerning the framing of Piero's picture at a period when Bellini might have seen it, there are no secure grounds for regarding Bellini's practice, or that of his Venetian contemporaries, as a guide to the significance of the ambiguity Piero built into the design of the *Montefeltro Altarpiece*. Moreover, since we do not know the visual context for which the picture was designed, it may be that Piero expected it to contain architectural elements that provided clues and thus effectively abolished the ambiguity that is now so prominent.

The Nativity of Christ

As we have seen, the comparative abundance of documentation relating to Piero's *Montefeltro Altarpiece* provides neither a definite dating nor a great deal of help in interpreting the nature of its three-dimensional composition. When we turn to the *Nativity* (Fig. 6.38) we are in the more usual condition of simply lacking documentation as to its origins. The work is not mentioned by Vasari, but its provenance can be traced back to the family of Piero's brother Marco, and there has been no dispute as to its authorship. The use of oil paint suggests a relatively late date, while the fact that the picture may be unfinished has

68 The images of orthogonals in the upper part of the picture are extremely short, but those in the square-tiled flooring are sufficient to establish that the meeting point of the images of orthogonals (which is the point opposite the eye of the ideal observer) is on the central vertical of the picture, at about the level of the shoulder of the angel sitting on the lowest step of the throne.

69 The reconstruction of Antonello da Messina's lost altarpiece for San Cassiano, dating from the late 1470s, in J. Wilde, 'Die "Pala di San Cassiano" von Antonello da Messina, *Jahrbuch der kunsthistorischen Sammlungen in Wien* N. F., 3, 1929, pp.57–72, makes an entirely acceptable case for the picture's having included a well-defined architectural setting, but the details of the setting are necessarily hypothetical.

sometimes been taken as an indication that Piero's work on it was interrupted either by his death or by the loss of his eyesight. As has often been pointed out, Vasari's account of Piero's blindness may merely be a standard literary way of explaining what Vasari perceived as the absence of pictures painted in Piero's later years. Vasari seems not to have known the Montefeltro portraits, the *Montefeltro Altarpiece* or the *Madonna di Senigallia*, which is perhaps rather strange in view of his stressing the importance of Piero's connections with Urbino. Nevertheless, there is an unpleasantly convincing degree of detail in Vasari's discussion of Piero's blindness.[70] However, if Piero did lose his sight he must have done so only in his last five years of life: there is nothing visibly unsteady about the handwriting in which he drew up a sketch of some of the provisions of his will in July 1487.[71]

In any case, it is clear that at least some of the areas of thin paint in the *Nativity* are probably due to overcleaning. Moreover, if the picture was as near completion as we now see it, there would seem to have been no obvious reason why a pupil should not have finished it and arranged for it to be delivered to the person who had commissioned it. Thus the picture's remaining with Piero's close family suggests that it was always intended for them, or even for Piero himself – a private picture not a public one.[72] Not that we have any need to point to whimsical self-indulgence to explain any of its characteristics, but much of it is so characteristic of Piero that there is a certain appeal in being able to suppose he was following only his own taste.

It has rightly been pointed out that the composition of Piero's *Nativity* appears to be related to that of the fresco of the same subject by Alesso Baldovinetti (*c.*1426–1499) in the cloister beside the main door of the church of the Santissima Annunziata in Florence (Fig. 6.39).[73] This fresco dates from the early 1460s. If it truly is the origin of some of the features of Piero's painting, then we have further evidence for his having made an undocumented visit to Florence, though this is not incontrovertible evidence, since Alesso is known to have been in Arezzo in 1482, and might have discussed his picture with Piero.[74]

The subject treated by Piero is, strictly, not the Nativity itself but the Virgin's Adoration of her newborn son – a scene presented to St Bridget of Sweden in a vision in Rome in 1370 and recorded in her canonization process of 1391.[75] The Adoration is sometimes particularly associated with Netherlandish works, but it was illustrated by several Tuscan painters

70 In the 1550 edition of his *Lives*, Vasari says the blindness was caused by 'un male di cattarro' and in the 1568 edition this becomes simply 'un cattarro'. In both editions Piero is said to have lost his sight at the age of sixty (and to have died aged eighty-six). The words 'un cattaro' are often interpreted as being a corruption of 'una cateratta' – that is, cataract. However, there is a more straightforward reading, namely that the word 'cattarro' means what it usually means, 'catarrh', and refers to a cold in the head. This would be a plausible history for the sudden, but not necessarily complete, loss of sight due to retinal detachment. Sharply increased pressure due to attempts to clear blocked passages is a well-known cause of detachment in old age.

71 See Battisti, vol.2, p.624, document 210, and p.635, fig.528.

72 For the possibility that the picture was a wedding present for a member of Piero's family and can consequently be dated to 1482, see Lightbown, *Piero della Francesca* (full ref. note 10), pp.273–4.

73 See Lightbown, *Piero della Francesca* (full ref. note 10), p.276, and M. A. Lavin, 'Piero's Meditation on the *Nativity*', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no. 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art, 1995, pp.127–41, esp. p.128. For possible Sienese antecedents of Piero's picture, see Marco Bussagli, 'Note sulla *Natività* di Londra: la gazza e l'asino, due motivi dissonanti', in *Città e Corte nell' Italia di Piero della Francesca. Atti del Convegno Internazionale di Studi Urbino, 4–7 ottobre 1992*, ed. Claudia Cieri Via, Venice: Marsilio, 1996, pp.233–41.

74 See Lavin, 'Piero's Meditation on the *Nativity*' (full ref. note 73), p.138, note 14.

75 For details of St Bridget's vision and an example of a treatment of the scene dating from about 1400, see Lavin, 'Piero's Meditation on the *Nativity*' (full ref. note 73), pp.127–8.

6.39 Alesso Baldovinetti (*c.*1426–1499), *The Nativity*, early 1460s, mural, 426 × 426 cm (without painted frame), Chiostrino dei Voti, Santissima Annunziata, Florence.

of Piero's time. A very pretty example by Filippo Lippi (*c.*1406–1469), dating from the mid-1450s and now in the Galleria degli Uffizi, Florence, has the Child lying on the Virgin's cloak.[76] A simpler version (*c.*1459), also by Filippo Lippi, in Berlin (Gemäldegalerie, Berlin, formerly Staatliche Museen, Dahlem), has the Child lying on the bare ground. The Adoration, with the Child lying on the ground, is the subject of the central panel of the highly influential *Portinari Altarpiece* (Galleria degli Uffizi, Florence) of Hugo van der Goes (died 1482), painted in about 1475 and brought to Florence in 1483. Since we do not know the exact date of Piero's *Nativity*, it is conceivable that Piero saw the *Portinari Altarpiece* before

76 This picture was painted for the Medici Chapel in what is now the Palazzo Medici-Riccardi, and was very well known in the fifteenth century.

painting his own panel, but there is no necessity to invoke such contact to explain the character of the *Nativity*. It is indeed usually described as the most 'Northern' of all Piero's pictures, but that is inevitably a slightly rhetorical judgement rather than any kind of detailed assessment. Detailed parallels can be drawn between features of many of Piero's pictures and Netherlandish models that he may have known.[77] There is, moreover, no dispute about his use of oil paint and his contact with the work of Justus van Ghent at the court of Urbino, where Justus arrived in person in 1472.

St Bridget's vision was only of the newborn Child and his Mother kneeling to worship Him, accompanied by the sound of angels singing. This part of the story Piero has told largely in blue and white, with very pale flesh tones and characteristic small touches of red. The contrast with the colouring of the seated figure of St Joseph and the two coarsely dressed standing shepherds has surely not been greatly affected by the damage suffered by the surface in these latter areas. The group from the vision is held together with some subtlety in the picture plane. The Child is lying on part of the Virgin's cloak, but the cloak has been twisted to allow Him to rest on its dark blue outside, and the cream lining makes only a weak contrast with the colour of the ground behind it, so that in the plane the cloak slightly separates the figures that it unites in the imagined actuality. Another of Piero's non-contrasts is to be seen in the way the colour and the pattern of fingers ensure that the Child's left hand is almost lost against the foot of the angel that is seen behind it, while a simple relationship in depth is established by the strong contrast where the toes of the same angel's other foot vanish behind the dark blue cloak. Small items of glittering jewellery and the repetition of reds make an additional visual link between the Virgin and the two angels who are singing. There is another weak colour contrast between the head of the lute of the angel on our right and the horn of the ox. The forms almost seem to blend into one another, but the position of the ox's feet, marked by a strong contrast of black hoof on light-brown ground, and the intense glance the animal directs towards the Child, make the spatial relationship sufficiently clear.

There are also a number of alignments that tie the various parts of the picture together. The sense of three-dimensional relationships conveyed by the picture is so strong that most of these alignments are hardly noticeable to the casual glance, since they belong entirely in the picture plane. For instance, the extended index finger of the hand of the shepherd who is pointing upwards is almost exactly aligned with the Virgin's nose, and the staff that he is carrying is close to alignment with the Child's right forearm. This line is parallel to the straight black line of the tail of the magpie and to the outer edge of the white arm of the angel on our left. These three parallel lines are at right angles to the line of the shadow of the stable roof that links the silent magpie to the muzzle of the braying donkey.[78] These alignments concern only the composition in the plane. As in Piero's *Baptism*, and in any number of other pictures by him, there are no alignments that allow us to deduce anything about a possible perspective scheme. As can be seen from the line of the top of its roof, the rough shelter has been placed so that its back wall is not parallel to the plane of the picture. However, Piero has made the roof itself into a pattern that is a tidier version of that of the piece of earth and vegetation in the foreground.

77 See Bert W. Meijer, 'Piero and the North', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no. 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art, 1995, pp.143–59.

78 For the possible narrative significance of the donkey and the magpie, see Bussagli, 'Note sulla *Natività* di Londra' (full ref. note 73), and Lavin, 'Piero's Meditation on the *Nativity*' (full ref. note 73), pp.131–2, and 135–6.

Not only is this a picture without visible mathematical perspective, it is also one without a middle ground. In this it resembles the Montefeltro portraits, the *Madonna di Senigallia* and the *Montefeltro Altarpiece*. There are, of course, many fifteenth-century pictures in which one suspects that the omission of the middle ground, or its artful abbreviation, is a tribute to the painter's determination to avoid attempting a task in which he recognizes himself as unlikely to succeed. Piero appears to be doing as a matter of choice what others may have done for reasons of limited competence. It is possible that custom had made the foreground and backdrop style acceptable in itself. Piero's composition can, in fact, be read in naturalistic terms, with the foreground scene placed on some kind of terrace or small escarpment.[79]

Piero's backdrop, as in the Montefeltro portraits, is highly naturalistic. On the right we have a detailed townscape that can be identified as showing Borgo San Sepolcro. On the left we have a landscape that could belong nearby and that fades away, with slight blurring in the distance, to small hills rather like those in the Montefeltro portraits. It is possible that the far distance may originally also have been intended to be bluer. The sharp focus in both of these deep vistas is reminiscent of Netherlandish art. Since the pictures hang next to one another in the National Gallery in London, one can hardly fail to notice the contrast with the dab-and-squiggle landscape behind the main figures in the *Baptism of Christ*. However, if the calligraphic part of Masaccio's influence seems to have waned, we nevertheless have an echo of Florence that almost certainly dates back to Piero's first visit in the late 1430s. The musical angels, closely grouped together, seem to be a reminiscence of a panel from the singing gallery (*cantoria*) that Luca della Robbia (*c*.1400–1482) made for Florence cathedral in 1431–8. (The gallery is now in the Museo delle Opera del Duomo.) Piero's angels even resemble Luca's singers – who are presumably not angels – in not having wings.

The Williamstown *Madonna and Child Enthroned with Angels*

The style of Piero's known works is highly consistent. The fact that there are rather few pictures whose possible attribution to Piero is a matter of dispute is a tribute to this consistency, but should probably also be seen as evidence that Piero did not run a large workshop. The non-existence of such a workshop would, moreover, be consistent with Piero's apparently rather small output, and with the likelihood that he drew some financial benefits from the family business and thus was not entirely dependent upon his painting for his living. The disputed attributions are those of the *Madonna Contini-Bonacossi* (private collection, Florence), four fresco figures of Evangelists and their symbols in Santa Maria Maggiore, Rome, the *Madonna and Child Enthroned with Angels* in the Sterling and Francine Clark Art Institute, Williamstown, Massachusetts (Fig. 6.40), and, to my mind, the panel portrait of *Sigismondo Malatesta* now in the Louvre.

The present study does not deal with matters that seem likely to help in regard to the ascription of the first two works or the fourth one. The structure of the pictorial space shown in the *Madonna Contini-Bonacossi* is not notably unsatisfactory, but the overall design seems too close to works by Filippo and Filippino Lippi to allow the work to be ascribed to a sufficiently early date for Piero to be painting in such an apparently tentative

79 Lavin, 'Piero's Meditation on the *Nativity*' (full ref. note 73), p.129, identifies this as a threshing floor.

way. The Evangelists, on the vault of a side chapel in Santa Maria Maggiore, are badly damaged, which may partly explain the apparent coarseness of some of the execution. In contrast, the design of the figures is relatively accomplished, but that does not constitute grounds for ascribing it to Piero rather than to one of his many competent contemporaries. The resemblance to Piero's style might be explained as emulation of his then recent, now lost, works in the Vatican. For the Louvre's portrait of *Sigismondo Malatesta*, the design is so close to Piero's portrait of him in the Rimini fresco that arguments must largely be concerned with the execution. It is unfortunate that the picture was not shown at the exhibition in the Ducal palace in Urbino in 1992, which provided an excellent opportunity for comparisons. However, even a cursory inspection shows infelicities in the drawing where the neck meets the clothing. The inadequacy is so blatant that we must surely be concerned with the work of a copyist. There is also, particularly in the rendering of the elaborate brocade, a degree of roughness in the execution that is not at all typical of the paint handling found in works securely ascribed to Piero, whatever their supposed date.

The *Williamstown Madonna* contains many elements that echo passages in works known to be by Piero. Indeed, no picture known to be by Piero contains a comparable quantity of recognizable repeats, except perhaps the *Flagellation of Christ* (Fig. 5.28), in which there are repetitions of elements from the Arezzo frescos, on a much smaller scale. For instance, the background panelling in the *Williamstown Madonna*, showing heavy swags against dark marble revetment enclosed by white framing, is a smaller version of the background in the Rimini fresco (Fig. 6.1). The angel on the far left, who has arms crossed, could well be an adaptation from a drawing also used for the angel of the *Annunciation* in the *Sant'Antonio Altarpiece* (Figs 6.29 and 6.34). The repetitions in the Williamstown picture are, moreover, different in scope from those in the *Flagellation*. In the latter, there is a basic similarity in the required architectural settings that makes the repetition comprehensible, and perhaps suggests a closeness in date for the works concerned. In the case of the Williamstown panel, the repeated motifs come from a variety of works, which seem to be of a variety of dates. Nor are the reminders of other pictures always felicitous. The angel on the far left in the Williamstown picture is considerably less accomplished than the angel in the lunette of the *Sant'Antonio Altarpiece*.

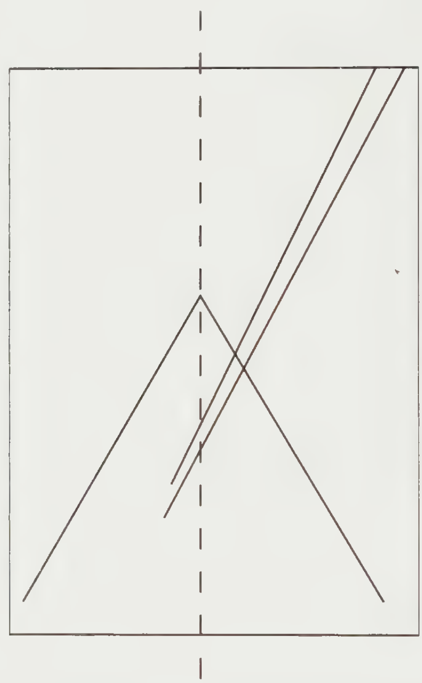
Since I am judging the work only from photographs, it may be that deformations of the colours, as well as ageing of the pigments, have made the modelling and the handling of light look less satisfactory, but it nonetheless hardly seems comparable in subtlety with what we find in, say, the *Madonna di Senigallia* (Fig. 4.15) or the *Nativity* (Fig. 6.38). The comparison with these works is justified by the similarity in the facial types that appear. In particular, the Christ child in the Williamstown picture, whose outstretched arms resemble those of the Child in the *Nativity*, is of a decidedly Netherlandish type. Indeed, the figure of the Child in the Williamstown panel might be an adapted copy of the Child in the *Nativity* reversed and turned through a right angle so as to sit upright instead of lying flat. Such an adaptation would account for the curious awkwardness of the Williamstown Child. Although some earlier scholars placed the *Williamstown Madonna* as early as the 1440s, the consensus now tends to favour a later date, which inevitably entails the comparisons just mentioned and seems to me to rule out the possibility that the picture was painted by Piero.[80] Whether he or his assistants and pupils had any hand in the design of the picture is,

80 Battisti, who does ascribe the *Williamstown Madonna* to Piero, discusses it together with the *Madonna di Senigallia*. His discussion is largely concerned with the latter picture. There may be an implied judgement of the quality of the execution in the fact that, in the course of considering the significance of symbolic elements, we are given plates of four

6.40 Follower(s) of Piero della Francesca, *Madonna and Child Enthroned with Angels* (*The Williamstown Madonna*), oil (and tempera?) on panel, 107.8 × 78.4 cm, Sterling and Francine Clark Art Institute, Williamstown, Massachusetts.

of course, a different question. In any case, one can avoid appeals to such inherently subjective criteria as considerations of the quality of the modelling. The picture contains a number of short orthogonals that provide objective evidence of the perspective construction.

As in the central panel of the *Sant'Antonio Altarpiece* (Fig. 6.29), orthogonals appear on the outside edges of the plinth supporting the Madonna's throne. In the Williamstown picture both these lines are short, and extending them is consequently a slightly hazardous procedure. The picture is not particularly large, so using transparent rulers placed over a roughly A4 photograph – the method employed here – is not likely to cause serious trouble. One might expect symmetry, and it does indeed seem that the two lines, when extended, will meet at a point on the central vertical, a little below the right hand of the Child. The extended versions of these two lines have been sketched in Figure 6.41, together with the central vertical, shown dashed.



6.41 Sketch of the *Williamstown Madonna*, not to scale, to show the images of orthogonals, extended to meet the central vertical, which is shown as a dashed line. Drawing by JVF.

The most straightforward reading of spatial relationships between the architectural setting and the figures in the *Williamstown Madonna* is that the figures are grouped in the vicinity of two colonnades that are at right angles to one another, probably forming the corner of a colonnaded courtyard. There seems to be some awkwardness in deciding exactly what is happening where the mouldings above the architraves meet at the corner, but this is not

details of the *Madonna di Senigallia* and none of the *Williamstown Madonna*. See Battisti, vol.1, pp.295–306.

For a summary of the various datings of the *Williamstown Madonna*, see M. A. Lavin's introduction to *Piero della Francesca and His Legacy* , ed. M. A. Lavin (Studies in the History of Art, no. 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art,1995, especially p.16.

obtrusive. It thus seems clear that the architrave and mouldings shown on the far right at the top of the painting are meant to run directly into the picture, which means that the lines defining them provide a set of images of orthogonals. The longest of them, and thus the one whose direction is most precisely defined, is the line marking the lower edge of the architrave. When extended, this line, which starts at a point close to the top right corner of the picture, meets the central vertical in a point a little below the level of the Child's right knee, that is at a point considerably lower than the 'centric point' that seems to be defined by the orthogonals in the dais. In fact, as can be seen in the sketch in Figure 6.41, the line of the base of the architrave (the line shown on the far right) is almost parallel to the line of the orthogonal edge on the left side of the dais.

Even making generous allowance for the awkwardness of extending short lines, it is difficult to believe that in drawing these lines a competent draughtsman took account of the fact that, when extended, they should meet on the central axis of the picture. Nor are matters greatly helped if we extend other lines from the right-hand entablature. The other line shown in Figure 6.41 is that of the edge between dark and light nearest the base of the moulding, chosen as being one of the longest of the available images of orthogonals. It is qualitatively correct, in being inclined downwards towards the line of the base of the architrave, but it will not meet this line until we reach a point considerably to the left of the central vertical. One cannot really avoid the conclusion that the perspective of the architecture is different from that of the throne.

On the other hand, that formulation may be too prudent to be helpful. The throne – of which we see only the dais, so it may be no more than a simple stool – has merely been made symmetrical. Without other evidence of a perspective scheme, it is a little rash to assume the drawing of the outline of the throne is the result of anything more than rule of thumb. However, the drawing of the architecture involves a standard application of the rule that images of orthogonals should converge to a single point, which is the point to which the eye of the ideal observer is supposed to be directed. My impression is that in the present example the images of the lines that a spatial reading of the picture suggests should be orthogonals do not converge to a single point. The inaccuracy has little effect upon the illusion of depth, but gratuitous departure from mathematical correctness in such a simple matter is not something that analysis of his attested works leads one to expect from Piero della Francesca. It is, indeed, so unexpected that attempts have been made to reconstruct the architectural setting, possibly with the throne, while positing two separate 'vanishing points'.[81] Occam's razor, which embodies a principle of economy and is thus always the most useful tool in any reconstruction, suggests one might do better to abandon the ascription to a painter notable for his concern with the mathematics of perspective.[82]

All the same, given the strong resemblances of various parts of the picture to parts of other pictures by Piero della Francesca, it does seem possible that the *Williamstown Madonna* had some connection with Piero himself, perhaps through a pupil or former assistant making use of extant drawings. Moreover, although the execution of the perspective is faulty, the composition is interesting in playing off a relatively symmetrical foreground

81 Battisti, vol.1, pp.304–5. The accompanying text does not discuss the diagrams, and their captions do not go far in explaining what has been done. In particular, it is not clear which lines in the picture are being used to find the two 'vanishing points' (curiously, the lines of the entablature on the right do not seem to be involved). It will be obvious from Figure 6.40 that the possibilities are numerous and the results are liable to be widely spaced.

82 The ascription of the *Williamstown Madonna* to Piero has been explicitly rejected by Creighton E. Gilbert, *Change in Piero della Francesca*, Locust Valley, New York: J. J. Augustin, 1968, who believes it to be a shop picture, and by Carlo Bertelli, *Piero della Francesca* (full ref. note 49). The literature is discussed in Battisti, vol.2, pp.522–5.

group of figures against a notably asymmetrical architectural setting. The disengagement between figures and setting, in this case by a total absence of middle ground, can be paralleled in pictures by Piero, but is by no means specific to him. Since we cannot see the bases of the columns, we have no way but guesswork to estimate the distance between the figures and the background colonnade. Also, Piero's willingness to scale down architecture, as he did in the *Flagellation*, should warn us against too heavy reliance upon intuitive estimates. This is a problem we encountered earlier in attempting to relate the figures to the architecture in the *Montefeltro Altarpiece*. However, for the *Montefeltro Altarpiece* we were presented with two well-defined squares, and could find a viewing distance as well as a centric point. No such information is available in the Williamstown picture.

After Piero's death, his drawings would presumably have been available to his pupils and former assistants, and it is conceivable that they also had access to a compositional sketch, which they used as the basis for this picture. It seems highly unlikely that the painters of the *Williamstown Madonna* – it seems to show the work of more than one hand – were working from a complete set of drawings by Piero. If they had been, the perspective would surely have been correct. It is important that the architectural setting should look correct – as, in the event, it does – because otherwise we should lose the play of symmetry against asymmetry that characterizes the composition, since the symmetry and asymmetry concerned are not simply on the picture plane but largely in the three-dimensional composition.

In rejecting the *Williamstown Madonna* on mathematical grounds there is, of course, an implicit claim that the earlier analysis of Piero's use of mathematical techniques presents a reasonably coherent account of his practice. Like the elaborate face-saving reconstructions, this claim might also be a candidate for death by Occam's razor. The assessment of the part played by mathematics, visibly and invisibly, in Piero's work as a painter is obviously to some extent subjective. The analyses set out in the present essay do, however, allow us to establish a certain number of ground rules.

The rules of the game

If one read Piero's perspective treatise, or indeed the first book of Alberti's *De pictura*, without any knowledge of the painting of the time, one might expect that such paintings strove to present a scene exactly as it would appear to an actor or bystander in it. However, a moment's thought tells one that, in this case, Alberti's prescription, that the height of the eye of the ideal observer should be the height of the eye of standing figures in the picture, implies viewing conditions markedly different from those that obtain for most works of art. Alberti seems to be proposing that the fictive scene should be designed to appear as an extension of the real scene on the viewer's side of the panel or wall. The roughly contemporary surviving work that best fits this prescription is Masaccio's *Trinity* fresco, though it of course antedates Alberti's text by about ten years. There is no evidence that this level of naturalism, in the sense of making it appear that the wall is pierced (as Vasari puts it in his *Life* of Masaccio[83]) was widely adopted. Whether this was for reasons of decorum or competence is open to conjecture.

Piero della Francesca's text, while much more technical than Alberti's, is also far less overtly prescriptive, except in the narrowly didactic laying down of procedures for obtaining the

83 G. Vasari, *Le Vite dei piu eccellenti architetti, pittori e scultori italiani da Cimabue a tempi nostri*, ed. P. Barocchi and R. Bettarini, Florence, 1971, vol.3, p.127.

desired effect. There is almost no discussion of the overall design of the picture. This silence is no doubt partly to be explained by Piero's explicit concern only with geometrical perspective, not with the other parts of painting. However, as we have seen in connection with the last proposition of the first book, Piero does not mention the height of the ideal eye in the picture even when, in this particular case, it would affect the mathematics. Instead he gives a general rule that allows one to circumvent the problem concerned.[84] It accordingly seems that he thought the choice of a height for the eye of the ideal observer was a matter of *disegno* rather than perspective. That is, to put it in the narrower terms of our own time, it looks as though Piero classed decisions about the height of the ideal viewpoint as matters not of optics but of pictorial composition. The choice of viewing distance was, however, considered a matter of perspective: it is the main subject of the last proposition of *De prospectiva pingendi*, Book 1. So we do know what Piero considered to be the rules of the game in relation to the viewing distance. Unfortunately, only two of Piero's surviving paintings actually provide us with the mathematical information required to reconstruct their viewing distance, namely *The Flagellation of Christ* and the *Montefeltro Altarpiece*. Comparing Piero's theory and practice is consequently a somewhat perilous undertaking, particularly since the illusion of depth is not highly sensitive to the actual viewing distance, as Piero assuredly knew.

Both paintings present problems. The *Flagellation* has a fairly long viewing distance, about two and a half times the picture width, which puts the ideal eye too far away to appreciate the detailed execution. However, since the height of the centric point is low, it is possible – if Piero was following the usage we have seen in other pictures – that *The Flagellation* was intended to be placed rather high up, perhaps inset into panelling or furniture, and that a close viewpoint was not attainable. As we have seen, the frescos in the cycle of *The Story of the True Cross* are also finished in great detail and include much that would not be seen by a normal viewer. Piero is by no means the only fifteenth-century painter who works in this way. For example Andrea Mantegna's panel *The Introduction of the Cult of Cybele in Rome* of 1505–6 (National Gallery, London) was almost certainly designed to be placed high up, as a frieze, but is finished with a degree of detail comparable with that of Piero's *Flagellation*.

The problem of the *Montefeltro Altarpiece* is entirely different, namely that the perspective does not allow an unambiguous reading of distances behind the picture plane. This might have been remedied by a suitable frame – as already mentioned, the *Montefeltro Altarpiece* in some ways resembles Giovanni Bellini's San Giobbe altarpiece in its exile in the Accademia, Venice – but the viewing distance of Piero's picture is in any case only twice the picture width, which seems short for a picture that is so tall. The likeliest explanation seems to be that even if, as seems probable, the picture was not originally intended as an altarpiece, Piero expected it to be subject to similar viewing conditions, and arranged for an ideal viewing distance that in practice acted as a minimum viewing distance.

Essentially, the difficulty in interpreting both *The Flagellation of Christ* and the *Montefeltro Altarpiece* is that we do not know what their original viewing conditions were. All that can really be said is that one can plausibly postulate viewing conditions in which Piero's design of the perspective scheme would have been a reasonable compromise between the theoretical ideal and the actual pictorial possibilities. We may note, however, that both pictures fulfil Piero's conditions for the maximum angle a picture may subtend at the eye. This is true also of the

84 Piero della Francesca, *De prospectiva pingendi*, Book 1, section 30: Parma MS, pp.16 verso–17 recto; BL MS, pp.18 recto–19 recto; Piero ed. Nicco Fasola, pp.96–9; see Chapter 5.

various scenes in *The Story of the True Cross*, in which, uncertain though the viewing distance is, one can feel the wall forcing one backwards if one tries to look at the whole tier from as close as the centre-line of the chancel, from where the angle at the eye is about a right angle.

We are in a rather happier situation in regard to the height of the ideal eye, since it is the other half that is lacking: we have almost no stated theory but quite a lot of evidence as to Piero's practice. The single piece of theory that Piero gives us in regard to the height of the eye is his stipulation, at the end of the section proving the correctness of his perspective construction (Book 1, section 13), that the point where (to put it in today's terms) the line of sight meets the plane of the picture (that is, the foot of the perpendicular from the ideal eye to the picture plane) must lie within the picture itself. He says this rather by the by, as if it were obvious. And indeed it is obvious if we remember that, in the standard optical formulation of the time, what we should now call the line of sight was the 'centric ray', the main vehicle of vision. Thus where that ray meets the picture plane is the point to which the viewer's attention is directed, and when the viewer is looking at the picture the point must, obviously, be in the picture.

One simple consequence of this is that it imposes limits on designing a picture to be seen from below. If the lower edge of the picture is above the natural eye height of a viewer, an exact correspondence of the viewer's eye with the ideal eye position is impossible, because it would take the 'attention point' (what Alberti calls the 'centric point') outside the picture. So in these circumstances mathematical correctness must necessarily be abandoned.

Logically, the choices would seem to fall into two categories. The first is to design the picture with the centric point outside the picture field. Masaccio has almost done this in the *Trinity* fresco, since almost all the substantial content of his picture is above the centric point. Much of the part below the donors' step, which contains the centric point, is now missing, a fact that probably reflects Vasari's judgement that the important part, the part he wanted to preserve, was above the step. The more extreme seen-from-below perspective of Mantegna's frescos in the Orvetari Chapel of the Eremitani in Padua, painted in 1454–7, found almost no imitators.

The second category of choices is to put the centric point inside the picture, thereby making the ideal eye height unattainable in practice. This category includes the Albertian practice of making the height of the ideal eye the eye height of a standing figure in the picture. Piero's preference seems to have been for what one might call a qualitatively correct eye height: since we are looking up at the pictures in the fresco cycle *The Story of the True Cross* on the walls of San Francesco, he arranges for the imagined ideal viewer to be looking up at what is shown in each picture. The ideal eye height is consistently lower than that of standing figures in the scene portrayed. There is no possibility of the ideal eye height being mathematically correct for a real viewer, but Piero's choice does allow for the glance being directed upwards. Moreover, as we have seen, at least two of the scenes in the narrative predella of the *Sant'Antonio Altarpiece* have been designed to be seen from above, presumably by the officiant. This would again be an example of qualitative correctness where mathematical correctness is impossible. The fresco cycle of *The Story of the True Cross* in Arezzo and the *Sant'Antonio Altarpiece* in Perugia provide our best examples for Piero's practice, since we can form a clear idea of how Piero expected people to look at them, or at least of how the pictures would be seen from the best viewing positions that could be envisaged under normal conditions – that is without erecting scaffolding.

Apart from this basic 'error' in the mathematics that is imposed by actual viewing conditions, the perspective in *The Story of the True Cross* seems to be correct, as far as it can be checked. That is to say that lines that look as if they are meant to be the images of orthogonals do indeed, when extended, meet at a 'centric point', whose position, as marking the point where the centric ray meets the picture, is consistent with the angles from which we feel we see other less simple elements, such as human figures, horses and trees. One must, of course, admit that such analysis contains a certain amount of possibly circular reasoning, since it relies upon the identification of lines as orthogonals in order to establish the positions of the centric points. However, we are concerned with a complete set of pictures and we have found a series of linked regularities. It seems unlikely that mere overextension of goodwill would permit the detection of the degree of orderly arrangement in the heights and alignments of centric points that we have been able to identify in *The Story of the True Cross*. Moreover, the straightforward but unobtrusive mathematical means of organization that Piero seems to have employed in his overall design of the cycle are entirely consistent with what we find in the composition of his other works.

There can, of course, be no reasonable doubt that Piero fully understood that the ideal viewing heights for the pictures in *The Story of the True Cross* were incorrect in geometrical terms, that is inapplicable to an actual viewer. There can be equally little doubt that he had looked carefully enough at the work of admired predecessors, such as Masaccio's *Tribute Money*, to know that such an 'error' did not have a fatally deleterious effect on the impression of naturalism conveyed by a work of art. Though they handle paint very differently, it could well have been from Masaccio as much as from Domenico Veneziano that Piero learned the importance of giving a convincing account of the flow of light as confirmation of three-dimensional relationships between the figures and objects portrayed in the picture. The care Piero devotes to this part of optics for painting is one of the leading characteristics of his pictures, and in most of them it is much more immediately noticeable than his use of the part of *perspectiva* dealt with in his treatise. The exception is *The Flagellation of Christ*, in which mathematical construction is unusually prominent and demonstrably correct overall (though with two small 'errors' of which Piero was certainly fully aware, see Chapter 5).

In some of Piero's paintings, for example the *Flagellation*, it is possible to trace shadows that have apparently been drawn, in accordance with the laws of geometrical optics, as cast by illumination coming from a precisely positioned light source. In most cases, however, the exact direction of the light must be more or less taken on trust. Further, we may remind ourselves that if Piero wanted the lighting in the picture to follow the direction of the lighting on the picture – which it seems he generally did – then the direction and intensity would necessarily be variable, since the main lighting would be daylight, from a moving Sun. It is thus a matter of near necessity that direct observation should be more significant than mathematical construction in the positioning of highlights on jewellery or even the reflections of light from armour.

The nearing of the quincentenary of Piero's death in 1992 stimulated a series of cleaning operations whose results have amply confirmed that Piero's art is notably 'Northern' in its careful rendering of details of texture. Vasari, who praised Piero's skill in showing the highlights on armour, said that Piero used clay models and pieces of cloth for making drawings of drapery, and one may surely wonder whether some of the gemstones mentioned in the *Trattato d'abaco* as examples of merchandise may have found their way into the painter's workshop as objects for a different kind of study.[85] It seems highly likely that he made some sketches of Battista Sforza's jewellery, since it is shown in great detail in her portrait. It is possible that

85 For example, in Piero's *Trattato d'abaco* we have the problem:

A gentleman has given his daughter in marriage, and to complete a necklace he needs 100 stones made up of

Vasari was well informed about Piero's working practices since Luca Signorelli, whom Vasari claimed as a relation, is known to have been Piero's pupil, and could have supplied information.[86] The suggestion of the use of clay models may be anachronistic – we know such models were used in Vasari's own time – but is not inherently implausible or at variance with what we see in Piero's work. Indeed, it may be the strong modelling of some of the drapery in the frescos of *The Story of the True Cross* – for instance, the sculptural rippling flow of the cloaks of the Queen of Sheba's ladies – that inspired Vasari's remark in the first place. In any case, it is clear that Piero was always concerned to use lighting and modelling to give the figures and objects shown in his pictures sufficient autonomous solidity to ensure that anomalies in the geometrical perspective, such as an unattainable ideal eye height, could pass unnoticed. Indeed, in some pictures, such as most of the panels of the *Misericordia Altarpiece*, the figures have negligible help from geometrical perspective of the kind described in *De prospectiva pingendi.*

Since *perspectiva* was the complete science of vision, everything discussed here, except the mathematical organization of the composition (either of individual pictures or of a cycle or altarpiece as a whole) would have been considered by Piero to belong to *perspectiva.* However, since he was a good mathematician he must certainly have known that he was in some respects departing from the science of *perspectiva* as understood by natural philosophers. The rules of his game, as a painter, were not the same as the rules of theirs. Given the social and intellectual values of Piero's time, it would be utterly anachronistic to try to see this division as one between his 'art' and their 'science'. As a painter, Piero was a craftsman.

The present chapter ends our detailed discussion of Piero's paintings. In the following one we shall examine the relationship between his craft practice and the world of learning, so the question is not how 'scientific' Piero's work is, but rather how far it can be seen as connected to the learned arts. The question is of interest because the gradual rise in social standing of painters and some other craftsmen seems to have been partly mediated by the increased interest the learned took in their works (and modes of working), as well as a function of the increased learning displayed by painters themselves. In the sixteenth century, the fact that Michelangelo wrote learned poetry presumably helped considerably in raising his social status. Seventy or so years earlier, it is reasonably clear that Piero's skill in 'abacus' mathematics would not have been by any means as effective a claim to consideration, since abacus mathematics was specifically associated with craftsmen. However, the translation of Piero's perspective treatise into Latin does suggest that his work in that area was of interest to scholars. Also, the Latin version of his work on the regular solids is an indication of the beginnings of that kind of acceptance of mathematics we see later in the dedication of Pacioli's much more lightweight vernacular work, *De divina proportione*, to the Duke of Milan in the 1490s. Pacioli more or less volunteers information about this link by the fact that he incorporated Piero's work into his own. Concerns with rightful ownership of texts are in this case much less interesting than what the theft, if theft it was, tells us about the perceived status of Piero's kind of mathematics.

pearls, rubies, sapphires and balas rubies. He summons his agent and gives him 100 ducats and says: go to Genoa and lay out these 100 ducats in buying pearls, rubies, sapphires and balas rubies; and make the total number 100 and do not spend more than $\frac{1}{5}$ of a ducat per pearl and $\frac{1}{2}$ per ruby and 1 per sapphire and 3 per balas ruby[.] I ask how many pearls, how many rubies, [how many sapphires,] how many balas rubies he will have. Piero ed. Arrighi, pp.69–70; BML MS, p.20 verso. This kind of problem is entirely standard in abacus books, and can be traced back to Islamic sources. The problem is solved by the method of double false position, giving the answer 51 pearls, 8 rubies, 22 sapphires, 19 balas rubies.

86 See Chapter 3, p.71.

7

But is it Art?

In earlier chapters we have examined the nature of Piero della Francesca's practice both as a mathematician and as a painter. The present chapter considers his work as a whole in the context of the intellectual life of his time, and in particular its relation to the learned arts taught in universities.

As a painter, Piero della Francesca would have been seen by his contemporaries as a craftsman. No doubt he saw himself that way too. However, it was not usual for craftsmen to write about their work. Evidence concerning literacy rates is hard to come by, but reading skills were presumably more widespread than writing ones, beyond the capacity to sign one's name on a legal document. It is perhaps revealing that Leonardo da Vinci, who was born about forty years after Piero, was willing to describe himself as illiterate, though in context this meant only that he knew no Latin and did not claim to possess the kind of learning associated with that language. All the same, Leonardo is probably the person Baldassare Castiglione (1478–1529) has in mind when, in his *Book of the Courtier*, published in Venice in 1528, he refers rather dismissively to a painter who not only took an interest in music, which Castiglione considers appropriate, but also concerned himself with natural philosophy.[1]

In Piero's case, his experience of writing more conventional mathematical treatises may have encouraged him to write about perspective. As we have seen, in its style, Piero's *Trattato d'abaco* closely resembles the abacus books used in schools. To judge by the number of surviving manuscripts, it seems to have been common for masters who taught in abacus schools to write such textbooks, which apparently also served to advertise their mathematical skills. Thus, although Piero seems never to have taught mathematics, the example of authorship was there for him to follow, and the fact that there was at the time no abacus school in Borgo San Sepolcro suggests he may have been asked to produce a model abacus book to show what kind of thing one learned in an abacus school.[2] All the same, it is tolerably clear that merely having written a vernacular textbook did not constitute a claim to be regarded as truly learned, which was to be learned in the subjects taught in universities. Furthermore, to judge by the nature of most of the abacus books that have survived, to have written one must normally have established nothing at all concerning the author's relation to the new learning of the humanists.

1 Baldassare Castiglione, *Il Libro del Cortegiano*, Venice, 1528, Book 2, Chapter 39; Baldassare Castiglione, *Il Libro del Cortegiano*, intro. Amedeo Quondam, notes Nicola Longo, Milan: Garzanti, 2000, p.179.

2 See James R. Banker, *The Culture of San Sepolcro during the Youth of Piero della Francesca*, Ann Arbor: University of Michigan Press, 2003; see also Chapter 3.

As we have seen, Piero's *Trattato d'abaco* is unusual in containing advanced examples in algebra, and extremely unusual in going into three-dimensional problems in geometry. It is the latter characteristic – which, as we have seen in Chapter 4, is connected with his style as a painter – that seems most relevant to examining Piero's relationship with the learned tradition of mathematics because, like writers in the learned tradition, Piero explicitly bases his work on that of Euclid, giving precise references to propositions in the *Elements*. However, noting that he used Euclid in this way does not help much in assessing Piero's relationship to specifically humanistic learning, because Euclid had always been available and had always been used. There is, moreover, a serious difficulty that may perhaps be removed by further historical research but undoubtedly bulks large at present. This difficulty lies in the unsatisfactory nature of current understanding of the learned tradition in the sciences of the fifteenth century, and to some extent of the sixteenth also.

The history of the mathematical sciences

The codification of the seven liberal arts – the *trivium* of grammar, rhetoric and dialectic and the *quadrivium* of arithmetic, geometry, music and astronomy – is usually traced back to Boethius (*c*.480–524). It becomes a recognizably 'medieval' feature of university teaching that persists into the sixteenth century and fades out, slowly, during the seventeenth. In particular, throughout this long period, 'mathematics' was recognized as consisting of the four sciences (or arts) of the *quadrivium*. For a number of reasons, one of the most significant being the important work of Nicolaus Copernicus (1473–1543), the development of astronomy has usually been taken as the pattern for understanding the history of science in the period from about 1350 to 1640. The historiographic force behind this may well contain elements of the art of the successful raconteur. Copernicus' *De revolutionibus orbium cœlestium*, published in Nuremberg in 1543, puts forward a model of the Universe in which we have a central, stationary Sun rather than a stationary Earth. The development that follows can be summarized fairly simply: the Copernican theory is at first regarded as implausible but, after astronomical work by Johannes Kepler (1571–1630) and work on terrestrial physics by Galileo Galilei (1564–1642), is gradually proved to be true, its final triumph being assured by the explanation of the heliocentric planetary system in terms of a universal force of gravitation, as given by Isaac Newton (1642–1727) in his *Philosophiae naturalis principia mathematica*, published in London in 1687. This makes a good story. And historians of science are generally agreed that it represents one strand in what was going on in mathematics and natural philosophy during the period concerned. They are also agreed that it is by no means the whole story even of the changes taking place in the theory and practice of astronomy. All the same, this part of the history of astronomy has provided a framework within which other developments are considered, so we shall need to take a closer look at it.

In this context, the first direct impact of the new learning that sought to recover ancient texts, and hence ancient skills and wisdom, is seen in the work of Regiomontanus (Johannes Müller of Königsberg, 1436–1476). The Romans relied upon the Greeks in astronomy, so in this case the texts to be recovered were Greek, the most obviously important being the summary of Greek astronomy in the *Almagest*, written by Claudius Ptolemy (flourished

A.D. 129–41).[3] Regiomontanus, who was educated chiefly at the university of Vienna, writes convincingly Ciceronian Latin. In learning Greek he relied upon the personal help of Johannes Bessarion (1403–1472), who took him to Italy. Regiomontanus' early death prevented him from completing his work on the *Almagest*, a task he had inherited on the early death of his astronomy teacher at Vienna, Georg Peurbach (1423–1461). However, since Regiomontanus was writing a kind of commentary on Ptolemy, the incompleteness of the work was not crucial. In fact, the book proved to be of great interest to astronomers. It was printed in Venice in 1496, with the title *Epytoma Ioannis De monte regio In almagestum ptolomei*, and seems to have become the standard advanced textbook on astronomy (that is, on geocentric astronomy) so that when Kepler wrote a textbook on heliocentric astronomy, he referred to the older work by calling his new one *Epitome astronomiae copernicanae*.[4]

Regiomontanus' *Epytoma* is not a translation of Ptolemy's text or even a reliable guide to its content. That is, the *Epytoma* is not a scholarly edition of the *Almagest*. Regiomontanus was chiefly engaged in extracting from Ptolemy's treatise what he thought would prove useful to astronomers of his own time. One might perhaps call the *Epytoma* a shortened recension of Ptolemy's work. The Greek text of the *Almagest* appeared in print in 1538, but there was no satisfactory Latin translation from the Greek.[5] The first printed edition of the *Almagest*, published in Venice in 1515, was of the Latin translation made from an Arabic text in 1175 by Gerard of Cremona (*c.*1114–1187). Since Copernicus tells us, in his preface, that he has been working on *De revolutionibus* for twenty-seven years – which may in fact be an understatement – the dates alone suggest that the recovery of the original Greek text of the *Almagest* did not play a significant part in his work, and closer inspection shows that this is indeed the case.

Historians have found it difficult to see Copernicus' theory as a response to any current astronomical concerns. It seems rather to have arisen from an interest in older planetary systems that had been rejected by Aristotle in *On the Heavens*. Copernicus, who had studied Greek at Padua, describes his system as 'Pythagorean'.[6] This provides a connection with humanist scholars, whose studies had given access to a wider range of ancient texts. It may be significant that one of Copernicus' fellow medical students at Padua, Girolamo Fracastoro (*c.*1478–1553) – now best remembered for his poem *Syphilis*[7] – revived another ancient astronomical system, that of the homocentric spheres of Eudoxus (*c.*408–355 B.C.).[8]

3 The original Greek title of the work translates as *Mathematical Construction in Thirteen Books*, but the Greek name became corrupted to *Greatest Construction . . .* , and the Greek word 'greatest' (μεγιστη), further distorted by its passage through Arabic, provided the Latin title *Almagestum*, by which, despite the efforts of neo-humanists, the work is still generally known. There is a learned and heavily annotated English translation by G. J. Toomer, *Ptolemy's Almagest*, London: Duckworth, 1984.

4 Kepler's *Epitome*, which is divided into several books, was published in Linz over the period 1618 to 1620, and the final book in Frankfurt in 1621. It is reprinted in KGW, vol.7.

5 An unsatisfactory one, by an otherwise unknown author, George of Trebizond, had been made in 1451 and was published in 1528. All competent judges were agreed that this translation failed to make sense of Ptolemy's mathematics. The *Almagest* is heavily mathematical, and it was indeed the mathematical parts that were of most interest to Renaissance readers.

6 For the significance of 'Pythagorean' in the Renaissance see Paolo Casini, *L' Antica Sapienza in Italia: Cronistoria di un Mito*, Rome: Il Mulino, 1999.

7 Girolamo Fracastoro, *Syphilis sive morbus Gallicus* [*Syphilis or the Gallic Disease*], Padua, 1530. Venereal disease had first appeared in Europe in the late fifteenth century.

8 Girolamo Fracastoro, *Homocentrica*, Padua, 1530.

Copernicus and Fracastoro were contemporaries at Padua from 1500 to 1501, so it is tempting to imagine they discussed the two chief world systems mentioned and rejected by Aristotle and Ptolemy, that they decided to reconstruct them, and that it then happened that Fracastoro chose the system of Eudoxus and Copernicus that of the Pythagoreans. Both Copernicus and Fracastoro no doubt had an unusual degree of interest in astronomy for its own sake, but it is worth remembering that they would have been taught the subject as part of their medical course; an understanding of astrology was required for the practice of medicine. Thus the university of Padua, which had a large medical faculty, was also prominent in the study of mathematics. The same was true of the university of Bologna, where the most famous mathematical humanist of the sixteenth century, Federigo Commandino (1509–1575), took a degree in medicine. This link persisted well into the seventeenth century, so that when Galileo taught mathematics at Padua, from 1592 to 1610, he would largely have been teaching elementary geocentric astronomy, and most of his students would have been prospective physicians.

The few surviving records of university curricula suggest that during the fourteenth and fifteenth centuries the usual introductory text on astronomy was the *Treatise on the Sphere* (*Tractatus de sphera*) written in the middle of the thirteenth century by Johannes de Sacrobosco, whose vernacular name was probably John of Holywood. In later years of study, students might also read the first two or three books of the *Almagest*. As this reading was in Latin, the translation of the *Almagest* was probably that of Gerard of Cremona – which, as we have seen, was the version that first appeared in print in the early sixteenth century. A similar degree of continuity is found in the study of arithmetic, where the standard text dated from late antiquity, and in geometry, where curricula prescribe the first few books of Euclid's *Elements* and the numbers of surviving manuscripts indicate widespread use of the mid-thirteenth century Latin translation by Campanus (Giovanni Campano da Novara, died 1296). This translation was used for the first printed edition of Euclid, published in Venice in 1482. The publication of a Greek text, in Venice in 1505, edited by Bartolommeo Zamberti, seems to have made little difference. Zamberti did re-ascribe Books 14 and 15 of the *Elements* to Hypsicles – he points out that this name is given in his manuscript[9] – but his edition otherwise merely showed that Campanus' version was thoroughly reliable. Thus the development of the mathematical sciences tends to expose a considerable degree of continuity in a period for which historians of other subjects have sought to emphasize the elements of change. Piero della Francesca's mathematics and his painting epitomize this contrast.

In learned mathematics, the impact of new scholarship began to make itself felt only in the sixteenth century. The most notable contributions were Commandino's editions of advanced geometrical texts, such as the works of Archimedes (1558), Apollonius of Perga (1566) and Pappus of Alexandria (1588). The printing of scholarly versions of ancient musical texts by Ptolemy and Aristoxenus dates from about the same time.[10] As a result of this pattern of development, which can be seen as also extending to non-mathematical sub-

9 Zamberti says the name of Hypsicles is given as author at the beginning of Book 14. Today's scholarship endorses this opinion of the authorship of Book 14 but ascribes Book 15 to an unknown later author.

10 The works of Archimedes were published in Rome, those of Apollonius of Perga in Bologna, and those of Pappus of Alexandria in Pesaro. Ptolemy's *Harmonica* and Aristoxenus' *Elementa musica* were published together, in translations by Antonio Gogava (Antonius Gagavinus), in Venice in 1562.

jects such as medicine,[11] historians of science tend to study the fifteenth century in terms of its continuity with the Middle Ages, and to place the beginnings of substantial appearances of discontinuity in the mid-sixteenth century at the earliest.

In practice, what tends to happen is that those historians of science who would describe themselves as medievalists concentrate their attention on the period before 1400, at which point they regard the stage as being more or less set for colleagues who specialize in the 'early modern' period, whose guiding concept is that of the Scientific Revolution, which is usually considered as beginning in earnest towards the end of the sixteenth century and getting properly under way in the seventeenth. (The name 'Scientific Revolution' will be explained below.) In this scheme, Copernicus appears somewhat isolated, as having produced a book whose serious import was not understood until more than fifty years after its publication. To state things thus baldly is not to imply that this picture is in any way unreasonable in regard to history of astronomy. Indeed, if we wish to make comparisons with history of art, we may note that Giotto can be made to appear isolated in rather the same way. In history of astronomy, the period from 1400 to 1500 is treated almost as a no man's land, and the parts of the Giotteschi – who all somehow fail to be Masaccio – are played by almost everybody active in the period from 1540 to 1590.

We can easily understand the uneasiness in classifying the fifteenth century as a whole as belonging either to the late Middle Ages or to the early Renaissance in regard to the mathematical sciences if we examine the work of Regiomontanus, which provides one of our earliest examples of humanist scholarship being applied to scientific texts. However, the previous generation provides an interesting example of a scholar who was concerned with mathematics for a rather different reason, namely Nicolaus Cusanus, whom we mentioned briefly in Chapter 4 for his handling of the notion of infinity.

Nicolaus Cusanus

Cusanus was born in 1401, in Kues (Latin: Cusa), a village near the city of Trier in southern Germany, and his family name was Khrypffs (there are several variant spellings). He was educated at the university of Padua from around 1417 to 1423, where he studied law, and at the university of Cologne, where he took a degree in theology. During his time in Padua he formed a lasting friendship with a fellow student, Paolo del Pozzo Toscanelli, who was to become one of the leading mathematicians of his generation. Cusanus' career was in the church: he was made a cardinal by Pope Nicholas V in 1448 and appointed Bishop of Bressanone (Brixen) in 1452. His interest in mathematics and in natural philosophy was that of the theologian. In particular, he regarded the study of mathematics as a way of leading the soul to think on higher things and thus to understand the nature of God. This is not to say that he regarded mathematics as itself divine. On the contrary, he seems to have believed it to be a human invention, but nevertheless one that provided means of illustrating divine truths.

There are plenty of historical examples to show that treating mathematics as a way of leading the soul upwards (a process described by the Greek term 'anagoge') is not neces-

11 See Vivian Nutton, 'Greek Science in the Sixteenth-Century Renaissance' in *Renaissance and Revolution: Humanists, Craftsmen and Natural Philosophers in Early Modern Europe*, ed. J. V. Field and F. A. J. L. James, Cambridge: Cambridge University Press, 1993, pp.15–28.

sarily conducive to doing good mathematics. Authors sometimes bend the mathematics to make a theological point. Cusanus, however, seems to have been a fairly competent mathematician, with an appreciation of mathematical rigour and some original ideas. In our own time, it seems to have been largely scholars with little interest in mathematics who have been inclined to describe Cusanus as a 'mystic' – an epithet that tends to the pejorative in a subject such as mathematics, where rigorous proof is the preferred method. Nearer Cusanus' own time, Johannes Kepler, an outstandingly competent mathematician who saw all natural philosophy as anagoge, thought highly of Cusanus' work. This is not to say that Cusanus' work is free from mathematical errors, but they seem to be the normal kind of error, internal to the mathematics rather than connected with its perceived theological implications.

In any case, it is clear that Cusanus' mathematical skills were taken seriously by his contemporaries, since he was asked to speak about the reform of the calendar at the Council of Basel in 1436. His oration, which was later printed, takes a long historical run-up on the matter, starting with the Chaldaeans and moving on to the Greeks, Persians, Egyptians and Romans.[12] There eventually follows a note explaining the details of the inadequacy of the calendar. These appear as difficulties with the dates of the equinoxes and thus, in Cusanus' terms, inaccuracies in the motions ascribed to the higher spheres of the heavens. This is entirely conventional, but in the midst of his technical discussion, Cusanus introduces an unexpected reference to the squaring of the circle: 'and so it is said that the higher motion is comprehensible to the human mind: as by the same intelligence the circle can be squared . . .'.[13] With hindsight, this seems less than felicitous. However, it points to a long-standing concern with this particular mathematical problem, which Cusanus does indeed firmly believe to be soluble.

In Cusanus' engagement with squaring the circle, theological motives seem to have played an important part, but he did also have an interest in mathematics for its own sake. In fact, he wrote five works specifically devoted to mathematics. One of these has a theological slant, but its mathematical content is far from negligible. Indeed, these works were of sufficient mathematical interest that, when a leading humanist of a later generation, Jacques Lefèvre d'Étaples (1455–1536), published them in the second volume of his edition of Cusanus' *Opera* in Paris in 1514, the text was accompanied by substantial mathematical commentaries.[14] Several other of Cusanus' works were also considered worthy of mathematical annotation. From the way Cusanus uses mathematics it is tolerably clear that he genuinely enjoyed the intellectual exercise it afforded, and he sometimes commends the subject as a worthy exercise. For instance, the second chapter of his *Complementum theologicum figuratum in complementis mathematicis* begins:

> No one is ignorant that in the mathematical arts truth is attained more securely than in the other liberal arts. So those who taste the discipline of geometry we see hold to it with wonderful love, as if a kind of food for intellectual life were contained within it in purer and more simple form.[15]

12 Nicolaus Cusanus, *Reparatio Calendarii*, in N. Cusanus, *Opera Omnia*, ed. Jacques Lefèvre d'Etaples, Paris, 1514, vol.2, pp.xxii–xxix. Cusanus' works were printed many times in the fifteenth and sixteenth centuries, which attests to his continuing high reputation.

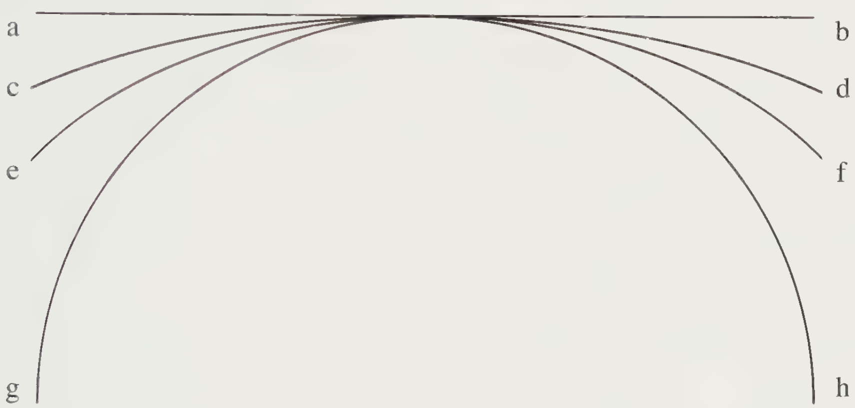
13 Nicolaus Cusanus, *Reparatio Calendarii*, in Cusanus, *Opera* (full ref. note 12), p.xxiii recto: '& ita dicunt motum superiorem per humanum ingenium compraehensibilem: sicut circulus per idem ingenium est quadrabilis . . .'.

14 The commentator, named as Omnisanctus in the text, is identified by Lefèvre in his list of contributors as 'our friend and brother Omnisanctus Vassarius, a member of an Augustinian order of canons'. Cusanus, *Opera* (full. ref. note 12), vol.1, sig.aaii verso.

15 'Nemo ignorat: in ipsis mathematicis veritatem certius attingi quam in aliis liberalibus artibus. atque ideo

Cusanus is accordingly inclined to warm to the use of mathematical imagery in the theology of others and to extend discussion on a mathematical level. A good example of his practice is to be found in chapters 12 to 15 of *De docta ignorantia*. Chapter 12, which is entitled 'How one should use mathematical symbols in this matter' ('Quomodo signis mathematicalibus sit utendum in proposito') begins:

> The most devout Anselm has compared the greatest truth to infinite straightness, and in following him let us agree to imagine the figure of straightness as a straight line. Other authorities compare the most holy Trinity to a triangle with three equal and right angles.[16] And because such a triangle necessarily has infinite sides (as will be shown) it could be said to be an infinite triangle. And these [theologians] also we shall follow.[17]



7.1 Copy of the diagram in Nicolaus Cusanus, *De docta ignorantia*, Chapter 13, showing how a circle with greater and greater diameter becomes closer and closer to being a straight line. Drawing by JVF.

Following on from this, Cusanus proceeds to show how a circle with greater and greater radius becomes more and more like a straight line. A copy of his diagram is shown in Figure 7.1. To a modern reader, the diagram may well seem largely to speak for itself. However, as we have seen in examining Piero della Francesca's mathematics, in the fifteenth century, the accompanying diagram is normally proposed merely as a supplement to a verbal argument, that is, the words have the greater authority in establishing the meaning.

eos qui ipsam geometricam degustant disciplinam in admirabili amore ipsi adhaerere videmus, quasi pabulum quoddam vitae intellectualis: ibi purius atque simplicius contineatur.' Nicolaus Cusanus, *Complementum theologicum*, Chapter II, in Cusanus, *Opera* (full ref. note 12), vol.2, p.xcii verso.

16 For a straight-sided triangle (which necessarily lies in a plane), the sum of the interior angles is two right angles (a result proved in Euclid, *Elements*, Book 1, Proposition 32). Thus, for an equilateral triangle, in which the three angles are of course equal to one another, each angle must be 60°. However, if we consider spherical triangles, that is triangles drawn on the surface of a sphere, their sides being parts of great circles, then it is indeed possible for an equilateral triangle to have three right angles. For example, imagine the equator and two meridians that are 90° apart. Theologians had presumably learned enough astronomy to come across this kind of triangle.

17 'Anselmus devotissimus: veritatem maximam rectitudini infinitœ comparavit, quem nos sequentes: ad figuram rectitudinis quam lineam rectam imaginor convolemus. Alii peritissimi: trinitati superbenedictae triangulum trium æqualium & rectorum angulorum compararunt. Et quoniam talis triangulus necessario est ex infinitis lateribus (ut ostendetur) dici poterit triangulus infinitus. & hos etiam sequimur.' Cusanus, *Opera* (full ref. note 12), vol.1, p.v recto.

In the particular case of Cusanus' discussion of the circle's becoming more nearly a straight line, there are serious problems in interpreting the text in a precise mathematical way. This is partly because of the use of the term 'infinitus' and cognates. As a theologian, Cusanus, unlike most mathematicians, was of course happy with the notion of infinity, which was one of the attributes of God, but we need to remember that in this period the adjective *infinitus* did not necessarily mean what would now be meant by 'infinite'. In principle, it designated things as indefinitely large or, in the case of a straight line, having no defined end point. Piero della Francesca speaks of 'a line without [an] end' ('linea senza fine'), meaning simply a finite line one of whose end points is not yet decided. Cusanus is clearly referring to something more like the modern 'infinite', but he is not using his terms in a way that truly allows one to decide whether he is, in the mathematical elements of his discussion, distinguishing between something that is merely indefinitely large and something that is actually infinite in the modern sense, for instance in being greater than any assignable quantity. There are several other uncertainties about Cusanus' meaning. However, what he says can be given a rigorous meaning that makes sense in modern mathematical terms. This is so not only in the discussion of the circles and the straight line but also in other passages that involve discussion of infinity – for instance the passages about triangles in chapters 14 and 15 of *De docta ignorantia*.

The great watershed in the mathematical understanding of the notion of infinity is the *Rough Draft on Conics* of Girard Desargues (1591–1661), published in Paris in 1639.[18] It is exceedingly rare for mathematical comments about infinity, made by a philosopher or by a mathematician writing before 1639, to make sense in modern terms. The most notable exception is Johannes Kepler, writing about optics in 1604.[19] Kepler is known to have admired Cusanus and probably did so on account of his blend of mathematics and theology. It is accordingly tempting to believe that we should have Kepler's support in giving Cusanus the benefit of the doubt and crediting him with truly having had a grasp of a mathematically usable notion of infinity. This is not, of course, to rob Desargues of the credit for not only having such a notion, but also expressing it with proper precision and going ahead to use it in a coherent manner, but the nature of Cusanus' handling of infinity does suggest that he is to be taken seriously as a mathematician.

This needs to be said because Cusanus repeatedly proposed solutions to a problem that is now notoriously, indeed proverbially, seen as insoluble: that of squaring the circle. The context of his proposed solutions is theological, and what interests Cusanus is not in fact the squaring of the circle, that is finding a square whose area is equal to that of the given circle, but rather the equivalent problem of rectifying the circumference, that is finding a straight line whose length is equal to the circumference of the circle (or the converse, that is finding a circle whose circumference is equal to a given straight line).[20] Solving this

18 Girard Desargues, *Brouillon project d'une atteinte aux evenemens des rencontres du Cone avec un Plan*, Paris, 1639 (annotated English translation in J. V. Field and J. J. Gray, *The Geometrical Work of Girard Desargues*, New York: Springer-Verlag, 1987). A summary of the mathematical content of the work is given in J. V. Field, *The Invention of Infinity: Mathematics and Art in the Renaissance*, Oxford: Oxford University Press, 1997.

19 In a work about optics, *Ad Vitellionem paralipomena quibus astronomiae pars optica trditur*, Frankfurt, 1604, reprinted in KGW, vol.2. See J. V. Field, 'Two Mathematical Inventions in Kepler's *Ad Vitellionem paralipomena*', *Studies in History and Philosophy of Science* 17(4), 1986, pp.449–68, and J. V. Field, *The Invention of Infinity* (full ref. note 18).

20 The equivalence of these problems is seen quite easily in today's notation, since for a circle of radius r the area will be πr^2 and the circumference will be $2\pi r$, so finding either the area or the circumference effectively means finding the value of π.

problem meant providing a 'demonstration', that is showing how the straight line in question could be constructed. It is now known that this problem is not soluble by the means allowed by Cusanus, that is the standard 'geometrical' means employed by Euclid, using only straightedge and compasses. In fact, the problem is not soluble by any means.[21] However, in Cusanus' time it seems to have been taken for granted that the problem could be solved. On the other hand, when their opinions were requested, learned professional mathematicians told Cusanus that each of his proposed solutions was faulty. Since Cusanus belonged to a highly respectable intellectual milieu, these mathematicians included not only his friend Toscanelli but also Regiomontanus.

Cusanus' works continued to be read after his death, and his final attempt at circle-squaring was published posthumously, being issued together with Regiomontanus' *De triangulis* in Nuremberg in 1533, as part of a short tract with the title *De quadratura circuli*. The editor of the volume, the astronomer Johannes Schöner (1477–1547), says in his introductory letter that he found this piece among Regiomontanus' books and considered it worth printing. As printed, the piece has Regiomontanus' name on the title page, and the lengthy title says we have a dialogue about Cusanus' *De quadratura circuli*. Regiomontanus acknowledges substantial contributions from Toscanelli, and the book includes a dialogue on the problem of the squaring of the circle in which the two participants are Toscanelli (called Paulus medicus Florentinus) and Cusanus. This is dated Brixen 1457.[22] There are several other short items, but the work concludes with a much longer piece, a detailed discussion of one of Cusanus' schemes, mainly in the form of a letter from Regiomontanus to Toscanelli. Near the beginning of this, Regiomontanus says that he of course will not send these comments to Cusanus until he has received Toscanelli's comments on them.[23] This letter is dated Venice, 5 July 1464. Cusanus died on 6 August of that year – and, in accordance with his title as cardinal, was buried in San Pietro in Vincola – so it seems likely that Regiomontanus' comments were never sent to Cusanus and the letter itself remained unpublished out of respect for the memory of the eminent theologian.

No such considerations held back Joannes Buteo (Jean Borrel, 1492–1572) when he published his *De quadratura circuli* in 1559.[24] The work is divided into two books: the first a general discussion of the problem, the second a series of refutations of methods proposed by various mathematicians, many of them the author's near contemporaries. Four of the methods are from Cusanus. Buteo's interest in the matter is purely mathematical and his motive in writing seems to have been to defend Archimedes' *On the Measurement of a Circle*, in which Archimedes obtains approximate values for the area and circumference of the circle. It is not clear to me why Buteo regarded the continued search for an exact construction as an insult to Archimedes (recognized as one of the greatest mathematicians of all time), but it seems he did. Archimedes' method allows him to set both upper and lower bounds on the area, and to obtain an approximate value to any required degree of precision.[25] As the method is rigorous, it is clear (today) that Archimedes has, in a sense, solved the problem. However, even after Archimedes' work became available in print in the six-

21 That is, π is not only an endless fraction but the kind of number now called 'transcendental'. This was proved in the nineteenth century.

22 Cusanus [and Regiomontanus], *De quadratura circuli*, Nuremberg, 1533, pp.10–12.

23 Cusanus [and Regiomontanus], *De quadratura circuli* (full ref. note 22), pp.29–43.

24 Joannes Buteo, *De quadratura circuli Libri duo, ubi multorum quadraturae confutantur, & ab omnium impugnatione defenditur Archimedes*, Lyon, 1559.

25 The method is described in more detail in Chapter 8, note 59, p.322.

teenth century, the search for an exact straightedge and compasses construction continued.[26] Buteo's firing line has Cusanus in thoroughly respectable mathematical company.

Cusanus' natural philosophy also makes use of rather more mathematics than might be expected given its Aristotelian basis, but here the mathematics is of a less innovative kind than what we find in his theology. As a natural philosopher, Cusanus follows the standard Aristotelian practice of relying heavily upon the evidence of the senses in coming to opinions about the world. From the famous line in the Book of Wisdom about God having made the world in number, measure and weight,[27] Cusanus is inclined to make the practical deduction that it is by counting, measuring and weighing that one comes to an understanding of the world. He does not, apparently, see mathematics as a tool for investigating the nature of the world, though his education and his friendly relations with Toscanelli and Regiomontanus must have ensured he was thoroughly familiar with the use of mathematics in making models of celestial motions.

The above account of some of Cusanus' uses of mathematics is not proposed as in any way complete. Its purpose is merely to show an example of the deployment of learned, university mathematics in a humanist context. Cusanus' mathematical imagery seems to indicate an aesthetic sense in geometry, and since he was a member of the court of Nicholas V, it is not surprising that attempts have been made to uncover possible connections with the new art of his time. These have proved largely unavailing. There is no evidence that Cusanus was a collector or that he ever commissioned paintings. He must certainly have seen some of the work that, say, Fra Angelico (*c.*1400–1455) executed for Nicholas V, but no trace of his personal opinions seems to remain. There is, however, a strong indication that he was personally acquainted with Leon Battista Alberti: among the volumes that survive from Cusanus' library there is a fifteenth-century manuscript of Alberti's *Elementa artis picturae*. The likeliest explanation is, of course, that the book was a gift from the author.[28] Nevertheless, the little we know of Cusanus' taste indicates a preference for Northern art: he names Rogier van der Weyden (1399–1464) as the best painter.[29] It seems that, even if he followed Alberti in perceiving a mathematical component in the Italian paintings of the early and mid-fifteenth century, this did not make him admire that style above all others.

Nor is there any evidence for Cusanus' being a channel for Albertian ideas in Brixen. At least, there is no direct evidence. There is, however, a tantalizing coincidence, namely that one of the first German artists whose works show Italian influence, Michael Pacher (born *c.*1430–35, active 1467, died 1498), lived and worked in Brixen in or, more likely, soon after Cusanus' time. Against this, pictorial evidence points to Pacher having had contact with the work of Andrea Mantegna[30] and it has been suggested that Pacher may have visited Padua in the period 1450–55.[31] There thus seems to be no reason to suppose that Cusanus

26 A brief history, paying scant attention to Renaissance work, can be found in E. W. Hobson, *Squaring the Circle: A History of the Problem*, Cambridge: Cambridge University Press, 1913.

27 Wisdom 11: 20.

28 See Eberhard Hempel, 'Nikolaus von Cues in seinen Beziehung zur bildenden Kunst', *Berichte über die verhandlung der Sächischen Akademie der Wissenschaften zu Leipzig, Philologisch-historische Klasse* Band 100, Heft 3, 1953, p.14.

29 Hempel, 'Nikolaus von Cues' (full ref. note 28), p.16.

30 Hempel, 'Nikolaus von Cues' (full ref. note 28), p.16.

31 See Nicolò Rasmo, *Michael Pacher*, trans. Philip Waley, London: Phaidon, 1971; and Artur Rosenauer (ed.), *Michael Pacher und sein Kreis* (Proceedings of Symposium, Brunico, Italy, 24–6 September 1998), Bolzano: Athesia, 1999. I am grateful to Christopher S. Wood of Yale University for this latter reference.

was a teacher of, or even an advocate for, an Albertian use of mathematics in art, particularly since his own taste seems to have been for Northern Late Gothic.[32]

Johannes Regiomontanus

Cusanus was a philosopher and theologian with a liking, and some talent, for geometry. Regiomontanus was a professional mathematician, and very much a mathematician's mathematician. Most of his work is extremely technical, being connected with the complicated mathematico–astronomical problems of the day.[33] His concern with the work of Ptolemy, which we have already mentioned, was characteristic of Renaissance humanism in being an attempt to return to the excellent system of the ancient Greeks, which it was assumed had been corrupted by later accretions and by translation into Arabic and Latin. Regiomontanus believed that a correct reading of the *Almagest* would enable astronomers to overcome current difficulties, such as those connected with the calendar. It is clear that at this time there was no groundswell of learned opinion calling for a radical reform of astronomy. It is, moreover, clear that this was still true in the following century when just such a reform was suggested by Copernicus.

The scholarship required to tackle the *Almagest* in Greek is by no means only linguistic. The text is not addressed to beginners, it is essentially an advanced textbook in celestial kinematics and much of the mathematics is far from elementary. On the other hand, as for the process of working one's way through Euclid, for someone who was very good at mathematics, as Regiomontanus was, following the mathematical argument would no doubt have helped in overcoming linguistic problems. Regiomontanus does indeed complain, in the dedicatory letter addressed to his patron Cardinal Bessarion, that the *Almagest* is difficult, both for its Greek and for its mathematics. Nonetheless, as we have already mentioned, the resulting book, the *Epytoma*, was highly successful in the sense that it proved to be useful to following generations of astronomers. It is essentially upon this work that Regiomontanus' historical reputation rests. For our present purposes, however, it is of greater interest to look at a work on geometry, *De triangulis omnimodis* (On triangles of all kinds), written about 1462 and conceived as an introductory work, to be read before the recension of the *Almagest*.[34]

Whatever the subject one is writing about, it is a fairly normal practice among scholars to write the book first and then write the introduction. Regiomontanus did this in a more than usually explicit manner: although he never actually completed work on his recension of the *Almagest*, he wrote another, separate, treatise as an introduction to it. This introductory work, *De triangulis*, was first published in 1533, and, as we have already mentioned, Regiomontanus' short discussion of Cusanus' circle-squaring, *De quadratura circuli*,

32 Hempel, 'Nikolaus von Cues' (full ref. note 28), p.30. For a detailed discussion of the possible connections between Cusanus' philosophy as a whole and the work of Leonardo da Vinci see Robert Zwijnenberg, *The Writings and Drawings of Leonardo da Vinci: Order and Chaos in Early Modern Thought*, trans. Caroline van Eck, Cambridge: Cambridge University Press, 1999.

33 The classic study of Regiomontanus has fairly recently been translated into English: Ernst Zinner, *Regiomontanus: His Life and Work*, trans. Ezra Brown (Studies in the History and Philosophy of Mathematics, vol.1), Amsterdam and New York: North-Holland, 1990.

34 For the date of *De triangulis*, see Menso Folkerts, 'New Results on the Mathematical Activity of Regiomontanus', in Zinner, *Regiomontanus* (full ref. note 33), pp.363–72, esp. p.365.

was issued with it.[35] The prefatory letter to the readers of *De triangulis* begins by explaining that, although this work was written second, it needs to be read before the other one:

> Although I wrote these short books on triangles after the epitoma, taking things in reverse order by composing the introductory work second, still I did present the [complete] art: for no one who omits to learn about our triangles will have an adequate understanding of the study of heavenly bodies.[36]

(My translation of this passage is a little free, since Regiomontanus' Latin is authentically Ciceronian in its accumulation of participles, gerunds and subordinate clauses, and thus defies easy rendering into readable English.)

De triangulis has a narrow focus: it is about 'solving' triangles, that is, given some sides and angles, finding the remaining ones. Regiomontanus' first two books deal with plane triangles; the remaining three consider spherical ones, which are indeed necessary for prospective students of astronomy. We do not seem to have an independent study of the properties of triangles. Rather, the whole work seems to be written with an eye to its usefulness in practical calculations. All the same, its systematic nature, and Regiomontanus' unfailing rigour and occasional elegance, make *De triangulis* interesting in its own right. Such merits probably explain why the work has sometimes been hailed as the first treatise on trigonometry. The claim seems to be something of an exaggeration. There is much about triangles in many earlier works: in Euclid, in Leonardo of Pisa's *Geometria* (thirteenth century), in many practical texts addressed to surveyors, and indeed in Leon Battista Alberti's *Ludi matematici*, which proposes what are essentially surveying problems in recreational form. Accordingly, to say Regiomontanus' *De triangulis* is the first text on trigonometry is somewhat misleading, except in the extremely narrow sense that the work does indeed, as its title suggests, deal only with triangles. Nonetheless, Regiomontanus does not really propose trigonometry as a subject in its own right. As we have seen, he says in his letter to the reader that the work is conceived as providing specialized know-how required for tackling problems in astronomy. The work does, all the same, have quite a lot in common with the trigonometry textbooks from our own time, because it works systematically through various kinds of triangle and, in the modern manner, gives general methods of solution as well as numerical examples.

Regiomontanus starts with definitions and simple theorems, first for right-angled triangles. There are references to Euclid and references back to his own earlier results, that is we have an orderly series of propositions. Most are phrased as problems. For instance, Book 1, Theorem 26 is 'In any right-angled triangle when two sides are known to deduce the third one'.[37] Each theorem is stated and proved in general terms, but there is often a numerical example by way of additional explanation. Numerical examples were sometimes used in explaining Euclid, par-

35 Regiomontanus' *De triangulis* is available in a modern photographic reprint with English translation on facing pages: Regiomontanus, *Regiomontanus On Triangles: De triangulis omnimodis by Johann Müller, otherwise known as Regiomontanus*, trans. Barnabas Hughes O. F. M., Madison and London: University of Wisconsin Press, 1967. This reprint does not include the treatise on squaring the circle. Although helpful, Hughes' translation is not completely reliable.

36 'Quamvis hosce triangulorum libellos post epitoma conscripserim præpostero fretus ordine, posterius quidem opus texendo introductorium, atque artem ipsam tradiderim: nemini tamen triangulos nostros prætereunti astrorum disciplina satis agnoscetur.' Latin text on A3 recto in the 1533 edition and Regiomontanus, *Regiomontanus On Triangles* (full ref. note 35), p.26. The translation is my own.

37 Regiomontanus, *De triangulis*, Nuremberg, 1533, p.24; Regiomontanus, *Regiomontanus On Triangles* (full ref. note 35), pp.64–5.

ticularly in school texts, so Regiomontanus' use of them here merely confirms the essentially pedagogical character of the work. What marks the treatise as being addressed to prospective scholars, rather than to prospective craftsmen, is that the references to Euclid are not only numerous but also in a form that suggests the reader is expected either to know the propositions of the *Elements* already, or to have easy access to a copy of the work. For instance, Piero della Francesca repeatedly simply assumes his reader knows that in a right-angled triangle the square on the hypotenuse is the sum of the squares on the two other sides; Regiomontanus states explicitly that this is 'by the penultimate of the first' ('per penultimam primi'), that is, Proposition 47 of *Elements*, Book 1. Regiomontanus' work is indeed 'practical' in the sense that, as we have seen, he envisages its having an application in astronomy, but he nonetheless writes as one who is aware of constructing an orderly system of mathematical results.

There are also overall differences in the content of *De triangulis* compared with works addressed to surveyors. One of the most striking is that, aware that astronomers measure angles, Regiomontanus is much concerned with the sizes of angles in triangles. In contrast, Piero della Francesca's problems ask only about sides, heights and areas. There are other characteristics of Regiomontanus' work that also point to its application in astronomy, such as in Book 1, Theorem 27, where the problem is to find the angles of a right-angled triangle when one knows the sides. Here it is suggested that the reader should use a table of sines.[38] Tables of sines were to be found in astronomy books (such as the *Almagest*) but, as far as I know, no writer of a 'practical mathematics' text ever refers to them. In his way, however, Regiomontanus is practical too: at the end of this section he gives fairly detailed instructions on how to carry out the calculation.

There is little overlap between Regiomontanus' problems on triangles and those found in the geometrical part of Piero della Francesca's *Trattato d'abaco*, but Regiomontanus does treat the two problems found at the end of Piero's algebra section. These are, in Piero,

> There is a triangle which squared is 100 *bracci* [that is, its area is 100], and its sides are in sesquitierce proportion [that is, in the ratio 4:3]. I ask for the size of its sides.[39]

and

> There is a triangle, which has base 12 and its height is 10, and the two other sides add up to 24. I ask what each is.[40]

Piero solves both of these problems by algebra, in each case setting one of the unknown sides as the 'thing' ('cosa'). Regiomontanus solves his corresponding problems by geometry.

The first of them is found in *De triangulis*, Book 2, Theorem 10, in the form

> Given the area of a triangle and the proportion of the sides, it is possible to find each of them, and to find its angles.[41]

The additional part about the angles is not of much mathematical import, since once all three sides have been found the triangle is completely defined, but, as already noted, the concern

38 Regiomontanus, *De triangulis*, pp.24–5; Regiomontanus, *Regiomontanus On Triangles* (full ref. note 35), pp.64–7.

39 Piero della Francesca, *Trattato d'abaco*: BML MS, p.79 recto; Piero ed. Arrighi, p.168.

40 Piero della Francesca, *Trattato d'abaco*: BML MS, p.79 verso; Piero ed. Arrighi, p.169.

41 Regiomontanus, *De triangulis*, 1533, Book 2. Theorem 10, p.50; Regiomontanus, *Regiomontanus On Triangles* (full ref. note 35), p.116.

with angles is typical of Regiomontanus, and one of the differences between his work and Piero's. Another difference, characteristic of the learned tradition in general, is that Regiomontanus states the theorem in general terms, rather than presenting it as a numerical example. As we have seen, Piero adopts this style in his treatise on perspective, but he almost never uses it in the *Trattato d'abaco*. This contrast is seen once more in Regiomontanus' version of Piero's final algebra problem, which appears in *De triangulis*, Book 2, Theorem 16:

> Given the base of any triangle, and the height from it, and knowing the sum of the sides, to find each of them.[42]

Regiomontanus' solution to this problem is, again, geometrical and given in general terms.

Thus far, what has been said of *De triangulis* portrays the work as entirely in the Latin geometrical tradition, providing geometrical solutions where Piero, writing in the vernacular, had recourse to the 'arithmetical' technique of algebra, which is typical of the practical tradition. However, there are a few problems in which Regiomontanus too uses algebra. One such is Book 2, Theorem 12, which states

> Given the height and the base, and knowing the proportion of the sides, to know the third side also.[43]

Regiomontanus' solution begins

> As yet, it has not been possible to solve this problem by geometrical means, but we shall try to do it by algebra.[44]

That is to say that he regards geometry as the method of preference, but is prepared to use algebra if baulked. We may notice, also, that he seems to assume that a geometrical solution will eventually be forthcoming. With hindsight, sharpened by work done by Carl Friedrich Gauss (1777–1855) at the end of the eighteenth century, we can see that Regiomontanus is right in principle. The equation he obtains (which he does not, of course, write out in symbolic notation but expresses in words) is merely a quadratic one, so the problem could be solved by means of straightedge and compasses. It was about a century after Regiomontanus' time that mathematicians first began to realize that algebraic methods could sometimes be more powerful than geometrical ones. It is thus somewhat anachronistic to see the works of the practical tradition, with their freer use of algebra, as in any sense more advanced than the learned works that employ only geometry. However, algebra was later to prove a very important acquisition by the learned tradition from the practical one. Regiomontanus' *De triangulis* is accordingly of interest in showing us an early example of the importation of algebra into a learned treatise. Unfortunately, there seems to be no trace of where and how Regiomontanus learned his algebra.

Like Piero della Francesca, Regiomontanus shows an interest in special triangles such as equilateral and isosceles triangles, where a modern textbook would provide a result for a general triangle and leave it as an exercise for the reader to see how the result could be used

42 Regiomontanus, *De triangulis*, 1533, Book 2, Theorem 16, p.53; Regiomontanus, *Regiomontanus on Triangles*, (full ref. note 35), p.122.

43 Regiomontanus, *De triangulis*, 1533, Book 2, Theorem 12, p.31; Regiomontanus, *Regiomontanus on Triangles* (full ref. note 35), p.118.

44 Regiomontanus, *De triangulis*, 1533, Book 2, Theorem 12, p.31; Regiomontanus, *Regiomontanus on Triangles* (full ref. note 35), p.118. Regiomontanus' name for algebra is 'ars rei et census', that is 'the art of the thing and the power'.

in special cases. However, despite this common element in their style, the overlap in particular problems in Piero and Regiomontanus is not extensive. All the same, some of the differences found in areas of overlap are rather revealing. For example, one of the problems found in both texts is that of finding the height of an equilateral triangle. This is Piero's first problem in the section of his work devoted to geometry, and it comes in the form of a numerical example:

> Let the triangle ABC have equal sides, and let each be 10 *bracci*. I ask what is its height.[45]

Regiomontanus comes to the same problem in *De triangulis*, Book 1, Theorem 34, and he states it in general form:

> The side of any equilateral triangle is, in square, in sesquitierce proportion [ratio 4:3] to its height. Hence if the side is known it follows that the height is known also, and conversely.[46]

The contrast here is not only in the style of exposition, which epitomizes the difference between vernacular and Latin works, but also in the importance given to the result itself. For Piero, this result is the starting point for his geometry problems, and he expects his readers to be interested in the heights of triangles because they will need to compute areas. Regiomontanus comes to the same problem much later in his work, and for him it is just one more case of finding a property of a given triangle. However, as we have seen, most of the propositions in the first book of *De prospectiva pingendi* are in as abstract a style as that of *De triangulis*, so once it was translated into Latin that part of the perspective treatise could easily fit into the learned tradition.

One of the characteristics of learned treatises was the habit of proving mathematical results, or at least giving exact references to where they had been proved by Euclid. *De prospectiva pingendi* conforms to this model, but on the whole abacus books, Piero's included, do not. For example, when Piero comes to consider the circle, he wishes to draw a circle passing through the three vertices of a triangle (such a circle is now called the circumcircle of the triangle). The problem of constructing the circumcircle of a given triangle (that is, of any triangle) is found in Euclid, *Elements*, Book 4, Proposition 5, but Piero is dealing with a particular triangle, and he of course requires not a construction but to know the size of the circumcircle:

> There is a triangle ABC, such that side AB is 14, and BC 12, and AC 10; around it I want to draw the smallest circle that is possible, such that it touches all the angles of the said triangle. I ask what will be its diameter.[47]

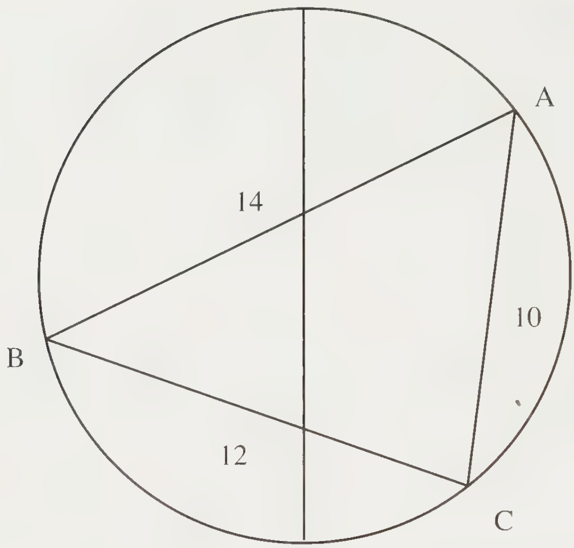
A copy of the accompanying diagram is given in my Figure 7.2. It turns out that Piero has a neat little formula for solving this problem. His solution begins

> You must know that multiplying two sides of a triangle one with another and dividing the result by the height gives the diameter of the circle that contains the said triangle.

45 Piero della Francesca, *Trattato d'abaco*: BML MS, p.80 recto; Piero ed. Arrighi, p.169. Piero's methods of finding the height are discussed in Appendix 2. Arrighi's transcription omits the dots that Piero uses in the BML manuscript, as in *De prospectiva pingendi*, to isolate Arabic numerals and letters used for points, lines and figures. I have chosen to follow Arrighi rather than the BML manuscript in this respect.

46 Regiomontanus, *De triangulis*, 1533, Book 1, Theorem 34, p.29; Regiomontanus, *Regiomontanus on Triangles* (full ref. note 35), p.74.

47 Piero della Francesca, *Trattato d'abaco*: BML MS, p.95 recto; Piero ed. Arrighi, pp.201–2.



7.2 Piero della Francesca, *Trattato d'abaco*, diagram for the circumcircle of a triangle, BML MS p.94 verso. Drawing by JVF.

The solution accordingly proceeds smoothly:

> So first find the height, which you will find is the root of 96; then take AB and AC as roots that is 14 by 14 makes 196 and 10 by 10 makes 100. Now multiply one with the other, 100 by 196 makes 19600, divide by 96 [which] gives $206\frac{1}{24}$, and the root of $206\frac{1}{24}$ is the diameter of the smallest circle that contains the triangle ABC.

This is, of course, entirely acceptable in the context of an abacus book, but it naturally irritates the historian that Piero gives no source for his neat formula for the diameter of the circumcircle, which is not in Euclid. In fact the result is fairly easy to prove (employing the typical fifteenth-century method of using the properties of pairs of similar triangles), but Piero is not likely to have expected his readers to ask for a proof. Neither his text nor his diagram gives any hint for finding one.

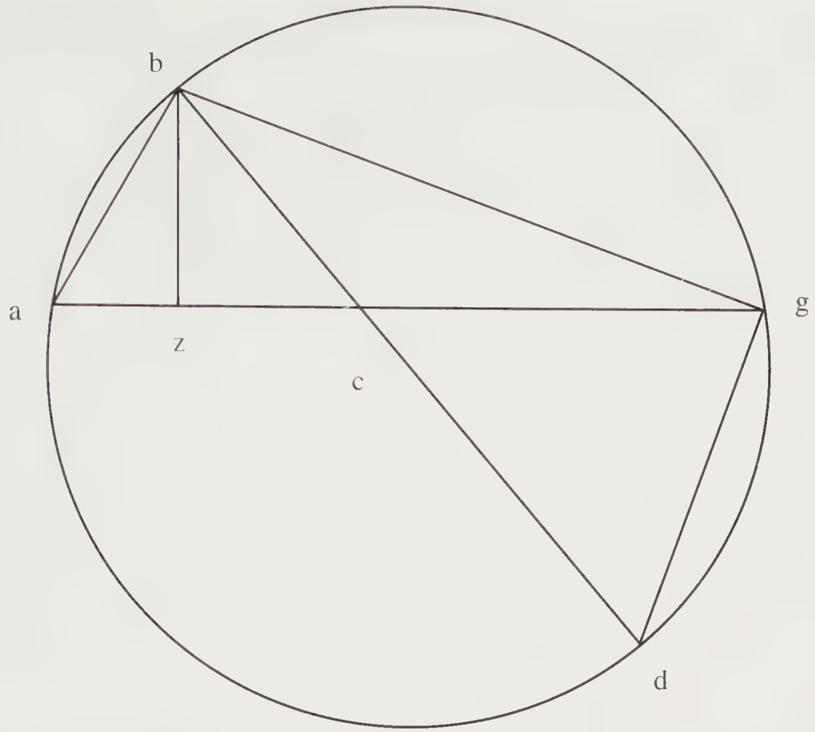
It so happens, however, that we fare much better if we turn to Regiomontanus. *De triangulis*, Book 2, Proposition 24, deals with the same problem (a copy of his diagram is given in Fig. 7.3):

> Given the three sides of a straight-sided triangle [that is a plane triangle], to find the diameter of the circle that circumscribes it.[48]

Regiomontanus feels the need to explain why this proposition has been included, and says

> Although this has nothing to offer in regard to finding the angles of the triangle, it will be seen to be useful in what follows.

48 Regiomontanus *De triangulis*, 1533, Book 2, Theorem 24, p.56; Regiomontanus, *Regiomontanus on Triangles* (full ref. note 35), p.128.



7.3 Diagram for finding the diameter of the circumcircle of a given triangle, after Johannes Regiomontanus, *De triangulis*, Book 2, Theorem 24, 1533 edition, p.56. Drawing by JVF.

This remark shows again that we are not reading a treatise on trigonometry, a study of triangles for their own sake. Regiomontanus' attention is narrowly focused and he is inclined to economy in attaining his mathematical purpose of providing the results needed by a prospective astronomer. His proof, which proceeds by using the similar right-angled triangles *abz* and *dbg*, makes several references to propositions in Euclid.

Regiomontanus' proof is straightforward, but that seems to be a consequence of the result itself being simple, rather than of any great ingenuity on Regiomontanus' part. This remark is not intended as a reflection on Regiomontanus' mathematical skill, which was undoubtedly great, but is rather by way of pointing out that it seems perfectly in order to suggest that had Piero della Francesca come across the result he would have had little trouble in devising a proof, which means that he would have been able to satisfy himself that the result was true.[49] There is, of course, no reason whatsoever to suppose that Piero read Regiomontanus' manuscript. We would not expect to find the formula that Piero quotes in many abacus books, for the simple reason that they usually contain so little geometry, but it does seem to have been known in the practical tradition also. It occurs, without proof, in a man-

49 In fact, I myself had worked out exactly this proof before I stumbled across it in *De triangulis*. Piero was certainly a better mathematician than I am.

uscript abacus book written in Florence about 1460.[50] The writer of the manuscript identifies himself as a pupil of Domenico d'Agostino Vaiaio. As the text contains passages in which Domenico speaks in the first person, it seems likely that the mathematical content should be ascribed to him. This is not, of course, to imply that the content is necessarily original to Domenico. He does, however, seem to lay claim to the formula for the diameter of the circumcircle. In any case, its occurrence in this manuscript indicates that the formula was known in a milieu in which Piero could have been personally acquainted with some of the people involved. There might even have been a commercial connection with Piero's family, since Domenico Vaiaio was probably, as his name implies, a dealer in skins. This suggestion is, of course, highly speculative, but the essential point remains: we do not have to seek far for a possible source for the formula that Piero quotes.

The place of Piero's mathematics

The fact that Piero della Francesca discusses the circumcircle of a triangle and gives a formula for its diameter may be ascribed to his having a livelier interest in geometry than most writers of abacus books. However, it also tends to locate Piero's *Trattato d'abaco* among the more advanced texts of its type. Another indication of this is Piero's introduction of Heron's formula for finding the area of a triangle directly from the three sides. As we have seen, it was usual to find the area by using one side of the triangle and the corresponding height, finding the latter by a preliminary calculation if necessary. This is what Piero has just done when he introduces this alternative method, saying 'You can know it [the area] without finding the height'.[51] If written out as one would write it now, Heron's formula looks much neater than the other method. It tells us that if we call the lengths of the sides a, b, c and define s (the semiperimeter of the triangle) as $\frac{1}{2}(a + b + c)$, then the area of the triangle is given by $\sqrt{s(s-a)(s-b)(s-c)}$. This formula, which in Piero's time was known in both the learned and the practical traditions,[52] looks a little less attractive in practice, however, if one considers the difficulty of taking the square root of a large number. This may well explain why knowing Heron's formula does not discourage Piero from also giving other methods of finding the area. In any case, even in his treatment of the standard problems of practical geometry Piero gives strong indications of being well up on the subject.

This sits well with historians' general judgement of Piero's algebra. Because the development of algebra was to have significant effects on the development of mathematics as a whole in the sixteenth and seventeenth centuries, Piero's contributions in this area have received considerable attention, in particular because his work was used in Luca Pacioli's *Summa de arithmetica . . .* , published in Venice in 1494, which became a standard textbook. Since algebra developed rapidly during the second half of the fifteenth century, Piero's work soon appeared dated, even in the reworked form in which some of it appeared in

50 *Alchuno caso sottile: La quinta distinzione della 'Pratica di Geometria' dal Codice Ottoboniano Latino 3307 della Bibloteca Apostolica Vaticana*, ed. Annalisa Simi, Quaderni del Centro Studi della Matematica Medioevale, ed. L. Toti Rigatelli and R. Franci, no. 23, Siena, *c.*1998, see p.33. I am grateful to Dr Simi for telling me about this manuscript and kindly giving me a copy of her publication.

51 Piero della Francesca, *Trattato d'abaco*: BML MS, p.80 verso; Piero ed. Arrighi, p.170.

52 See Menso Folkerts and Richard Lorch, 'Some Geometrical Theorems Attributed to Archimedes and their Appearance in the West', in *Archimede: Mito Tradizione Scienza*. ed. Corrado Dollo (Proceedings of a conference held in Syracuse, October 1989), Biblioteca di Nuncius, Studi e Testi IV, Florence: Olschki, 1992, pp.61–79, esp. p.73ff.

Pacioli's *Summa*,[53] but it is clear that in his own time Piero was at least thoroughly up to date, though there is no evidence for his having made any original contribution to algebra.[54]

Historians' concentration on algebra, which is understandable within the framework of the history of mathematics (that is, mathematics as defined in our own time) has tended to diminish Piero's stature as a mathematician by distracting attention from his contributions to geometry. These are, of course, significant in the present context since, as we have seen, they allow us to relate Piero's work as a mathematician to some aspects of his work as a painter. Moreover, once we see Piero in the context of his craft skills, as mathematician and painter, it may be possible to place him not simply in the story of the development of mathematics, but also in the much more historically significant developments in the mathematical sciences as a whole, which led to changes in their relation to natural philosophy, and to the emergence of a mathematized world picture such as we find in the work of Newton. The period during which this development took place is generally given the title 'the Scientific Revolution', a name derived from the title of a book first published in 1954.[55] There has been a certain amount of dissension among historians of science about this title, essentially on the grounds that what comes out of the revolution should not be called science in today's sense. I am happy to concede this last point. It in no way alters the fact that the title of Newton's *Mathematical Principles of Natural Philosophy*, made sense when the book was published in 1687 but would have seemed bizarre two centuries earlier. In this context, Piero's geometry is important because it links him to the learned tradition.

In the fifteenth century, the discipline of geometry was seen, entirely correctly, as having an impeccable ancient pedigree. The ancient pedigree of algebra did not emerge until Diophantus' *Arithmetica* (*c*.250 A.D.) was rediscovered and published in the late sixteenth century (first printed edition Bologna, 1572), though the first treatise on algebra to be written in Latin, thus claiming rights of citizenship among the learned, had appeared some decades earlier, in the form of Girolamo Cardano's combatively titled *Ars magna*, published in Basel in 1545. (We may note that the same publisher, Petreius, issued this work and Regiomontanus' *De triangulis*, as well as Copernicus' *De revolutionibus*.) In the fifteenth century, arithmetic was either traced back to Euclid's *Elements*, where it is clearly subordinated to geometry – in the sense that arithmetical results can be deduced from geometrical ones, but not vice versa – or it was taught from the humbler textbook of Nicomachus of Gerasa (active 100 A.D.). Algebra was regarded as an outgrowth of arithmetic, and taken to be Islamic in origin.

As we have seen in the work of Cusanus and Regiomontanus, geometry provides the pattern for what is considered best in mathematics. It is noticeable, also, that Alberti's borrowings from the practical tradition are connected with geometrical matters, largely to do with surveying. This is perhaps to be expected in view of his interest in building, but it is clear that some arithmetic might have been useful too, for instance in calculating the cost

53 See Enrico Giusti (ed.), *Luca Pacioli e la matematica del Rinascimento. Atti del convegno internazionale di studi. Sansepolcro 13–16 aprile 1994*, Città di Castello: Petruzzi Editore, 1998.

54 See R. Franci and L. Toti Rigatelli, 'Towards a History of Algebra from Leonardo of Pisa to Luca Pacioli', *Janus*, 72, 1985, pp.17–82; for an unusually dismissive view, see Enrico Giusti, 'L'algebra nel *Trattato d'abaco* di Piero della Francesca: osservazioni e congetture', *Bolletino di Storia delle Scienze Matematiche* 11.2, 1991, pp.55–83.

55 The author of the book was Rupert Hall. On his choice of the title, see A. R. Hall, 'Retrospection on the Scientific Revolution', in *Renaissance and Revolution: Humanists, Craftsmen and Natural Philosophers in Early Modern Europe*, ed. J. V. Field and F. A. J. L. James, Cambridge: Cambridge University Press, 1993, pp.239–49.

of materials, which was assuredly an important consideration in deciding upon a design. Assessing the quantity of materials was also a significant part of the planning of a building since, for instance, bricks were fired to order in batches of a certain size, the kiln being broken to pieces after the firing to get the bricks out.[56] Given the priority generally accorded to geometry by the learned, it is not surprising that it was the geometrical parts of Piero della Francesca's mathematics that were translated into Latin.

In *De prospectiva pingendi*, whose Latin title – used in the vernacular texts as well as the Latin ones – must be seen as staking a claim to a connection with the learned tradition, the style of the theorems is more or less that of Euclid. The worked examples of finding perspective images of course belong strictly to a practical tradition, consisting as they do of long and detailed drawing instructions. As we have seen, in their bulk these examples dominate the treatise. They provide a jarring contrast from the mathematical point of view, but since they were included in the Latin translation, one must suppose that the learned reader, perhaps with appetite whetted by the writings of Alberti, took an interest in the practical side of the craft of designing pictures.

However, the learned reader would surely have felt less at home with the *Libellus de quinque corporibus regularibus*, whose style is entirely that of practical treatises, worked example succeeding worked example. The content of the *Libellus* is largely drawn from the *Trattato d'abaco*, but the differences are sufficiently extensive, and of sufficient mathematical import, to make it certain that – unless we have some inexplicable fakery – Piero was involved in the task of producing the *Libellus*, probably in the vernacular. It is conceivable that if the work was composed specifically for Guidobaldo da Montefeltro – and, as we have seen, it is certain only that the single surviving copy was written for him, which is not at all the same thing – then whoever had the idea of commissioning it, apparently as a companion piece to the perspective treatise already in the Ducal library, expected that the geometrical parts extracted from the *Trattato d'abaco*, and the additional sections specially written for the new work, would be cast or recast in a style close to that of the mathematics at the beginning of the perspective treatise, that is in the learned manner of Euclid's *Elements*. Given the nature of the content of the *Libellus*, this could indeed have been done for almost all of the work. However, it was not done. The result is a work that looks rather anomalous in Latin.

We know that the Latin version of the perspective treatise was copied several times, since there are differences between the four surviving copies. This is a strong indication that the work found a readership. The fact that we know of no copies of the *Libellus* except the one made for Guidobaldo da Montefeltro suggests that this text was not read as widely as the other one. It is probably significant that when Pacioli published the *Libellus* in his *De divina proportione* of 1509, he did so in the vernacular, despite his volume as a whole having a Latin title that must, presumably, be construed as a bid for attention from the learned.

The learned craftsman

It seems that, by the standards of craftsmen of the time, Piero della Francesca was rather well educated, particularly in mathematics. This distinction as a mathematician is certainly

56 See R. Goldthwaite, *The Building of Renaissance Florence*, Baltimore and London: The Johns Hopkins University Press, 1980.

partly due to his temperament; as Vasari implies at the beginning of his *Life*, Piero seems to have had a natural aptitude for mathematics, for arithmetic as well as for geometry. Nevertheless, Piero's social circumstances are also relevant. In some places, municipal schools made it possible for a limited number of poor boys to receive elementary instruction, but it is obvious that in general wider opportunities were open to the well-to-do. So Piero started with an advantage. On the other hand, it perhaps would not have mattered much. The guilds themselves sometimes ran schools, and apprentices of sufficient aptitude must also have acquired some intellectual skills as well as practical ones in the workshop. [57] The system must have been fairly effective because the social origins of skilled craftsmen were diverse, but it is worth noting that at least three of the early practitioners of perspective come from relatively privileged educational backgrounds.[58] The fathers of Filippo Brunelleschi and Masaccio were both notaries, and Donatello was educated by his rich cousins in the banking branch of the Bardi family, who presumably would not have taken on the task unless the boy seemed clever enough to profit from such an education.

Piero is by no means alone among his fellow craftsmen in being capable of writing books and showing signs of humanist culture, such as some acquaintance with the writing of Vitruvius. However, for Brunelleschi, the evidence is somewhat ambiguous. His earliest biographer, Antonio di Tuccio Manetti, does not refer to his having written any books. Some engineers of his time did do so, but their books were addressed to individual noble patrons and most have a partly military content, which is to say that they came from social situations notably different from Brunelleschi's.[59] It is far from clear that Brunelleschi would have had any strong motive to write about his perspective method, since the panels he painted demonstrated it, and the technique would presumably have been passed on in the usual workshop manner – though the mind rather boggles at the idea of anyone giving Donatello or Masaccio a drawing lesson.

Alberti, on the other hand, while writing not only about matters of taste and judgement, but also about matters concerned with the practice of a craft, dedicated *De pictura* to his patrons, and in addressing the learned Gonzagas he naturally wrote in Latin. His descriptions of perspective are too schematic to be regarded as a substitute for a technical account such as Brunelleschi might have written and such as Piero did write.[60] Thus Alberti's dedication of the vernacular version of *De pictura* to Brunelleschi is presumably to be construed as a gesture that invites him to become involved in the humanist programme that Alberti associates with the new style in art. As noted in Chapter 2, in *Della pittura* no less than its Latin original Alberti expects the reader to have some detailed knowledge of classical literature and mythology. If Alberti really expected Brunelleschi to read *Della pittura*, he was in principle assuming that Brunelleschi knew quite a lot about ancient literature, but it is, of course, possible that Alberti did not really expect Brunelleschi to follow the classical allusions. Alberti was, however, definitely making a show of treating Brunelleschi as if he were

57 On guild activity in Borgo San Sepolcro, which did not include running an abacus school, see James R. Banker, *The Culture of San Sepolcro* (full ref. note 2).

58 See Peter Burke, *The Italian Renaissance: Culture and Society in Italy*, Cambridge: Polity Press, 1986.

59 Some of these engineering books are discussed in Paolo Galluzzi, *The Art of Invention: Leonardo and Renaissance Engineers*, trans. M. Mandelbaum, M. Gorman, L. Otten and K. Singleton, Florence: Giunti, 1999; see also Pamela O. Long, 'Power, Patronage, and the Authorship of *Ars*: From Mechanical Know-how to Mechanical Knowledge in the Last Scribal Age', *Isis* 88/1, 1997, pp.1–41.

60 See Chapter 2 and J. V. Field, 'Alberti, the Abacus, and Piero della Francesca's Proof of Perspective', *Renaissance Studies* 11/2, 1997, pp.61–88.

an educated man, albeit one who does not read Latin (or perhaps merely does not read it fluently). In a later generation, painters and sculptors certainly did need to find out about parts of classical mythology, such as the story of the birth of Venus or of the labours of Hercules, but their paintings and sculptures cannot tell us how deep such knowledge went. After all, it is not customary for art historians to regard artists as experts in theology, whatever subtleties of meaning may be read out of their works.

Alberti's knowledge of ancient art, as displayed in *On Painting*, seems to be derived from Pliny's *Natural History*, Books 35 and 36. Pliny, who describes some actual works, is still our most detailed source on the matter, and there is no reason to doubt the soundness of Alberti's scholarship. In contrast to Alberti's, Piero's references to ancient painters, in the introduction to the third book of his perspective treatise, seem to be almost entirely taken from Vitruvius, who merely supplies a list of names.[61] Further, part of the prefatory letter to Piero's *Libellus de quinque corporibus regularibus* in the manuscript version in the Vatican Library is a paraphrase of a passage from Vitruvius.[62] Both Piero's uses of Vitruvius could be explained away. The second may have been inspired by the translator of the *Libellus*, who was probably Maestro Matteo di Ser Paolo d'Anghiari (see Chapter 3), for no such preface is found in the vernacular text printed by Pacioli in *De divina proportione*. The first, the list of names, is not truly proven to be a use of Vitruvius since Piero has an additional name, that of Apelles, which is not found in the later accepted text of Vitruvius, and this may point to his having made use of an intermediate source. However, from looking at his pictures one may reasonably deduce that Piero was interested in ancient buildings.[63]

It seems perfectly possible that his friend and translator may have been influential simply by lending Piero a copy of *De architectura*. Maestro Matteo had a substantial library, though what we know of its contents suggests that most of the volumes related to his work as a lawyer.[64] As we have seen in Chapter 3, though Piero did not make his own Latin translation of his perspective treatise, he seems to have been capable of reading the translation – perhaps checking the mathematics had been rendered correctly – and could write a few words of Latin in the margin. It thus seems likely that he could also read other Latin texts to some extent. The amount of linguistic skill required for understanding Vitruvius would, however, be considerably greater than that needed for understanding a Latin Euclid. In any case, given that Piero obviously had a serious interest in architecture, it may be worth entertaining the conjecture that the essay on architecture in Pacioli's *De divina proportione*, which seems unconnected with Pacioli's other work – but may of course be connected with his friendship with Leonardo da Vinci – might, like the last part of the volume, contain unacknowledged borrowing from a manuscript by Piero that Pacioli acquired after Piero's death.

De prospectiva pingendi is mathematical, which is to say that most of the optical results Piero uses are geometrical, derived from Euclid's work on optics. As we have precise references to specific propositions, it seems likely Piero was actually using Euclid. It is only in

61 Vitruvius, Introduction to *De architectura*, Book 3. On Piero's knowledge of this text see Chapter 3.

62 For details see Appendix 7. The Vitruvius passage is *De architectura*, Book 3, preface, §2.

63 On his use of surviving buildings as models, see Christine Smith, 'Piero's Painted Architecture: Analysis of His Vocabulary', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no. 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art, 1995, pp.223–53.

64 See James R. Banker, 'Piero della Francesca's friend and translator: Maestro Matteo di Ser Paolo d'Anghiari', *Rivista d'Arte*, 4th series, vol.8, 1992, pp.331–40.

the final proposition of his first book, where Piero is considering the maximum angle that the eye can take in, that we find something that is not in Euclid. Wording very like Piero's is found in the widely read optical treatise of John Pecham, written in the thirteenth century and known in the fifteenth as *Perspectiva communis*. This work is accordingly the likeliest source for Piero's discussion of the eye. However, it was by no means the only work on optics known in the fifteenth century.

In optics as in astronomy, most of the works used in the fifteenth century dated from the thirteenth. Moreover, like that of astronomy, the historiography of optics also shows a tendency to leap over the fifteenth century, in this case cutting from the thirteenth to the late sixteenth century, paying scant attention to what came between.[65] Accordingly, the same difficulties that we met in attempting to establish a context for Piero della Francesca's mathematics are encountered again in relation to the optical writings of another practitioner of the visual arts, from the generation preceding Piero, Lorenzo Ghiberti. The optical treatises used in the fifteenth century included those by Roger Bacon and by Witelo. As has been mentioned in Chapter 2, both of these treatises are important for transmitting the sophisticated optical theory of Ibn al-Haytham, whose work is notable both for the high level of its mathematics and for proposing that vision is by the eye receiving light ('intromission') rather than by the eye sending out beams of its own to make some kind of contact with objects in the outside world ('extramission'). In the fifteenth century it seems that it was the latter theory, extramission of eyebeams, that was by far the more widely accepted one, but the intromission theory is of interest in our present context because it makes visual apprehension of spatial relationships among the objects of the world a matter of external geometry, subject to the laws that describe the behaviour of light, rather than something to do with the transmission of sensation through eyebeams, which make contact with the objects of vision, thus making the sense of sight somewhat similar to that of touch. The intellectual shift that goes with the intromission theory does not change the actual geometry of the 'cone of vision' but, in altering the balance between what would now be called geometrical and physiological optics in describing the process of vision, it may be seen as facilitating the introduction of 'artificial perspective'.[66]

Unfortunately, our admittedly scanty evidence merely suggests widespread acceptance of the physical truth of the extramission theory. It seems also to have been widely accepted that the two theories were entirely equivalent geometrically. It was not until the early seventeenth century that it was recognized that this was not true, and a decision was made in favour of intromission.[67] It was probably on account of its mathematical interest that Ibn al-Haytham's optical work was printed in the sixteenth century,[68] and it is, of course, conceivable that, rather than his writing facilitating the development of artificial perspective, it was the widespread use of perspective that helped to facilitate the eventual triumph of Ibn

65 As for astronomy, there is some justification for this attitude – though hardly, I think, for the extreme form it takes in an otherwise deeply scholarly edition and translation of the work of Roger Bacon: D. C. Lindberg, *Roger Bacon and the Origins of 'Perspectiva' in the Middle Ages: A Critical Edition and English Translation of Bacon's 'Perspectiva' with Introduction and Notes*, Oxford: Oxford University Press, 1996.

66 See Gérard Simon, 'Optique et perspective: Ptolomée, Alhazen, Alberti', *Revue d'Histoire des Sciences* 53–4, 2001, pp.325–50; and Gérard Simon, *Archéologie de la vision*, Paris: Éditions du Seuil, 2003.

67 See J. V. Field, 'Two Mathematical Inventions in Kepler's *Ad Vitellionem paralipomena*', *Studies in History and Philosophy of Science* 17(4), 1986, pp.449–68.

68 Alhazen [Haṣan ibn Haṣan ibn al-Haytham], *Opticae thesaurus*, ed. Friedrich Risner, Nuremberg: Petreius, 1572.

al-Haytham's theory of vision. This suggestion is highly speculative, and it must be pointed out that the author of the book widely seen as establishing the physical truth of intromission, Johannes Kepler, never (as far as I know) shows any interest in the visual arts, except to use a *camera obscura* for making topographical sketches, apparently as a trial of the apparatus.

Witelo's treatise, which includes more of Ibn al-Haytham's mathematics than Roger Bacon's does, became a standard work in the sixteenth century. It is, however, Roger Bacon's treatise that Ghiberti seems to have largely relied upon in writing his *Commentaria*, completed after 1447,[69] though since Bacon himself made use of earlier sources it is naturally difficult to be sure exactly what Ghiberti had been reading.[70] In fact, a vernacular version of the work of Ibn al-Haytham, probably fourteenth century and probably made from the Latin, is preserved in the Vatican Library and may have been available to Ghiberti.[71] The difficulties in establishing an optical context for the *Commentaria* have tended to lead to its third book, the part that considers optics, being analysed as if it actually were a general treatise on optics. While this is not entirely unreasonable, it seems likely to be rather unfair, particularly in neglecting the additional context that is provided not only by the rest of the *Commentaria* but also by Ghiberti's art.

The relief panels of the second set of doors Ghiberti made for the Baptistery in Florence, which were completed about 1445, do not all make noticeable use of a mathematical form of perspective. Only three of the ten panels include sufficient architecture to present a reasonably visible perspective scheme. A fourth panel has a single small straight-edged structure, a skeletal stable, that looks convincingly perspectival but on closer inspection proves not to supply enough information for one to check whether it is formally correct. It is, of course, always difficult to assess the correctness of the perspective in a relief, since the surface is not flat. However, the perspective schemes of two of the three panels that show architecture are sufficiently well defined for it to be clear that the height of the eye of the ideal observer is that of a standing figure in the scene, and the same seems to be true of the remaining scene, in which a round building in the background is seen slightly from below, as the foreground figures would see it. That is, Ghiberti seems to be adopting Alberti's notion for setting the eye height of the ideal viewer. There is no attempt to take account of the actual position of the eye of a real-life viewer. All three of the panels in question are placed so that one looks down at them.

In reality, when one looks at the shimmering surfaces of the original panels rather than at photographs, the perspective effects in the 'Doors of Paradise' are not particularly pronounced. This may well have been a matter of decorum. However, though it is in the event rather discreet, Ghiberti's use of perspective is perfectly accomplished. As he was a colleague

69 Lorenzo Ghiberti, *Commentaria*, ed. Ottavio Morisani, Naples, 1947, from Florence, Biblioteca Nazionale, MS Magliabechiano XVII, 33.

70 Graziella Federici-Vescovini, 'Il problema delle fonti ottiche medievali del Commentario terzo di Lorenzo Ghiberti', in *Lorenzo Ghiberti nel suo tempo. Atti del Convegno Internazionale di Studi (Firenze 18–21 ottobre 1978)*, Florence, 1980, pp.349–87.

71 Ibn al-Haytham, *Alchaten figliuolo de alchaichen de li aspecti* (*Liber Alacen in scientia perspettiva*), Vatican Library, MS Vat. no. 4595. See Federici-Vescovini, 'Il problema delle fonti ottiche medievali' (full ref. note 70); Graziella Federici-Vescovini, 'Contributo per la storia della fortuna di Alhazen in Italia: il volgarizzamento del ms. Vat. 4595 e il commentario terzo del Ghiberti', *Rinascimento*, 2nd series, 5, 1965, pp.17–49; and Graziella Federici-Vescovini, 'Alhazen vulgarisé: *Le De li aspecti* d'un manuscrit du Vatican (moitié du XIVe siècle) et le troisième Commentaire sur l'optique de Lorenzo Ghiberti', *Arabic Sciences and Philosophy* 8(1), 1998, pp.67–96.

7.4 Lorenzo Ghiberti (1378–1455), *Jacob and Esau*, from the north doors of the Baptistery, Florence, *c.*1445, gilded bronze, 78.7 × 78.7 cm, Museo delle Opera del Duomo, Florence.

of Brunelleschi's there is no difficulty in finding a possible source of the relevant technique, and Ghiberti presumably shared the interest in the effect of natural optics in the perception of buildings that seems likely to have been the source of Brunelleschi's concern with perspective. In any case, the way Ghiberti handles perspective in the panels of the doors suggests not only that he had an adequate grasp of the relevant mathematics but also that he had given an adequate amount of thought to the way the finished doors would look, and had thus deliberately avoided giving too strong an effect of depth.

It is rather interesting here to compare the mathematically fairly similar perspective vistas in Ghiberti's panel of *Jacob and Esau* and the relief of *The Feast of Herod* that Donatello made for the font in Siena about twenty years before (Figs 7.4 and 2.11). Even in the artificial viewing conditions imposed by photography, one can see how much more strongly

Donatello, working on a relief to be seen in dim light, has insisted on the spatial effect, including prominent human figures that relate the more distant space to the foreground one. Ghiberti, working on a panel to be seen in daylight, not only uses much shallower relief but also allows the architecture to exist mainly as background. These differences no doubt partly reflect the two sculptors' markedly different attitudes to decorum, but they are nonetheless indicative of Ghiberti's understanding what he was about when making a relief in perspective. Ghiberti seems not only to have a grasp of perspective as a mathematical technique but also to be relating it to its source in natural optics. As he too had made sculpture to be placed on buildings, Ghiberti must surely have been aware, as Donatello was, of the effects of work being seen at an angle. Usually, of course, this was a matter of sculpture being seen from below, but it seems highly probable that Ghiberti allowed for such effects in designing the panels of the 'Doors of Paradise' and that his discreet use of perspective is entirely deliberate – making it his equivalent to the adjustments that Piero della Francesca made to the ideal eye heights in the frescos of *The Story of the True Cross* (see Chapter 5).

All in all, the evidence points to Ghiberti having a considerable understanding of optics, based upon a reading of sources whose treatment of the subject is rather more sophisticated than the standard one found in Pecham. That is, whereas Ghiberti is almost certainly a less accomplished mathematician than Piero della Francesca, he is better read in optics. We do not know anything about Ghiberti's social and educational background, but his being on friendly terms with the noted humanist and bibliophile Niccolò de' Niccoli (1364–*c.*1437) suggests that he was seen by his contemporaries as an educated man. In these circumstances, it seems preferable to read the third book of the *Commentaria* not as a treatise on optics but rather as something designed to inform the reader how one must take account of optics in making sculpture. Just as Vitruvius composed a text to include what an ideal architect would need to know about a wide range of subjects, so Ghiberti is trying to do the same for the sculptor.

Like Piero's *De prospectiva pingendi*, Ghiberti's *Commentaria* combine elements from the learned and the craft traditions. Merely writing about craft practice was a relatively new activity in itself, so the authors were not constrained to follow any established norm. Both do, however, seem to be conscious that they are showing connections between their craft and the learning of the universities, as exemplified by Euclid and Roger Bacon. They are also consciously demonstrating connections with the new classical learning of humanism. This is, of course, the programme put forward by Alberti in *De pictura*, but neither Piero nor Ghiberti writes in Alberti's manner. Whereas in *De pictura* it is the university and humanist elements that dominate, in the writings by Ghiberti and Piero it is the craft applications that determine the nature of the text.

Piero and the Albertian programme

It is beyond the scope of the present book, and indeed beyond the competence of the present author, to consider the relation of Ghiberti's sculpture to the writings of Alberti. Piero's practice of his craft as a painter has repeatedly been described as 'Albertian', with the meaning of the adjective left blessedly vague, though sometimes it apparently refers largely to the use of perspective. That both Alberti and Piero are concerned with perspective is obviously true, but we need to bear in mind that Piero's method of perspective construction was rather different from Alberti's, and, specifically, does not construct positions on the ground plane in relation to a horizontal grid. Alberti recommends constructing a grid and then finding intermediate positions by subdividing squares;[72] Piero finds general positions by direct construction.[73]

Piero's mathematical method is thus less constraining than Alberti's. He does not construct even a notional *pavimento* in setting out the ground plan of the scene shown in a picture. However, the historian's attempts to reconstruct the ground plan from a picture are much more likely to meet with success if there is a visible grid, which explains why Piero's pictures, while conveying a perfectly convincing illusion, prove so elusive when subjected to mathematical analysis. The most notable exception to this rule is, of course, *The Flagellation of Christ* (Galleria Nazionale delle Marche, Urbino) (Fig. 5.28), in which Piero has indeed provided not only a *pavimento* underfoot but also, in the coffered ceiling, a grid overhead as well. As was remarked in Chapter 5, there is something ostentatious in the degree to which this picture presents its space for analysis. An almost equally ostentatious grid is found in an unfinished Mantegna engraving showing the Flagellation which, like Piero's painting, has large foreground figures of people who take no part in the main action, which is seen behind them. Such a prominent *pavimento* is not usual in Mantegna's work, but more discreet *pavimenti* are common and form a natural concomitant to the rigorously 'classical' architecture that Mantegna seems to prefer to use as a setting wherever possible. In the use of perspective Mantegna is much more 'Albertian' than Piero.

The reason for the perspective construction of Piero della Francesca's pictures sometimes being identified as 'Albertian' is probably that Alberti's *De pictura/Della pittura* has been, and still is, much more widely read than Piero's perspective treatise. Alberti's writings on the visual arts are an important guide to the opinions and attitudes of his time, but it would certainly be rash to see these writings as the models for all practice or the faithful reflection of all responses to works of art. Nevertheless, Alberti has become a symbolic figure in our understanding of the early Renaissance. Perhaps because of its influence upon Vasari, Piero della Francesca's painting and the sensibility associated with it have also come to be regarded as epitomizing the 'new' style. It is thus inevitable that Alberti's work has repeatedly entered into the present discussion of Piero's work. But Alberti's writings have usually appeared for comparison, and never as a source for Piero's ideas or practices. As we have already noted, Piero and Alberti shared a taste for the ancient style in architecture, but this taste was too general at the time to be considered particularly significant.

Once we move away from perspective, it is generally Piero's style of composition that is labelled 'Albertian'. This seems to include a certain kind of legibility, particularly in depth, that implies that the objects shown have clear spatial relationships one to another. In general terms, this is probably fair, but it breaks down on the detail since Alberti does not mention abstract qualities in regard to composition in the plane, and for composition in depth he proposes to use a grid, thereby introducing the concept of measurement in terms of a single defined unit, whereas Piero imposes no such order.

The divisions become clearer still if we abandon the modern idea of pictorial composition and turn instead to concepts that are – in absolute terms – related to it, but nevertheless concepts Alberti and Piero would have recognized. We can then examine the elements

72 Alberti, *De pictura*, §33.

73 See Chapter 5.

of what Alberti calls the 'historia' of a painting. Alberti, who is drawing on Pliny, describes paintings of historical and mythological subjects, which in fifteenth-century terms are secular, but what he says of the different physical characteristics that are to be associated with, say, the twins Castor and Pollux, who are described as markedly different from one another in appearance,[74] can easily be adapted for the portrayal of a warrior or anchorite saint in contrast to a young martyr. Piero does make such contrasts in physique when occasion demands it, but on the whole the figures in his paintings are more notable for their degree of uniformity than for their variety. This uniformity enables the figures to be used as measures, so that, for instance, the slight difference in size between one of the Queen of Sheba's attendant ladies and her apparently mirror-image companion allows us to read the second, smaller, figure as being slightly further away.[75] As we have seen, this use of human figures as measures is an important element in Piero's construction in depth, sometimes quite explicitly, as in *The Baptism of Christ* (National Gallery, London) and the scene of *The Exaltation of the Cross* (San Francesco, Arezzo).

Another non-Albertian characteristic of Piero's art has frequently been remarked upon by historians, namely that he rarely shows emotion. The adjective most used to describe his figures is surely 'impassive'. There are a few exceptions, such as the highly expressive look of wonderment exchanged between two young people present at the death of Adam in the fresco cycle of *The Story of the True Cross* (San Francesco, Arezzo), but on the whole Piero's figures interact with one another so little that art historians have rightly not known how much account should be taken of the fact that the three foreground figures in the *Flagellation of Christ* appear to be ignoring one another's presence. This is not at all Alberti's prescription for how human beings should be shown. For example, he echoes Pliny's praise of the variety of emotions shown in Timanthes' painting of *The Sacrifice of Iphigenia*.[76] In religious paintings, the expression of emotion is often restrained as a matter of decorum. Such restraint is common to all painters, but it seems to come entirely naturally to Piero. The variety among the figures that Alberti demands and describes in terms of poses and emotions becomes in Piero a matter more connected with colour and lighting. These are not only qualities that Piero explicitly refrains from discussing in his perspective treatise, they are also matters that are intrinsically difficult to put into words in a reasonably precise way. There can be little doubt, all the same, that the colour was subject to the same kind of rigorous intellectual control as the drawing, for we see the same kind of use of colour again and again. The curiously wide variety of pigments that the restorers have found in analysing the paint used in the frescos of *The Story of the True Cross*, where different pigments are sometimes used for different parts of an area that is all the same colour, indicates that what Piero had in mind was to achieve some particular colour, using whatever pigment was to hand. The unity was from Piero's mind, not from using particular pigments.

As we have seen, comparison of their handling of perspective shows Piero to be less 'Albertian' than Mantegna. In his treatment of the *historia* Mantegna is again the more

74 Alberti, *De pictura* §38.

75 On the different sizes, see Roberto Bellucci and Cecilia Frosinini, talk given at the Courtauld Institute on 21 November 1997, and Roberto Bellucci and Cecilia Frosinini, 'Ipotesi sul metodo di restituzione dei disegni preparatori di Piero della Francesca: il caso dei ritratti di Federigo da Montefltro', in *La pala di San Bernardino di Piero della Francesca. Nuovi Studi oltre il restauro*, ed. Emanuela Daffra and Filippo Trevisani (Quaderni di Brera 9), Florence: Centro Di, 1997, pp.167–87.

76 Alberti, *De pictura*, §42.

'Albertian' of the two. The drawing is prominent, while the colour sometimes seems to subvert the draughtsmanship, acting against the forms, and occasionally the use of bright colour pulls a background figure forward. Further, while Mantegna does show impassive figures he also makes use of theatrical gestures and facial expressions that show strong emotions. This emotional expressiveness is less visible in most of Mantegna's religious works than it is in his secular ones, and it is of course on the former that any comparison with Piero's works must rest. All the same, in regard to painting it seems surprising that it is Piero rather than Mantegna who is most often cited as epitomizing the Albertian programme. Nor is it difficult to suggest possible explanations of Mantegna's apparent affinity with the Albertian model: Mantegna and Alberti seem to have shared an archaeological interest in sculpture, the only form of ancient art then known in quantity. Their common connection with the court of the Gonzagas at Mantua is also presumably relevant.

In visual terms, the work of Mantegna seems to come closest to Alberti's ideal. However, the work of Piero della Francesca seems to exemplify what we may call the 'scientific' side of Alberti's treatment of painting, that is considering painting as a matter of exact representation, 'scientific' in the sense of calling upon the mathematical part of optics. Mantegna might have agreed about this as well, but he left no writings on the subject. As we have seen, Piero's writings confirm that he took precisely the scientific attitude that Alberti ascribes to ancient painters and therefore, in echoing Pliny's praise of ancient works, implicitly prescribes to the painters of the fifteenth century. This connection of painting with matters of concern to learned people such as mathematicians and natural philosophers is part of Alberti's claim for a higher status for painting. Alberti also cites from Pliny some names of artists who were of high social rank.[77] The fact that Piero wrote mathematical works that appeared in Latin, and thus presumably circulated among the learned, would seem to make Piero a good example of the kind of 'new' painter Alberti had in mind. He may indeed have seemed so to Alberti.[78]

There is certainly a case to be made for regarding Piero della Francesca's work as linking the world of the craftsman and that of the scholar, and this is particularly true of his writing on perspective, which shows the concern with generality and rigour that is characteristic of scholarly mathematics. However, this is by no means all the story. While in his painting Piero is clearly striving for accurate representation, we have seen in the preceding chapters that there are many departures from the scientific correctness that Alberti seems to prescribe. Some departures are inevitable. In designing the various parts of the fresco cycle *The Story of the True Cross* it would have been impossible to set an eye height for each scene that would allow the picture to be read from the floor of the church in the manner assumed in the perspective constructions described in Piero's treatise and, apparently, employed by him in practice. In the treatise, the eye is assumed to be looking straight at the centric point, that is the line joining the eye to the centric point is at right angles to the picture plane. This is impossible to achieve for pictures placed well above eye level, since a perpendicular from the eye to the picture plane (the plane of the wall) would meet it below the actual picture, leaving one with the nonsensical situation that the correct way of looking at the

77 Alberti, *De pictura*, §28, from Pliny, *Naturalis historia*, XXXV, 133 and elsewhere.

78 See, for instance, Michael Baxandall, 'Alberti's Cast of Mind', in *Words for Pictures: Seven Papers on Renaissance Art and Criticism*, New Haven and London: Yale University Press, 2003, pp.27–38.

picture would be to direct the eye to a point outside the picture. Alberti glosses over this problem and simply recommends, without explanation, that the eye height be made that of a standing figure in the picture. As we have seen, Piero comes to terms with the problem by making the ideal viewpoint qualitatively correct, which is to say that we feel we are looking up at the scene shown in the picture.

It does not seem at all likely that Piero would have regarded painting as being in practice as scientific, and thus exact, a business as Alberti implies. Alberti makes his case using a certain amount of humanist rhetoric or, to put it unkindly, by hand waving. As a practising craftsman, and as an able mathematician, Piero was in the business of finding reasonable solutions to mathematically insoluble problems of representation. There are indeed scientific elements in his art, but they do not add up to convincing evidence that he believed painting truly was a science. Which is to say that had Piero been asked whether he practised an art – that is, a completely legitimate extension of one of the branches of mathematics or natural philosophy – his answer would presumably have been 'no', though one may strongly suspect he would have greatly preferred to be able to say 'yes'. His competence as a mathematician, no less than his skill as a painter, stood in the way of his regarding his craft as an art.

While Alberti's association of the new art with the new humanist learning probably had some effect in raising the social standing of artists, in the event their rise in status was rather slow.[79] As skilled craftsmen they could be well paid, and some of them, probably including Piero della Francesca, may have been regarded as in some sense learned, but artists' achievement of a notably higher social standing than other craftsmen seems to have been associated not with the scientific learning they displayed in the fifteenth century but rather with the literary skills that come to the fore in the sixteenth century. As we have already noted, in this respect Michelangelo's poetry is a more significant landmark than Piero's mathematics. Thus, although Piero does seem in certain respects to be a good example of the learned painter that Alberti envisaged, an examination of Piero's work as a whole, his writings as well as his paintings, suggests that even in his case Alberti was overestimating the degree to which representation could be a learned art. This element of misjudgement does not, of course, diminish the interest and importance of *De pictura* as a record of a well-educated man's view of contemporary events.[80] However, hindsight suggests that what Alberti was seeing was, in reality, the action in the craft of painting of a wider movement involving the increased use of mathematics in many different walks of life. In the next, and final, chapter we shall explore the relation of Piero della Francesca's work to these wider changes.

79 The connection of humanism with a particular 'classical' style in art seems to have been made from the first among Florentine humanists, but Alberti's opinions were certainly influential. See E. H. Gombrich, 'From the Revival of Letters to the Reform of the Arts: Niccolò Niccoli and Filippo Brunelleschi', in *The Heritage of Apelles*, Oxford: Phaidon, 1976, pp.93–110.

80 There is, to my mind, an echo of Alberti's view in the connection between fifteenth-century art and later science made in S. Y. Edgerton, *The Heritage of Giotto's Geometry: Art and Science on the Eve of the Scientific Revolution*, Ithaca: Cornell University Press, 1991.

8

From Piero della Francesca to Galileo Galilei

What Leon Battista Alberti says in *De pictura* concerning connections between painting and the learned arts, particularly the partly mathematical science of sight (*perspectiva*), tells us how things seemed to an educated person in the 1430s. However, this does not mean that we should necessarily accept, that is accept as true, statements that were really no more than an expression of Alberti's opinion or an interpretation of what he saw around him. While contemporary testimony is valuable, historical hindsight may still have a useful part to play. We know, as Alberti could not, something of the developments that were to take place in the following centuries. Prominent among these changes were the rise in social standing of painters and sculptors and the emergence of the profession of the architect. Such developments may indeed be seen as being foreshadowed in what Alberti says in *De pictura* and elsewhere. He is repeatedly at pains to associate the visual arts with high social standing. However, it appears that his analysis does not stand up at all well in the light of today's hindsight. Among the various occupations – of painter, sculptor and so on – the one that shows most traces of the usefulness of mathematical skills is the newly emerging profession of the architect.

Here we have, of course, a rather obvious connection with the art of the surveyor, whose skills were required for laying out the shape of the foundations of the building. And, as we noted in the last chapter, there is also a connection with the skill in handling numbers that was required to carry out calculations concerning the quantities of building materials that would be required.[1] This learning associated with designing and constructing buildings does not represent a great social change, since in the fifteenth century the learned had often interested themselves in architecture, as Alberti's own activity confirms.

Painters' practice did not become increasingly mathematical, though one may perhaps deduce that there was an increased use of preliminary studies, which in our terms (though not necessarily in Alberti's) may be regarded as 'scientific', since they involved the observation and recording of natural appearances. Thus, not to put too fine a point upon it, it seems that while Alberti's programme for raising the status of the visual arts did indeed prosper, it did not do so for the 'scientific' reasons that Alberti suggests in *De pictura*. Since, as we have seen, Piero della Francesca seems in some ways so good an exemplar of the learned painter – or more properly what Alberti regarded as the learned Plinian painter –

1 See R. Goldthwaite, *The Building of Renaissance Florence*, Baltimore and London: The Johns Hopkins University Press, 1980.

it seems fitting to end a book about Piero with some account of how his skills fit in with the revival of mathematical learning that characterizes the period from about 1400 to about 1650. In view of the developments in mathematics and in natural philosophy that took place in this period, it seems that what Alberti was seeing in the practice of the painters of his day was in fact the local effect of a much larger phenomenon.

Galileo Galilei (1564–1642) has been chosen as providing a note on which to end the story not only because it is reasonable to see Piero and Galileo as in some sense belonging to the same intellectual tradition – and Galileo's writings, like Piero's, have something to tell us about the relationship between Latin and vernacular learning – but also because Galileo's work is widely seen as an important contribution to the Scientific Revolution.

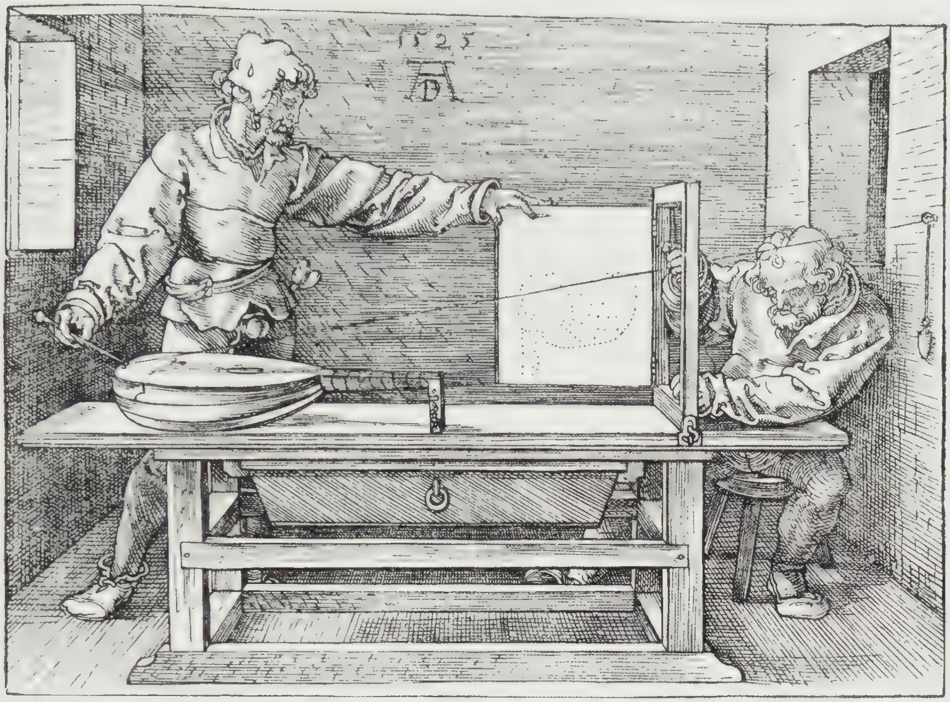
Perspective: treatises and practice

One obvious place to look for the further development in the mathematical skills considered useful to artists is the perspective treatise. Treatises are indeed plentiful, but anyone with the desire to discern overall mathematical progress in this type of publication during the sixteenth and early seventeenth centuries will be disappointed. Apart from the use of the 'distance point method' which, as we have seen, appears only in passing in Piero's perspective treatise, the later treatises rely heavily upon examples exceedingly close to his. The greatest differences are found in omissions. Most notably, if the 'more difficult' bodies in Piero's third book are not completely omitted, Piero's elaborate point-by-point construction is replaced by the use of a sighting device, which works in much the same way as Piero's method but starts the construction not from drawings, as in Piero, but from the actual body that is to be drawn.[2] A common, and no doubt useful, example shows how to obtain the image of a lute (see Figure 8.1, in which the lines of sight have been given the material form of string).

The differences in treatment in the later treatises are largely that, where Piero shows an interest in the basic mathematics that establishes a connection with geometrical optics, later writers on the whole do not. Nor do they share Piero's concern with proving that the constructions they recommend are mathematically correct. Further, in some cases the constructions turn out not to be correct. On the other hand, they are perfectly adequate for a situation where the proof of the pudding is in the eating. Theatrical success has visibly taken precedence over mathematical precision. Such a situation, or rather the way in which writers exploit it, makes learned theoreticians wince. As we have seen, under the normal conditions for wall paintings, it is impossible to construct pictures that are, in mathematical terms, completely correct. The ideal described in *De pictura* is attainable only under highly unusual circumstances, such as Brunelleschian peep-shows. By the late seventeenth century, as is shown by the spectacular ceiling that Andrea Pozzo (1642–1700) painted in Sant'Ignazio, Rome (1685), viewers were prepared to accept a restriction in viewing position in exchange for dramatic *trompe l'œil*. An accommodation had been reached with the science of optics.[3]

2 On perspective instruments see M. J. Kemp, *The Science of Art: Optical Themes in Western Art from Brunelleschi to Seurat*, New Haven and London: Yale University Press, 1990.

3 For detailed analysis of the optics involved see H. Pirenne, *Optics, Painting and Photography*, Cambridge: Cambridge University Press, 1970.

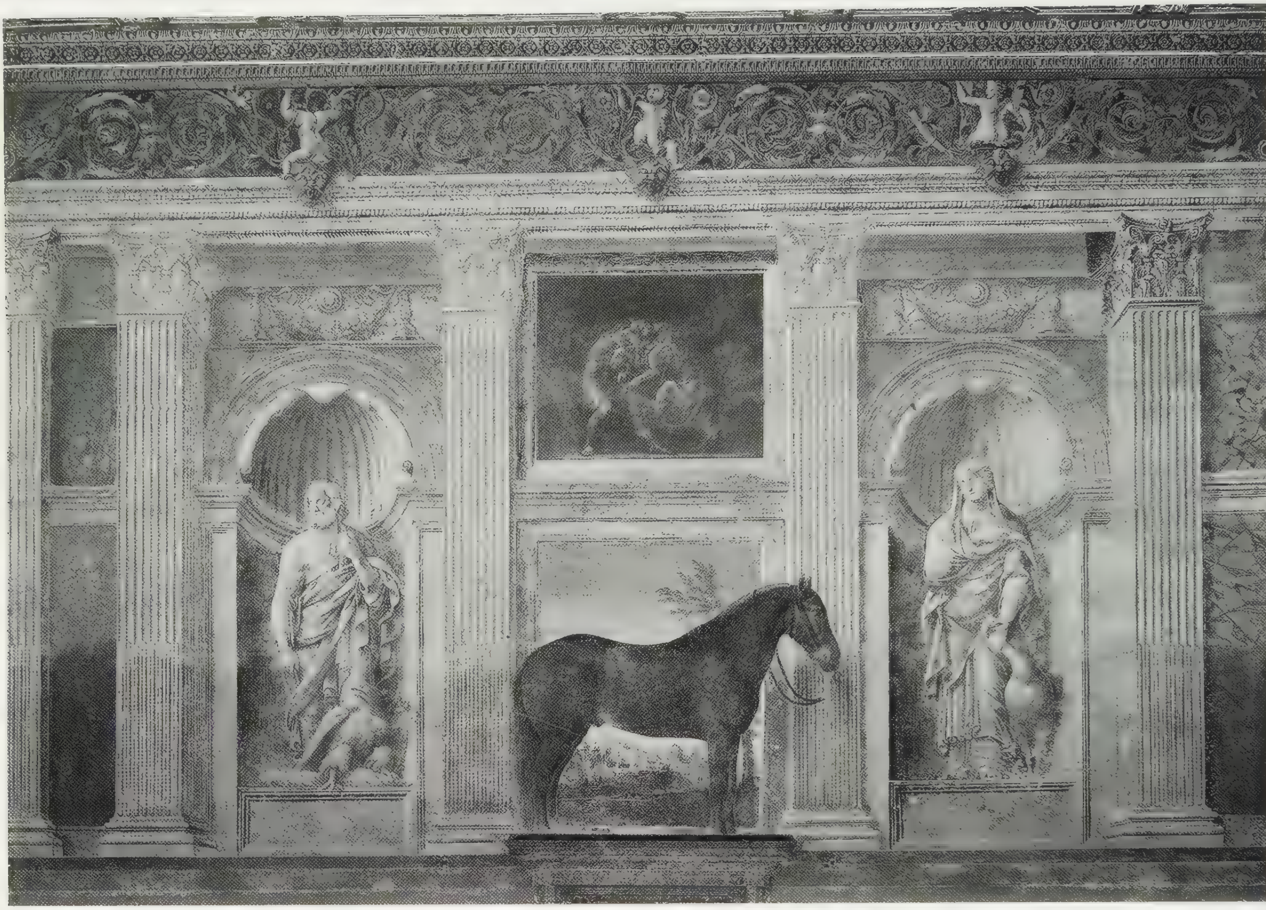


8.1 Albrecht Dürer (1474–1528), *Underweysung der Messung mit dem Zirkel und Richtscheyt*, Nuremberg, 1525, Book 3, Figure 64, sig. Niii recto. A perspective instrument is being used to obtain the image of a lute.

No such accommodation is visible in the work of an earlier exponent of illusionism, Giulio Romano (?1499–1546). His work in the Sala di Costantino in the Vatican Palace, where framing strategies include treating the main scenes as tapestries, can be regarded as merely a reflection upon the status of representation, without there necessarily being a strong intention to deceive the eye.[4] Matters become more serious, in optical terms, in some of the works Giulio painted after he moved to Mantua in 1524, for example in the false architecture shown in the Ducal palace, and still more so in the frescos in the Palazzo del Tè (after 1526). In some of these latter pictures particularly, the deception is undoubtedly a significant part of the effect – even if it is quickly corrected by the common-sense certainty that there cannot, for instance, be a horse standing on the mantelpiece (see Fig. 8.2). The viewer is invited to play at exclaiming 'April fool' to himself. It is this relatively overt playing with illusion that makes these secular decorations of Giulio's suitable examples for my present purpose, and their context also allows one to think in terms of theatrical effects. Most art of the time was not concerned to play with illusionism to this degree.

It is clear that a large part of the initial impact in the Camera dei Cavalli in the Palazzo del Tè depends upon the horses looking 'real', and we have every reason to suppose that

4 On these influential works see S. Sandström, *Levels of Unreality: Studies in Structure and Construction in Italian Mural Paintings during the Renaissance*, Uppsala: Almqvist and Wiksell, 1963.



8.2 Giulio Romano (?1499–1546), detail of the decoration in the Camera dei Cavalli, *c.*1528, fresco.

the effect was achieved as far as the original viewers were concerned. This particular component of contemporary response is one that is exceedingly difficult for present-day viewers to recapture for themselves, accustomed as we are to the accurate reproduction of appearances by means of photography, and the optically convincing 'special effects' made possible in moving images by the use of models and computer 'rendering'. As has already been noted, in Chapter 2, the impossibility of recovering the period eye almost certainly vitiates our understanding of the art of Uccello. The case is even clearer in regard to the generally less visibly mathematical illusionism of the sixteenth century. There can be no doubt, for instance, that in another room in the Palazzo del Tè, the Camera dei Giganti (completed 1534), where the plasterwork rounds off all corners, Giulio intended the effect that the true surfaces of the walls and ceiling would not be visible as such, and the enclosed space would apparently be surrounded by structures in collapse. The effect is still decidedly unsettling, but one simply has to guess by what factor it ought to be multiplied to arrive at the impact it would have had on a sixteenth-century viewer. The same naturally applies to the less ambitious illusions in the Camera dei Cavalli.

The horses on the mantelpieces provide some clear hints about how the trick is pulled. Each mantelpiece, or at least the extension of it under the horse's hooves, is fictive, but is a simple and rather robust perspective illusion, merely involving images of a few orthogonals. The horses themselves, apparently given a high gloss by diligent grooms, are rendered as convincingly solid forms, largely by careful attention to shading and highlights. In fact, the horses look so solid that the eye is effectively compelled to create some pictorial space for them to inhabit. Many earlier painters, among them Masaccio and Piero della Francesca, had also exploited strongly modelled living figures in this way. The effect is robust, which makes it useful when the actual viewpoints from which the work will be seen are widely scattered. Using living figures in this way does not necessarily depend upon mathematical competence, for instance in calculating the positions of highlights, but allows the painter to work simply by eye, perhaps also using small models, as Giorgio Vasari says Piero did for studying drapery.[5]

By the 1520s, the prominent display of rectilinear elements as a means of creating a pictorial space was rapidly becoming unfashionable in paintings such as altarpieces. Which is to say that the kinds of construction shown in perspective treatises play an increasingly inconspicuous part in most works of art. The change in fashion did not apparently free the draughtsman from the burden of drawing such things correctly – there are several disputes about the correctness of pictures that show it was a matter of concern to patrons – but the change in fashion may include a tacit recognition that the eye is usually satisfied with small perspective clues. In any case, artists themselves had known almost from the first that the eye was tolerant of mathematical inaccuracy.

One may distinguish two forms of such tolerance. The first consists of seeing the work as correct, that is as conveying an illusion of depth, even if viewed from a point at some distance from the ideal viewpoint built into the construction of the perspective scheme. The second is to see an effect of depth even if the construction itself is not exactly correct. Recognizing the first form of tolerance explains why Brunelleschi abandoned the peep-show set-up. Such recognition is implicit also in the perspective schemes that Piero della Francesca employed in his pictures, in which, as we have seen, it is generally impossible to view the picture from the theoretically correct viewpoint. Recognizing the second form of tolerance – of mathematical inexactness inherent within the picture – is by far the most plausible explanation for the large number of visually successful works that prove to be noticeably incorrect if examined in mathematical terms. Masaccio's *Trinity* fresco and Piero's *Resurrection* fresco provide striking examples of reliance on this second form of tolerance. Among older artists, one would be hard-pressed to find any perspective relief work by Donatello that did not make use of this tolerance (except perhaps the Lille *Feast of Herod*), and in the generation after Piero we find a notable quantity of works by Andrea Mantegna whose perspective schemes contain internal inconsistencies that are clearly intentional but not intended to disrupt the overall impression that we are seeing into a coherent pictorial space.

In the sixteenth century, settings dominated by straight-edged shapes were not common in paintings such as altarpieces, but were a regular feature of stage designs, which naturally often included architecture. Indeed, in the sixteenth century it seems to have been part of

5 Since Giulio worked as a sculptor, he would no doubt have had ready access to suitable materials. On his art see Frederick Hartt, *Giulio Romano*, 2 vols., New Haven and London: Yale University Press, 1958.

the normal duties of anyone who designed architecture to design stage sets as well, as can be seen from the careers of such prominent practitioners as Sebastiano Serlio (1475–1554) and Bernardo Buontalenti (1536–1608). Serlio's series of books on architecture includes not only a treatment of perspective (to which we shall return below) but also a discussion of stage sets.[6] Another prominent architect, Vincenzo Scamozzi (1552–1616), seems to have been responsible for the impressive three-dimensional perspective scenery for the Teatro Olimpico, Vicenza (1585), in which, it should, however, be noted, only the main shapes are made in perspective and much of the detail, such as mouldings and the modelling of the round shafts of columns, is painted.[7]

Many of the architectural details that are considered in perspective treatises were well adapted to the work of the specialists who painted small false decorative features such as fictive coffering and frames, like those found, for example, in the elaborate decoration of the interior of the San Marco Library, Venice, completed in 1560.[8] However, as in many other schemes for interiors in the sixteenth century, Giulio Romano's decorations in the Palazzo del Tè include not only fictive architecture but also fictive vistas over verdant countryside whose spaciousness is conveyed largely by changes of tone and colour. Much more ambitious perspective views had been painted about ten years earlier by Serlio's teacher, Baldassare Peruzzi (1481–1536), in his famous decorations of the Sala delle Prospettive (1517–19) in the Villa Farnesina, Rome (see Fig. 8.3). Giulio certainly knew these decorations, but was presumably not inclined to copy them too closely. In Peruzzi's decorations in the Farnesina we have a rare case of the situation actually being as implied, but not stated, in all accounts of perspective construction, namely that the picture goes right down to floor level and will be viewed by someone standing in front of it. However, since the pictures are on the walls of a room, there is no way of ensuring that the viewer will stand at an ideal viewpoint.

Photographs of illusionistic pictures tend to draw attention to the distortion caused by their being viewed from an incorrect position. A photograph shows what the eye really sees, removing the context that contributes robustness to the illusion in the presence of the actual pictures. Such subversion of the eye's habitual tolerance makes it difficult for historians to refresh their memories of their perceptions of Peruzzi's frescos, on which opinions seem to be somewhat divided. Luckily, we have plenty of evidence to show that contemporaries found the decoration highly impressive.

It is, of course, obvious that the pictures cannot be mathematically correct for all the positions from which they are seen by normal viewers. Thus Peruzzi must rely upon the eye being tolerant of being in the wrong place. It is consequently not surprising to find that there is much reliance upon modelling to convey solidity of such things as columns, and reliance on simple scaling to convey the great distance of items in the landscapes seen through the imaginary loggias. However, there is also enough evidence of mathematical construction of straight-edged forms to allow the deduction that the mathematical rules Peruzzi was applying are mathematically incorrect. This is confirmed by the mathematical inconsistencies in what Serlio writes about perspective construction, in the second book of his

6 Sebastiano Serlio, *Libro di geometria e di perspettiva*, *L'architettura*, Books 1 and 2, Paris, 1545.

7 A more detailed discussion of the Teatro Olimpico scenery is given in J. V. Field, *The Invention of Infinity: Mathematics and Art in the Renaissance*, Oxford: Oxford University Press, 1997, Chapter 6.

8 See Kemp, *The Science of Art* (full ref. note 2).

8.3 Baldassare Peruzzi (1481–1536), detail from the Sala delle Prospettive, Villa Farnesina, Rome, 1517–19, fresco.

series of volumes on architecture, published in Paris in 1545. It is somewhat disconcerting that the perspective should be mathematically incorrect in such a famous piece of illusionism, but the fact does help to explain the phrasing of the comment that Egnazio Danti (1536–1586), who was a competent mathematician, made in 1583 about Serlio's writing on perspective, namely that he passed on the rules of perspective in the form in which he had learned them from Peruzzi.[9]

Serlio's error is the more surprising since his writing shows a good grasp of the mathematics relevant to architecture, and indeed suggests that he took some interest in mathematics beyond results that were immediately applicable in his craft.[10] In any case, Serlio's publications were

9 Egnazio Danti, preface to *Le due regole della prospettiva pratica*, by Vignola [Giacomo Barozzi], Rome, 1583. Danti gives a long list of perspective treatises; see J. V. Field, 'Giovanni Battista Benedetti on the Mathematics of Linear Perspective', *Journal of the Warburg and Courtauld Institutes* 48, 1985, pp.71–99.

10 See J. V. Field, 'Why Translate Serlio?', in *Thomas Gresham and Gresham College: Studies in the Intellectual History of London in the Sixteenth and Seventeenth Centuries*, ed. F. Ames-Lewis, Aldershot: Ashgate, 1999, pp.198–221.

highly successful, running through several editions in the sixteenth century and being translated into several languages, including Latin.[11] There is no reason to doubt that his perspective constructions were used in practice and found to be satisfactory in visual terms.

It seems that by the mid-sixteenth century patrons were sometimes inclined to take an interest in perspective, and it appears that in consequence professional mathematicians were sometimes asked for their opinions. An explicit trace of this process is to be found in a short tract, 'On the Reasons for the Operations of Perspective' (*De rationibus operationum perspectivae*), written by Giovanni Battista Benedetti (1530–1590), who was mathematician to the Duke of Savoy. Benedetti begins his explanations by dealing with an error in construction that is part of the incorrect method given by Serlio.[12]

Serlio's account of perspective is typical of treatises of the time in more or less following the series of examples treated by Piero della Francesca, with a number of omissions. The chief difference is that Serlio's examples seem to be developed in a way that makes them applicable in designing stage sets. Indeed a few of them show a direct relationship to his illustrations of the Vitruvian sets for comedy and tragedy (the one for pastoral contains mainly landscape and thus shows no sign of formal construction). The Vitruvian connections of an interest in perspective are seen equally clearly in the work of Daniele Barbaro (1513–1570), whose treatise on perspective, *La prattica della perspettiva*, published in Venice in 1568 and 1569, is described in the author's preface as deriving from his work on Vitruvius. Barbaro's Italian and Latin editions of Vitriuvius' *De architectura*, published in Venice in 1556 and 1566, were to become the standard ones for the following century or so. While the bulk of the part of Barbaro's text that is concerned with perspective is taken, largely verbatim, from Piero – Barbaro seems to have had a vernacular manuscript of *De prospectiva pingendi*[13] – the examples of Vitruvian perspective stage sets are taken from Serlio. Indeed, the correspondence appears to be absolutely exact, which makes it seem possible that a Venetian printer of Serlio simply handed over blocks for use in the printing of Barbaro's book.

Barbaro is entirely explicit in his reliance upon Piero's text, but there is no indication that he ever saw any of Piero's paintings. He does, however, complain in his preface of a lack of perspective in the paintings of his own day. This may seem curious in view of the family villa at Maser having been decorated by Paolo Veronese (1528–1588), who often places the action of his paintings in an architectural setting that looks very like a stage set. Moreover, Barbaro's native Venice provides many examples of fictive architectural elements in interior decoration from Barbaro's period. The explanation seems to be that, by this time, to say a picture contained perspective implied it contained an architectural vista, such as we find in Domenico Veneziano's predella panel of *A Miracle of St Zenobius* (Fig. 3.11) or the receding colonnade in many Annunciation scenes – for example, Piero's version in the

11 For an account of the printing and translation history of Serlio's writings see Sebastiano Serlio, *Sebastiano Serlio On Architecture*, vol.1, *Books 1–5 of 'Tutte le Opere d'Architettura e Prospetiva' by Sebastiano Serlio*, trans. Vaughan Hart and Peter Hicks, New Haven and London: Yale University Press, 1996.

12 Giovanni Battista Benedetti, 'De rationibus operationum perspectivae', in *Diversarum speculationum mathematicarum et physicarum liber*, Turin, 1585. Although it was not published until 1585, Benedetti's text seems to have been written about twenty years earlier; see Field, 'Giovanni Battista Benedetti' (full ref. note 9).

13 See M. J. Kemp, 'Piero and the Idiots: The Early Fortuna of his Theories of Perspective', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no. 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art,1995, pp.199–211.

Sant'Antonio Altarpiece (Figs 6.29 and 6.34).[14] Just such a perspective vista is shown on the title page of one of the most successful treatises of the later sixteenth century, that by Vignola (Giacomo Barozzi, 1507–1573), which was published posthumously, in Rome in 1583, with extensive annotations and additions by Egnazio Danti (see Fig. 8.4).

This treatise, which was reprinted several times in the next forty years, is a kind of hybrid because, although its overall content is provided by Vignola, which fits in with the pattern of treatises being written by architects, the editor's contribution is of considerable importance and is distinguished by the text being set differently. Egnazio Danti was well qualified to write on perspective: his career was as a mathematician, but he came from a family of practising artists – its most distinguished member was his elder brother, the sculptor Vincenzo Danti (1530–1576) – and he must have been familiar with the techniques used in workshops. His editorial additions, which bring the work closer to his own work on optics, make the book more interesting to the learned, but the examples of perspective constructions tie the treatise to the specialized skills of making stage sets or painting fictive elements of architecture.

Egnazio Danti may have had a personal motive in the matter, but he was not alone among mathematicians of his time in taking an interest in perspective. The mathematics of the subject had already attracted attention from scholars of greater weight than Danti, namely Federigo Commandino and Giovanni Battista Benedetti.[15] This kindling of interest among mathematicians was to lead to the invention of a new kind of geometry, not known to the ancient Greeks, namely the projective geometry of Girard Desargues (1591–1665) published in Paris in 1639.[16] Formally, perspective is equivalent to conical projection, with the eye as the centre, but projective geometry is entirely different in concept from what we find in perspective treatises. As we have seen, in perspective the object is described as 'perfect' and the image as 'degraded'. That is, we are looking at what is changed by the procedure of making the image. In projective geometry, object and image have equal status. We look at what remains unchanged, which enables us to carry results over from object to image, for instance, from the circle to an ellipse or the parabola. The technicalities of this are of no importance in the present context, though the story behind the invention of projective geometry is significant as an instance of the practice of craftsmen coming to influence the development of a well-established learned discipline – and, moreover, leading to a viable mathematical definition of the previously slippery notion of infinity.[17]

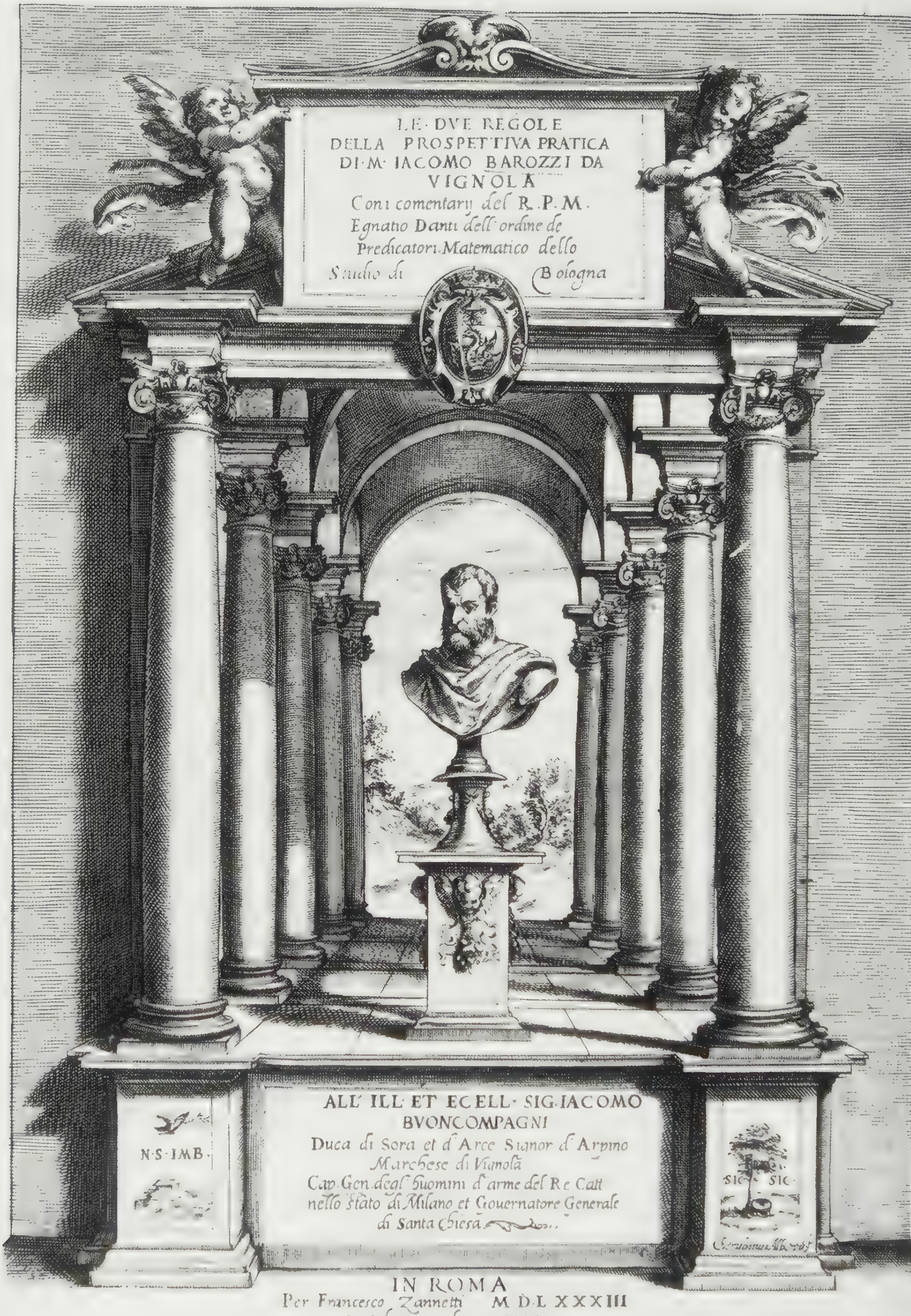
Having a precise and self-consistent definition of infinity within mathematics became important in the later seventeenth century when, with the increasing acceptance of a heliocentric model of the Universe, it became clear that in the current state of astronomical observation it was not possible to set bounds to the size of the Universe. This meant, for instance,

14 On this usage of the term 'perspective' see Thomas Frangenberg, *Der Betrachter: Studien zur florentinischen Kunstliteratur des 16. Jahrhunderts*, Berlin: Gebr. Mann Verlag, 1990.

15 Federigo Commandino, *Commentarius in Planisphaerium Ptolomaei*, Rome, 1558; on Benedetti's tract, see note 12.

16 Girard Desargues, *Brouillon project d'une atteinte aux evenemens des rencontres du Cone avec un Plan*, Paris, 1639 (annotated English translation in J. V. Field and J. J. Gray, *The Geometrical Work of Girard Desargues*, London and New York: Springer-Verlag, 1987). A summary of the mathematical content of the work is given in Field, *The Invention of Infinity* (full ref. note 7), esp. pp.190–206. See also Chapter 7, p.272.

17 On this part of the history of mathematics, see Field, *The Invention of Infinity* (full ref. note 7).



8.4 Title page of Giacomo Barozzi [Vignola], *Le due regole della prospettiva prattica*, ed. Egnazio Danti, Rome, 1583.

that in considering the universality of his postulated inverse square law of gravitation, Isaac Newton had to consider the possibility that the Universe was infinite.[18] Thus there was an indirect effect of studies of perspective upon the development of natural philosophy in the later seventeenth century, but this was mediated by professional mathematicians who came to take an interest in artists' practice.

It has sometimes been suggested that the invention and use of perspective in the Renaissance led to an increased concentration upon measurement, which furthered the progress of the sciences that we see in the Scientific Revolution.[19] There is something to be said in favour of this idea in the sense that it is indeed widely agreed among historians of science that mathematics played a part in the Scientific Revolution. However, to isolate one particular piece of mathematics as being of crucial importance seems to come dangerously close to importing into the history of ideas the dated style of historiography in which Great Men cause Great Events. Moreover, as has already been made clear, one should not overestimate the scientific elements in either the understanding or the use of perspective in the Renaissance. It appears that artists set no one an example in the matter of exactitude. In any case, their results were not to be assessed in mathematical terms. Painters could, and did, get the science seriously wrong while attaining satisfactory results. If the roots of the Scientific Revolution lie in the Renaissance, they are surely more complicated and intricate than this and we must look for them at least partly elsewhere.

Much has sometimes been made of Galileo's connections with the visual arts.[20] He does indeed seem to have been active in the Florentine Academia del Disegno.[21] Moreover, it is clear that Galileo had a substantial concern with optics, since he had an interest, practical and financial as well as intellectual, in making telescopes and in improving their performance. There is, however, no direct evidence of his being interested in perspective. On the other hand, there is some indirect evidence by association: Galileo was on good personal terms with the painter Cigoli (Ludovico Cardi, 1559–1613), whose work is in a naturalistic style such as Galileo is known to have admired.

Cigoli's paintings are not distinguished by prominent use of perspective, but he did invent an instrument to help artists draw correct perspective images. This is one of the many such instruments that had been a feature of perspective treatises addressed to artists since the beginning of the sixteenth century. Cigoli's instrument is, like most others, essentially a sighting instrument, but it is well adapted to making images of objects seen under rather extreme conditions, and to constructing anamorphic images, that is images which at first seem unreadable but can be read if viewed under special conditions, such as by looking at them with the eye very nearly in the plane of the picture. These highly distorted images cannot easily be drawn without the aid of a machine of some kind. Cigoli wrote a treatise on perspective, but it remained in manuscript. However, the design of his perspective instrument was described and illustrated in Jean François Nicéron (1613–1646), *La perspective curieuse*

18 See A. van Helden, *Measuring the Universe: Cosmic Dimensions from Aristarchus to Halley*, Chicago and London: University of Chicago Press, 1985. Newton proposed the inverse square law of gravitation in *Philosophiae naturalis principia mathematica*, London, 1687.

19 See, for instance, S. Y. Edgerton, *The Heritage of Giotto's Geometry: Art and Science on the Eve of the Scientific Revolution*, Ithaca: Cornell University Press, 1991.

20 For instance, in Edgerton, *The Heritage of Giotto's Geometry* (full ref. note 19).

21 See T. B. Settle, 'Egnazio Danti and Mathematical Education in Late Sixteenth-Century Florence', in *New Perspectives on Renaissance Thought: Essays in the History of Science, Education and Philosophy in Memory of Charles B. Schmitt*, ed. J. Henry and S. Hutton, London: Duckworth, 1990, pp.24–37.

(published in Paris in 1638).[22] Niceron, who worked in Rome, was an enthusiast for anamorphism. Galileo did not share this enthusiasm but, given his mathematical ability and his interest in optics, it seems rather likely that when Cigoli was writing his treatise on perspective he would have discussed its contents with Galileo.

In any case, it appears that Galileo and Cigoli did discuss what Galileo had seen through his telescopes, because the Moon that Cigoli shows under the Virgin's feet in his *Immaculate Conception* in the small dome of the Cappella Paolina in Santa Maria Maggiore, Rome, is not the smooth 'perfect' Moon of conventional natural philosophy but instead is manifestly based upon Galileo's drawings.[23] When illustrated in books, this can look like a spectacular gesture of solidarity with Galileo concerning his telescopic observations, but the picture is exceedingly high up, and the markings on the Moon, while detectable once pointed out, hardly spring to the eye, so the picture might just as well be construed as a private joke. However, it does indicate that Cigoli and Galileo at least discussed the appearance of the Moon as seen through a telescope.

Galileo's book about his observations, *Sidereus Nuncius*, published in Venice in 1610, contains illustrations of the Moon, and some of his original drawings of the Moon survive.[24] Some changes were made for the published versions, which suggests that one should be a little careful in interpreting both Galileo's original drawings and the printed drawings. All the same, the drawings themselves carry great conviction and bear eloquent testimony to what must have been a tremendously exciting series of observations. My use of the literary metaphor here is deliberate: Galileo's written text is at least as powerful as the drawings. For instance, he describes the movement of shadows in long and careful detail before drawing the conclusion that what he has been seeing is the shadows of mountains. The heavy emphasis on the fall of light does indeed, as scholars have claimed, link these pictures with artists' sketches of similar date, but the objective reality was almost entirely light and shadow too. It accordingly seems to me to be rather rash to conclude that Galileo's understanding of art played a particularly large part in his interpretation of what he saw. On the other hand, it is not clear, either, quite how much is training in optics and how much is imagination, in deciding to interpret the changing pattern of light and dark as the shadows of mountains. Galileo's long verbal descriptions seem to be designed to confront the reader with apparently raw visual evidence. One could make comparisons with the art of the time, but experience teaches that in the history of science all observations must be recognized as theory laden, and in Galileo's encounter with the Moon there is certainly an influence from his ideas about natural philosophy that is at least as powerful as the effect of his having been taught to see by looking at the art of his time.

Practical geometry and technical drawing

We have already noted that in the sixteenth century much of the writing on perspective was the work of architects. The connection between perspective and architecture is already apparent in the art of the previous century, since formal construction designed to produce

22 The Latin edition is Jean François Niceron, *Thaumaturgus opticus*, Paris, 1646.

23 See Kemp, *The Science of Art* (full ref. note 2), pp.93–4.

24 See Mary G. Winkler and Albert van Helden, 'Representing the Heavens: Galileo and Visual Astronomy', *Isis* 83, 1992, pp.195–217.

an effect of depth almost always involves straight-edged objects, of which buildings are convenient examples. Any treatise on perspective is accordingly likely to include specimens of architecture, and thus to act as a mirror of the architectural style of its time and place. So treatises on perspective were always liable to be read as manuals of architectural style, particularly by foreigners. To take a late example, Andrea Pozzo's perspective treatise, a lavishly illustrated folio, includes Pozzo's designs for illusionistic ceiling paintings, which incorporate numerous architectural features in the style then current in Rome.[25] The preface to the equally elegant English edition, published in London in 1707, lays much emphasis upon the usefulness of the work in making the advanced style of continental architecture known in England, a sentiment made more interesting by the fact that the sponsors of the edition include Christopher Wren (1632–1723), Nicholas Hawksmoor (1661–1736) and John Vanbrugh (1664–1726). It was almost certainly in a similar spirit that Diego Velázquez (1599–1660) acquired so many treatises on perspective, which, to judge by their places and dates of publication, he apparently bought on his visits to Italy.[26] From the sixteenth century on, heavily illustrated perspective treatises became a vehicle for transmitting style as well as a certain amount of useful geometry.

In a complementary process, treatises whose main subject was architecture tended to transmit a certain amount of the kind of mathematics that seems to have been standard among architects in Italy but may have been less familiar elsewhere. For instance, Serlio's writings contain much information about classical building styles and decoration, as well as acceptable modern adaptations of them, for example in the designs of churches. However, they also contain a considerable amount of mathematics. This is all of the 'practical' kind, dealing with such matters as setting out the proportions of the opening made for a door. The problems are geometrical and are posed as problems of construction. That is, they resemble the construction problems of Piero's perspective treatise rather than the calculation problems of the abacus tradition. Most of the problems are essentially qualitative, for instance to draw an oval shape, or to draw a smooth curve of given height and given width (which is a problem that Serlio indicates as arising in designing the arch of a bridge or the profile of an urn). However, even in the few cases where the problem is more closely defined, Serlio is not concerned to prove that the constructions are mathematically correct.

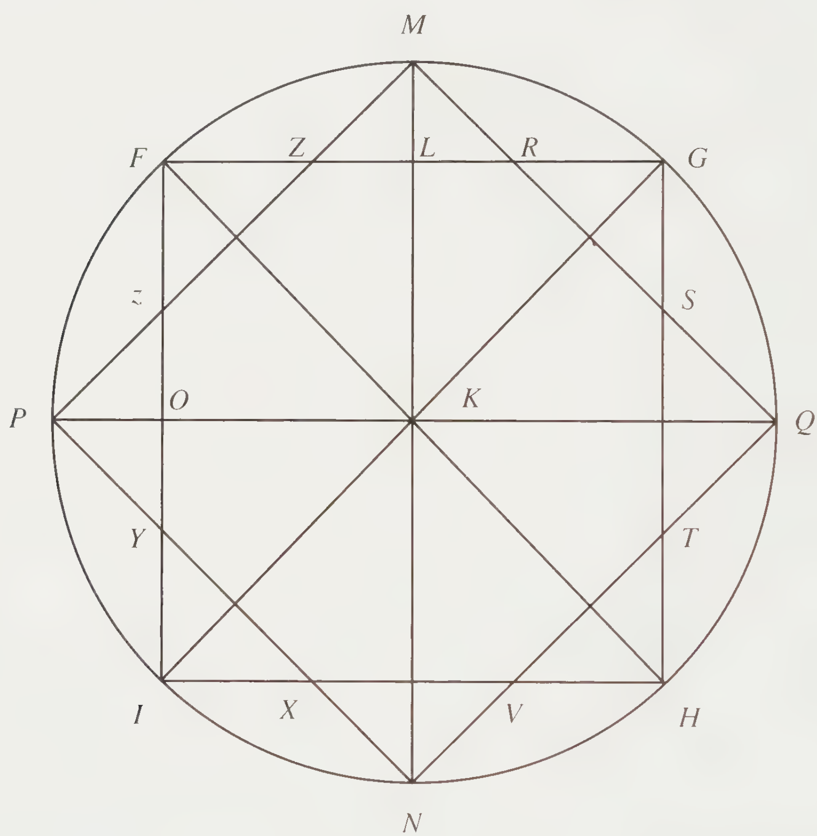
There are, nonetheless, various small indications that he knows rather more about mathematics than he considers he needs to tell his readers. For instance, in the case of the curve of given width and height, Serlio notes that it cannot be drawn with compasses but must be constructed point by point. This distinguishes it from the ovals that he draws immediately afterwards, which suggests that Serlio realized his curve was an ellipse. The construction he gives looks as if it were derived from the mathematics of constructing sundials – a matter with Vitruvian connections – so there is one obvious possible source from which he may have learned of it, but his remark does suggest his interest in mathematics was not merely superficial. There are several other passages that give similar indications.[27]

25 *Perspectiva pictorum et architectorum*, in Latin and in Italian, two parts, Rome, 1693, 1700.

26 See F. J. Sanchez Cantón, 'La librería de Velázquez', in *Homenaje ofrecido a Menéndez Pidal*, vol.3, Madrid, 1925, pp.379–406; and J. V. Field, 'Mathematical Books in the Library of Diego Velázquez (1599–1660)', forthcoming.

27 See Field, 'Why Translate Serlio?' (full ref. note 10); on the ellipse see esp. pp.211–13.

No doubt many of Serlio's drawing problems, and the methods of solution, are traditional. Many are indeed found in other books. One of Serlio's problems, that of cutting the corners off a square to turn it into a regular octagon, is also found in *De prospectiva pingendi*. Piero includes this construction as a preliminary to finding the perspective image of an octagon in Book 1, section 26.[28] The steps of his procedure can be seen in Figure 8.5, in which the original square is *FGHI*. Piero first draws the diagonals *FH* and *GI*, which intersect at the centre of the square, *K*. He then draws the circumcircle. Through *L* and *O*, the mid points of *FG* and *FI*, he draws lines parallel to *GH* and *FG* respectively, which meet the circle in the four vertices of another square. The intersections of the sides of this second square with those of the original one define the vertices of the octagon. Piero does not give a proof, but this construction is in fact mathematically correct. He gives a numerical treatment of the problem of constructing an octagon, in the context of cutting the corners off a

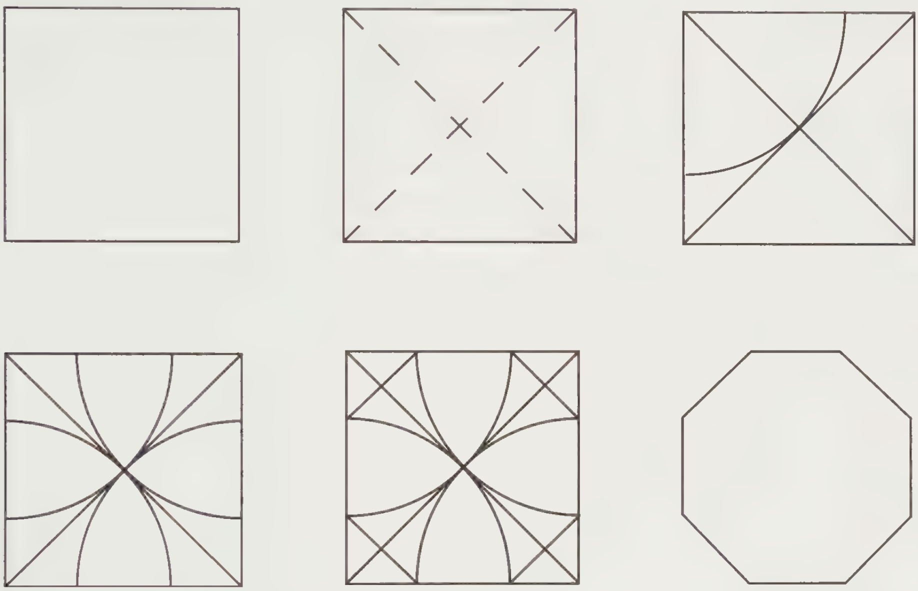


8.5 Cutting the corners off a square to turn it into a regular octagon. Diagram to illustrate Piero della Francesca, *De prospectiva pingendi*, Book 1, section 26. There is no corresponding diagram in manuscripts of the treatise. Drawing by JVF.

28 Piero della Francesca, *De prospectiva pingendi*: Parma MS, p.12 recto; BL MS, p.13 recto; Piero ed. Nicco Fasola, p.89. Translation in Appendix 4 below.

cube, in his *Libellus de quinque corporibus regularibus*. Earlier in the same work he had constructed an octagon outside a square (in Figure 8.5 the octagon would be *FMGQHNIP*), and this passage is derived from his *Trattato d'abaco*.[29]

Serlio gives a different method from Piero. His procedure is summarized in Figure 8.6. This method too is mathematically correct, though no proof is given. Serlio draws the diagonals of the square, then, with a point of the compasses at one corner of the square, and compass opening equal to the distance from the corner of the square to the point of intersection of the diagonals, he strikes off an arc to cut each of the sides adjoining the corner. The eight points so obtained will be the vertices of the required regular octagon.



8.6 Stages of the figure for drawing a regular octagon in a square, after Serlio ed. G. D. Scamozzi, 1584, Book I, fol.14 recto. Drawings by JVF.

Most of Serlio's geometry is set out in terms that are appropriate for practical application. A similarly practical attitude is seen in his descriptions of ancient architecture, where he almost invariably gives dimensions. Indeed, on one occasion he even provides a line the length of half the 'foot' used in the measurements. In some of the descriptions he refers the reader to the drawing for additional measurements, which is to say that the drawing is 'to scale' in the manner with which we are now familiar. The concept may not have been new in Serlio's time – it is naturally awkward to explore the history of things so ephemeral as technical drawings – but Serlio's illustrations are certainly among the first scale drawings

29 Piero della Francesca, *Libellus de quinque corporibus regularibus*, Tractatus 4, casus 5, in Luca Pacioli, *De divina proportione*, Venice, 1509, part 3, p.22 recto; Piero ed. Mancini, p.563. For the octagon outside the square, see *Libellus*, Tractatus 1, casus 39–42, in Pacioli, *De divina proportione*, Venice, 1509, part 3, p.7 recto–7 verso; Piero ed. Mancini, pp.512–14; and Piero della Francesca, *Trattato d'abaco*: BML MS, pp.92 recto–93 recto; Piero ed. Arrighi, pp.196–8.

to appear in print. This has a wider historical importance, showing illustrations assuming some of the authority earlier associated only with words. However, in regard to Serlio's books, a down-to-earth explanation at once suggests itself. It is clear that the buildings are being shown not only for straightforward archaeological reasons, but also – and perhaps for most readers this was the principal interest – so that the buildings could be copied, either in simulated form in stage scenery or in the more robust form of temporary architecture, such as a triumphal arch for a formal entry by an imperial visitor, or perhaps even as an actual habitable building. Serlio himself shows great talent for easygoing syncretism, so one can imagine he would have been happy for readers to adopt not complete buildings but only certain 'classical' features that, following his own example, could be adapted and added to structures whose main design was in the local vernacular style. That is to say the illustrations were serving as potential *modelli*, standing in for the three-dimensional models that were customarily presented to patrons, and possibly becoming a first stage for the construction of such models.

There is an earlier indication of this process. In his introduction to *De divina proportione* (1509), Luca Pacioli tells the reader that the original copy of his work, presented to the Duke of Milan, was accompanied by solid models of polyhedra, but that in the preparation of further manuscripts a substitute for models was provided by Leonardo da Vinci's drawings. This remark of course fits in well with the clear indications that Leonardo made his drawings from real models.[30] The substitution of perspective drawings for models of regular and semiregular polyhedra is acceptable because we know enough about the 'perfect' original shapes – for instance, that all their edges are of equal length – to allow us to read the 'degraded' perspective images without any ambiguity. However, perspective drawings of architecture could not be read unambiguously. So Serlio provides detailed dimensions in plans, and in drawings that are not exactly elevations, but more like front views with attenuated indications of depth. He also sometimes provides tidy cutaways, which perhaps should remind us that when Serlio was in the service of the King of France, Benvenuto Cellini (1500–1571) lent him a copy of a manuscript by Leonardo, who is among the earliest known users of the method of showing structures by means of what we now call cutaway and exploded diagrams.[31]

What we are seeing in Serlio's books, and in almost all subsequent treatises on architecture, is not a simple substitution of a perspective picture for a solid model, but rather a group of measured or scale drawings, sometimes supplemented by a perspective picture, serving as a significant element in the transmission of architectural designs. The use of these three types of representation is, of course, prescribed by Vitruvius. Serlio's treatise is notable for its inclusion of every last detail of the building. We find scale drawings showing profiles of mouldings, patterns of coffering for ceilings, patterns of panels for doors, shapes of hinges, and detailed layouts for gardens that include shapes of flowerbeds.

The situation is less clear-cut in regard to drawings of machinery. Heavily illustrated manuscript treatises of ingenious mechanical devices, presented to actual or potential patrons,

30 See J. V. Field, 'Rediscovering the Archimedean Polyhedra: Piero della Francesca, Luca Pacioli, Leonardo da Vinci, Albrecht Dürer, Daniele Barbaro, and Johannes Kepler', *Archive for History of Exact Sciences* 50 (nos 3–4), 1997, pp.241–89.

31 It is not known which manuscript this was. Cellini tells the story in his book on architecture (Venice, 1776). See Leonardo da Vinci, *The Literary Works of Leonardo da Vinci*, ed. Jean Paul Richter, 3rd edn, London: Phaidon, 1970.

antedate the use of perspective in Europe and are indeed found in the Islamic tradition.[32] Their ancestry obviously includes manuscripts of ancient writings on engineering, such as the *Pneumatica* of Heron of Alexandria (active 62 A.D.).[33] When, with the rise of illustrated printed books, we come to the 'machine books' of the sixteenth century, we find not only that the drawings are generally not to scale but also that the machines are often shown in landscape or accompanied by human figures whose sizes sometimes give a highly misleading impression of the size of the machine.[34] The explanation for this apparent neglect of exactness may simply be that for machinery a solid model was always going to be necessary, if only in order to establish that the parts would move as required. In any case, scale drawings seem to play a much less significant part in Renaissance engineering books than they do in architectural ones. There is no evidence that the introduction of more naturalistic illustrations in perspective had any significant effect upon engineering practice or the development of machines.

Serlio's multi-volume work on architecture contains quite a lot of geometry, all presented as applicable. Similar work on geometry is found in many other books on architecture, for instance in *L'Idea dell'Architectura universale* by Vincenzo Scamozzi, published in Venice in 1615. However, even in books such as the *Quattro libri di architettura* of Andrea Palladio (1508–1580), published in Venice in 1580, where there is no explicit instruction on geometry, the multitude of drawings that are marked with measurements, and the detailed discussions of proportions, suggest that the writer had some facility in handling mathematics and expected the same of his readers. The publication of so many treatises that assumed the reader would be able to handle this kind of mathematics is strong evidence for a general rise in the level of mathematical education in the sixteenth century, not only among the classes who could afford to buy such expensive books, but also among the workmen who would need to carry out instructions given in mathematical form.

It is worth noting, also, that much of the mathematics presented in these texts is not a mere recapitulation of results found in Euclid, though the methods used are indeed those familiar from the *Elements*. For instance, Euclid does not give a construction for deriving an octagon from a square by cutting off its corners. This problem has no importance in the orderly series of Euclid's propositions but, as we have seen, it was of interest to Piero della Francesca and to Serlio. Moreover, Euclid does not consider any curve except the circle, whereas Serlio is concerned with drawing more complicated curves, such as ovals made up of circular arcs struck from several centres. It is accordingly interesting to note that the workmen for whom Serlio provides such patterns to follow include not only masons but also carpenters (for doors), metalworkers (for door fittings), and gardeners (for flowerbeds). Books on fortification provide similarly mathematical instructions for laying out the plans of defensive walls. Reading Serlio and the many similar texts on architecture leads to the conclusion that what Alberti was seeing in painters' practice was only one manifestation of a general increase in the amount of mathematics used in the crafts.

32 See, for instance, Donald R. Hill, trans., *Al-Jazzari: The Book of Knowledge of Ingenious Mechanical Devices*, Dordrecht: Reidel, 1974.

33 The translator responsible for the *editio princeps* of Heron's *Pneumatica*, Venice, 1589, Giovanni Battista Aleotti (1546–1636), who was among other things an architect, once owned the manuscript of the Latin version of Piero's *De prospectiva pingendi* now in the British Library (Add. MS 10366).

34 See A. G. Keller, *A Theatre of Machines*, London: Chapman and Hall, 1964.

There is plenty of further evidence that points in the same direction for crafts less closely linked with architecture. One particularly striking instance is recorded by Niccolò Tartaglia (*c.*1499–1557). Tartaglia is now best remembered as an algebraist, but his magisterial *General Trattato di numeri et misure*, published in Venice in six parts in the years 1556–60, shows he was also a highly competent geometer. He says in the preface to his book on the mathematics of gunnery, *La nova scientia*, published in Venice in 1537, that he wrote it at the request of Giulio Savorgnan, then the commander of the military forces of Venice. Savorgnan seems to have been interested in mathematics and was probably on good personal terms with Tartaglia, since he appears as one of the people setting problems in Tartaglia's *Quesiti e inventioni diverse* . . . , published in Venice in 1554. In requesting Tartaglia to write on the mathematics of gunnery, which, as the title *The New Science* implies, was a new subject, Savorgnan was expressing confidence in the power of mathematics to make predictions about the behaviour of objects in the real everyday world.

Many, probably most, philosophers of the time would have had reservations of principle about this. These reservations related not to the quality of any specific theory that might be devised, by Tartaglia or anyone else, but to the capacity of any mathematical theory whatsoever to give an adequate account of the matter. There is no sign that Savorgnan and Tartaglia were consciously engaging in a philosophical debate on this principle. They may well have simply been taking a more hopeful view, presumably based upon experience of mathematics solving other practical problems. Tartaglia's theory of projectile motion was not completely satisfactory, but it did confirm the gunners in their belief that to get maximum range one raised the cannon barrel to an angle of 45°. The problem is generally regarded as having been solved, in theoretical terms, by Galileo, in his *Discorsi e dimostrazioni matematiche intorno a due nuove scienze*, published in Leiden in 1638, but gunners did not find Galileo's solution much more useful than Tartaglia's.[35]

It seems that in the fifteenth and sixteenth centuries mathematical education steadily became more easily available, and its general level rose. Mathematics is one of the subjects in which the availability of elementary instruction is particularly likely to be influential, because it is a subject in which children can show talent very early and thus find themselves being encouraged to continue their education. With this growth in mathematical education, it seems that mathematics slid into craftsmen's practice in rather the same way that silicon chips slide into artefacts today.

Taking algebra seriously

Tartaglia's theory of the flight of the cannonball was based on Aristotelian ideas about 'natural' and 'forced' motion, and the mathematics involved was geometry. As we have already mentioned, it was highly original to attack this particular problem in mathematical terms; but the actual mathematics was of the conventionally learned kind. The immediate model was probably the work of Archimedes (*c.*287–212 B.C.), whose theoretical writing on statics survives, though his activities as a military engineer, apparently never written up by Archimedes himself, are recorded merely in non-technical secondary litera-

35 See A. R. Hall, *Ballistics in the Seventeenth Century*, Cambridge: Cambridge University Press, 1952; and A. R. Hall, 'Gunnery, Science and the Royal Society', in *The Uses of Science in the Age of Newton*, ed. John G. Burke, Berkeley and Los Angeles: University of California Press, 1983, pp.111–41. The chief difficulty is that even at muzzle velocities as low as those in the seventeenth century, the motion is strongly affected by air resistance, which Galileo neglected.

ture such as Plutarch's *Lives*.[36] Tartaglia was to produce the first, partial, printed edition of works by Archimedes, published in Venice in 1544.[37] However, as we have already noted, even in his own time, Tartaglia was best known as an algebraist. Although his *Quesiti e inventioni diverse* contains problems on fortification and other military matters, the parts that proved most significant were those dealing with algebra.

With hindsight, we can see that the development of algebra in the fifteenth century and, at quickening pace, in the sixteenth, has an importance that reached well beyond the then confines of mathematics. Piero della Francesca's work is showing us merely the beginning of this process in the spread of mathematical education among prospective merchants and craftsmen. Piero's direct contribution to the work of the next generation was that almost all of the arithmetic and algebra of his *Trattato d'abaco* were incorporated, sometimes in developed form, in Luca Pacioli's *Summa de arithmetica, geometria, proportioni e proportionalità*, published in Venice in 1494. As we have already mentioned, this publication is further evidence, with Pacioli's use of Piero's *Libellus de quinque corporibus regularibus* in his *De divina proportione* (1509), that Pacioli had access to Piero's manuscripts, though it seems most likely that this happened only after Piero's death. Pacioli's *Summa* rapidly became the standard elementary textbook for algebra, and it was repeatedly cited by subsequent authors, though because of the speed of development in algebra in the following decades these references rather often take the form of complaints about Pacioli's errors.

As we have seen, in the fifteenth century it was possible to earn a living as a teacher of mathematics in an abacus school. As well as the schools set up by town councils and guilds, there were also private schools. In their competition to attract pupils, teachers sometimes engaged in public contests to show their skill in solving problems, which usually meant, in today's terms, solving equations. One of the places in which such contests were held was Bologna, a city that, having a university with a large medical school, had a demand for instruction in mathematics: physicians needed to know some astrology.[38] The eminent humanist mathematician and translator Federigo Commandino took a degree in medicine at the university of Bologna, and Serlio's good grasp of mathematics may be connected with his having received his earliest education in the city.

It seems that one of the leading mathematicians of Bologna, Scipione dal Ferro (1465–1526), invented a general method of solving cubic equations. Islamic mathematicians had methods for solving particular cubics, and some of these had found their way into the Western abacus tradition, where some of them were in turn (erroneously) claimed to be general. This is where the habit of explaining everything by worked examples shows its weakness: worked examples do not demonstrate that the method would work for absolutely any example. However, it seems Scipione's method truly was general, that is it worked for all cubic equations. Scipione apparently regarded this method as a trade secret – not unreasonably, since winning equation-solving contests was a means of attracting more pupils, and thus earning a better living – and he passed the method of solution on to a pupil, Antonio Maria Fiore.

It was probably in the 1530s that Fiore encountered Tartaglia in a contest, and Tartaglia, by his own account, managed to work out how his opponent was solving the cubic equa-

36 See D. L. Simms, 'Archimedes the Engineer', *History of Technology* 17, 1996, pp.45–111.

37 Archimedes, *Opera Archimedis . . . per Nicolaum Tartaleam . . . multis erroribus emendata . . . ac in luce posita, etc.*, Venice, 1544.

38 See Chapter 7.

tions. Tartaglia eventually published his method in *Quesiti e inventioni diverse* (1554), in the form of a rhyme, presumably intended to be committed to memory. Even in this simplified form, the story might lead to some reflections on the emergence of ideas of intellectual property, but in reality things were complicated by the fact that, before publishing his solution, Tartaglia discussed it with Girolamo Cardano (1501–1576). Cardano, who was born in Gallerate, near Milan, had a successful career as a physician, with some additional astrology on the side, and he also taught mathematics. The combined bulk of his printed works speaks for itself: Cardano believed in publishing. And he published Tartaglia's general solution for cubics in his *Ars magna* in Basel in 1545 – that is, before Tartaglia published it himself. Cardano did mention in his preface that he had obtained the solution from Tartaglia – adding that he felt free to publish because he had learned Tartaglia had obtained it from Fiore – but it was nonetheless taken that Cardano was claiming the solution as his own. Posterity has, quite unfairly, adopted the name 'Cardano's solution'. We cannot of course be perfectly sure the solution is exactly the same as that of Scipione dal Ferro or Antonio Maria Fiore, but it certainly is the same as that of Tartaglia. Cardano's publication was followed by a priority dispute in which each side accused the other of various forms of improper conduct. The controversy was carried on in print, no doubt enriching booksellers, and it in effect preserves an acrimonious version of an equation-solving contest.

Cardano's *Ars magna* was the first work on algebra to be written in Latin and thus explicitly addressed to the learned. Cardano himself seems to have considered the work important since he ended it with the lordly *envoi*: 'It was written out five times, may it last the same number of millennia' ('quinquies exscriptus maneat tot millibus annis'). All the same, it seems that neither Tartaglia nor, at first, Cardano grasped the full mathematical significance of the invention of a general method of solving cubics. To put it simply: the geometrical methods of Euclid allow one to solve quadratic equations, but nothing beyond that. In providing a general solution for cubic equations, mathematicians had made algebra more powerful than geometry. However, even as the consciousness of this new-found power sank in, algebra remained less rigorous than geometry, or at least its rigour could not be proved. This fact has, I believe, a serious historical significance that has largely been neglected by historians.

The lack of rigour takes a flagrant form in the Tartaglia–Cardano method of solving cubics. This may, of course, at least partly account for Tartaglia's hesitation over publishing. The method involves taking the square root of a negative number. Indeed even to put it that way is to make the matter less worrying than it would have been to a contemporary, since at the time 'number' (*numero*, *numerus*) meant a natural number, that is in today's terms, a positive integer. The concept of a negative number came to be accepted at much the same time as it became accepted that one could speak of taking the square root of a negative number. The objection was the same in both cases, namely that one could not draw such a thing. Following ancient practice, as found in Euclid, numbers were seen as the measure of lines. The idea that the line from *B* to *A* might be treated as the negative of the line from *A* to *B* appears only in the seventeenth century, notably in the work of René Descartes (1596–1650) and Girard Desargues.[39] In the sixteenth century, subtraction was seen as possible only if the number to be subtracted was smaller than the one from which it was to be subtracted. Constructing a line whose measure was a square root was straight-

39 René Descartes, *Geometrie*, Leiden, 1637; Desargues, *Brouillon project d'une atteinte aux evenemens des rencontres du Cone avec un Plan* (full ref. note 16).

forward. For example, Pythagoras' theorem allows a line length $\sqrt{5}$ to be constructed as the hypotenuse of a right-angled triangle whose other sides are of length 1 and 2. However, there was no way of constructing a line so that its length was $\sqrt{-5}$, that is, so that a square with that line as side would have a negative area of 5.

In the 1540s algebra was still seen as an extension of arithmetic and, since Euclid presents arithmetic as subordinate to geometry, algebra too was seen as subordinate to geometry. Accordingly, this impossibility of construction entitled a mathematician to say that the entity in question did not exist. It is clear that this did not deter the algebraists. In fact, although their method for solving cubic equations in some cases involves taking square roots of negative numbers, these roots may disappear again before one comes up with the actual values of the unknown that constitute the solution. So the final answers are in acceptable form. (One can obtain answers containing square roots of negative numbers, but these come in pairs, of the type $a \pm \sqrt{-b}$, so, since a cubic equation has three roots, at least one of them must be in acceptable form, that is, not involving the root of a negative number.) Moreover, one can put these values of the unknown back into the equation and see that they satisfy it. So one can check that the solution is indeed correct.

As we have seen, in abacus books it was usual to check the answer. Most of the relatively advanced arithmetic problems and algebra problems in abacus books are susceptible to having answers checked in this way. That is, checking does not involve reworking the complete calculation, but is a rather simple process. If the aim is merely to get the right answer – as it may well be in real life – then this kind of problem allows one to make reasonable guesses and then check to see whether they are correct. Indeed, the method of 'false position' involves doing exactly this: trying one or more possible answers, noting how far out they are, and then using this information to calculate the correct answer (see Appendix 1). The important point to notice here is that the rigour of the method used to find the 'false position' or the candidate answer is not important, because one can easily check whether the final answer is actually correct. It is only when there is no means of checking the answer that one has to be certain of the logical and mathematical rigour of the procedures by which one arrived at it.

Here we are, of course, discussing finding answers to individual problems, not establishing general theorems. One clearly cannot escape the need for a rigorous method in establishing a general truth such as that in every isosceles triangle the angles at the base are equal to one another. However, sixteenth-century algebraists do not encounter this difficulty because, except sometimes in passing, when stating rules for calculation, they do not deal in general results. Throughout the sixteenth century, algebraic texts continued to present their material as a series of worked examples, that is a series of specific problems, as in the abacus books of the fifteenth century. Any general rules were tacitly deemed proved by their success in providing solutions to examples – and we may note that the words used for proof, such as the Tuscan 'prova', which Piero della Francesca uses, also denote evidence or confirmation, that is they have a much wider meaning than the corresponding term in present-day English. Today's style of writing equations out in general form, with letters from the beginning of the alphabet standing for constants (that is, known but as yet unspecified numbers) and x, y and z for the unknowns, is first found in the writings of Descartes. However, systematic use of letters for unknowns appears in the work of François Viète (1540–1603), which seems not to have been widely disseminated until after his death. Employing Viète's notation, one could prove general results in algebra. Viète had made an equation something that had properties in the same way that a triangle had.

Some of the above account is based on historian's hindsight, but most of it does seem to be implicit in the practice of the time. Mathematicians could not fail to notice Tartaglia's and Cardano's use of the square roots of negative numbers, but it seems not to have excited loud objections of principle. The matter was settled, however, in 1572, in a vernacular but very learned volume written by Rafaelle Bombelli (1526–1572), an engineer who lived in Borgo San Sepolcro and worked on drainage schemes in the Val di Chiana. His *L'Algebra*, published in Bologna in 1572, whose manuscript draft Bombelli revised radically after the recovery of Diophantus' *Arithmetica* provided some insight into ancient Greek algebra, sets out a body of axioms for handling the roots of negative numbers. It lays down how they can be added, subtracted, multiplied and divided, and whether the results of these operations come out in terms of roots or of negative or ordinary numbers. In other words, Bombelli gives a formal protocol for handling roots of negative numbers in the same way that Euclid provides protocols for the entities handled in the *Elements*. Historians see Bombelli's treatise as both impressive and important, marking the attainment of intellectual maturity in the study of algebra. Contemporaries seem not to have reacted at all. However, as the work of the next generation shows, algebra was increasingly being taken seriously by mathematicians within the learned tradition. Descartes completed the process by making algebra an equal partner with geometry in his *Geometrie* of 1637.

Vernacular and Latin

Cardano's publication of a book on algebra in Latin is rightly seen as marking an epoch in the relationship between the practical and learned traditions in the mathematical sciences, particularly since the date is close to that of Tartaglia's mathematical book on gunnery. However, Cardano seems always to have published in Latin, thereby addressing his works on philosophy, medicine, astrology, and so on, to an international learned readership. Although Cardano is now best remembered as a mathematician, only a small proportion of his published works have to do with mathematics. This is not to belittle the historical significance of his publishing the *Ars magna* in Latin, but one perhaps needs to bear in mind that part of his reason for doing so may have been that he took himself for a learned man, as indeed he was, and as such felt it natural to publish in Latin – whatever the subject. All the same, it is clear that later in the sixteenth century such reasoning was not seen as inescapable. In Italy, at least, the balance between Latin and vernacular seems to have been changing during this period. And this change too is a process whose results become visible in the work of Galileo.

As we have seen, even in the fifteenth century the Latin and vernacular traditions were not completely disjunct. Rather, there was a recognizable area of overlap, at least in subject matter, and some translation took place in both directions. Thus Alberti, in addressing his essay on painting to his patrons, naturally used Latin; but in the following year, when he translated the text into the vernacular, apparently for the benefit of craftsmen, he did not make any substantial changes, and the work's learned humanist connections are as prominent in the vernacular text as they are in the Latin one. In this regard, the printing history of the work is instructive. The essay was first printed in 1540, in Latin.[40] Alberti's own vernacular text had presumably by then disappeared from view, because the first printed vernacular edition, which appeared in 1547, was of a new translation, made from the printed

40 Leon Battista Alberti, *De pictura*, Basel, 1540.

Latin text.[41] Another new vernacular translation appeared in 1568, as part of an edition of Alberti's *Opuscoli morali*.[42] The translator responsible for the 1547 edition, Lodovico Domenichi (*c*.1500–1564), also published vernacular translations of works by ancient authors such as Polybius (Venice, 1545), Boethius (Venice, 1563) and Xenophon (Venice, 1547). This puts Alberti's book into a world of classical learning in the vernacular.

At the time there was an ongoing debate among scholars, largely centred on the work of Petrarch (1304–1374), about the writing of poetry in the vernacular (*volgare*). In this debate, the main point at issue was whether the *volgare* could reach to the rhetorical heights required for serious literature. For technical treatises, the difficulty is rather that of vocabulary. Nor is the advantage always with Latin. In matters of architecture, for example, a problem may arise not from the poverty of the vernacular, but rather from changes in architectural styles or the use of elements not known to the ancients, or at least not recorded in ancient literature. For instance, in *De prospectiva pingendi*, Book 2, section 11, Piero proposes to construct the perspective image of a cross-vault on a square base – 'una volta in crociera sopra a muraglia quadrata' ('a cross-vault over a square of wall') – which in the Latin version is rendered as 'testudinem sive fornicem' ('a tortoise [probably meaning a formation of interlocked shields] or an entrance arch') and the reader is referred to the accompanying diagram, which shows the plan of the vault. This diagram is present in the Parma manuscript, which is vernacular and autograph, but there it is first used in the main body of the proposition, not in its initial statement.[43] The Latin text's calling upon the help of a diagram to make the meaning clear is a patent admission of linguistic inadequacy.

The same type of difficulty was liable to arise in treatises about music. The descendants of the rebec were called by names like 'lira da braccio' that associated them with the ancient world, while the lute, derived from the Islamic *oud*, was sometimes identified with the lyre. However, when it came to matters of practical detail the lack of ancient texts no doubt helped to encourage writing in the vernacular. In the sixteenth century both Gioseffo Zarlino (1517–1590) and his erstwhile pupil Vincenzo Galilei (*c*.1520–1591) wrote learned works on music theory in the vernacular.[44]

In writing on other mathematical sciences, however, the difficulty was likely to lie with the vernacular. As in the statement of the proposition just quoted, Piero della Francesca regularly uses 'quadrato' to mean 'square', but sometimes the mathematical context shows that he is using the word to mean what we would now call a rectangle. Presumably the loose usage of everyday life was finding its way into writing. This is inconvenient for an inattentive reader. However, Piero does usually take care to explain terms, in the manner of a mathematical definition. A slightly comic example appears when he introduces the circle in his *Trattato d'abaco*. He says: 'A circle [*tondo*] is a circular figure [*figura circolare*] enclosed by a single line which is called the circumference [*circumferentia*] . . .'[45] This reappears in the *Libellus de quinque corporibus regularibus*, as printed by Pacioli, in the rather neater form: 'A circle [*tondo*] is a surface enclosed by a single line which is called the circumfer-

41 Leon Battista Alberti, *La Pittura . . . tradotta . . . per M. Lodovico Domenichi*, Venice, 1547.

42 Leon Battista Alberti, *Della pittura*, trans. Cosimo Bartoli, in *Opuscoli morali*, Venice, 1568.

43 Piero della Francesca, *De prospectivà pingendi*, Book 2, section 11: Parma MS fol.29 recto; BL MS, fols 33 recto–34 verso, diagram on fol.31 recto; Piero ed. Nicco Fasola, p.122.

44 Gioseffo Zarlino, *Istitutioni harmoniche*, Venice, 1558; Vincenzo Galilei, *Dialogo della musica antica e della moderna*, Florence, 1581; reprinted 1602.

45 Piero della Francesca, *Trattato d'abaco*: BML MS, p.93 recto; Piero ed. Arrighi, p.198.

ence [*circumferentia*] . . .'[46] The phrasing of the Latin version of the *Libellus* in the single known manuscript copy is the same as in Pacioli's text.[47] The awkwardness of the wording in the *Trattato* seems to arise from the vernacular use of the word *tondo*, which is not from the same root as the Latin word *circulus* used in the learned tradition. In texts dealing with more advanced technical matters, the poverty of vernacular vocabulary was to prove an ongoing problem. For instance, in an astrological tract published in Frankfurt in 1610, written in German but given a Latin title, *Tertius interveniens*, Johannes Kepler repeatedly lapses into Latin for astronomical terms. This is highly visible because the printer sets the Latin in roman type whereas the German text is in blackletter.

Such problems of vocabulary are similar to those faced by any translator, or, in a less formal manner, by anyone reading in a foreign language. In a mathematical text they are not likely to prove a serious obstacle to the reader, who can generally rely upon the logic of the argument to guide guesswork about meanings of particular terms. The same effect would operate in regard to any technical text, though no doubt usually somewhat more weakly. However, when we consider the readership of works written in Italian, there is surely also another important factor to take into account: a good knowledge of Latin is very helpful indeed. Thus anyone with a conventional learned education was likely to be able to understand something of an Italian text. We know, for instance, that Kepler made a serious study of vernacular treatises on music by Gioseffo Zarlino and Vincenzo Galilei and he also read Galileo Galilei's *Saggiatore*, published in Rome in 1623.[48] There is no sign that Kepler's grasp of the Italian language was good enough for him to notice that the *Saggiatore* was written not in Tuscan but in Paduan rustic dialect.

Exceptional though Kepler was in many ways, it seems unlikely that he was unusual in being so fluent in Latin that he could, if sufficiently highly motivated, make sense of a text in Italian. In any case, it seems that, almost anywhere in Europe in the sixteenth century, an upper-class education regularly included the inculcation of a certain amount of Tuscan. It seems likely that Italians did not really need to write in Latin rather than in their local vernacular in order to reach an international readership. All the same, highly technical learned writings were almost necessarily addressed to the whole community of scholars, *urbi et orbi* as it were, which presumably encouraged a preference for Latin; the total potential readership was otherwise likely to be, or at least seem, too small to tempt a publisher. This rule seems to have applied in the mathematical science of astronomy. However, as we have seen, it seems not to have applied in the mathematical science of music. Maybe there were simply more musicians than astronomers.

There was presumably also an adequately large number of possible readers of the books on architecture and on perspective. These books were usually more heavily illustrated than the musical ones – though Zarlino's *Istitutioni harmoniche*, published in Venice in 1558, has notably handsome woodcut diagrams. For all these books, most of which were folio, the question should really be not about possible readers but first of all about possible purchasers. Prefaces and letters to the reader regularly claim that the books on architecture and per-

46 Piero, *Libellus*, Tractatus 1, paragraph immediately above casus 43, in Pacioli, *De divina proportione*, Venice, 1509, p.7 verso.

47 Piero ed. Mancini, p.515.

48 See J. V. Field, 'Kepler's Rejection of Numerology', in *Occult and Scientific Mentalities in the Renaissance*, ed. B. W. Vickers, Cambridge: Cambridge University Press, 1984, pp.273–96; and Johannes Kepler, *Harmonices mundi libri V*, Linz, 1619, reprinted in KGW, vol.6 (English translation *The Harmony of the World*, trans. E. J. Aiton, A. M. Duncan and J. V. Field, Memoirs of the American Philosophical Society, vol.209, Philadelphia, 1997).

spective will be useful to the practitioner, but the books were far too expensive for the normal run of artisans to buy. The production of these works as folios, and the occasional large quarto, makes it tolerably clear that the sale of the books was really to the class of persons that employed artisans and supervised their activities, probably also including some people who only daydreamed about doing so. The existence of this kind of amateur readership seems to have been recognizable in the next century, since in 1639, when writing to Desargues about possible readers of a learned work in the vernacular (in this case on mathematics), Descartes refers to readers who 'read works on Heraldry, Hunting, Architecture, etc., without any wish to become hunters or architects but only to learn to talk about them correctly.'[49]

Such amateurs were the sixteenth- and seventeenth-century counterparts of the people who, in the fifteenth century, read the Latin versions of Piero della Francesca's treatises, people whose interest in the subject matter was intellectual and perhaps social but not practical. In sixteenth-century Italy this kind of reading could be done in the vernacular. For the history of science, it is of particular significance that in this period learned books on music theory were being written in the vernacular, because Galileo Galilei was the son of one of the leading music theorists of the time, Vincenzo Galilei. As we have already mentioned, Vincenzo's learned treatise on music theory, *Dialogo della musica antica e della moderna*, published in Florence in 1581, was written in the vernacular. Moreover, his ensuing controversy with Zarlino, an impeccably learned exchange on both sides, and heavily embellished with appeals to the few surviving ancient texts on music as well as to what could be deduced of ancient practice, was nevertheless carried on in the vernacular.[50] The controversy was also notably acerbic, in which respect it may have set Galileo an unfortunate example. In any case, Galileo too wrote most of his books in the vernacular, though we may note that his brief account of what he had seen through his telescope, *Sidereus Nuncius* (1610), was published in Latin – and instantly became an international best-seller, with pirated editions appearing within months.

Despite this huge success in Latin, Galileo turned to the vernacular. It is sometimes said that this was to appeal for support for his controversial views over the heads of learned philosophers (Latin readers) to the wider public, but it may just as well have been to assert that his treatises on various subjects relating to terrestrial physics were connected with everyday experience in the same way as books on other subjects commonly treated in the vernacular. Such an assertion would indeed be highly defensible in the case of Galileo's last book, the *Discourses on Two New Sciences*, published in Leiden in 1638, since the sciences in question – sciences because they are treated mathematically – are the study of motion (including, as we have already mentioned, a treatment of the flight of the cannonball) and the investigation of the strength of materials. These sciences have obvious connections with fortification and architecture, both subjects generally written about in the vernacular. Galileo indeed claimed to have learned much from the workmen in the Arsenale in Venice, so there is nothing incongruous in supposing that his writings intend to exemplify rather than hide the author's connection with craftsmen.

49 Descartes to Desarges, 19 June 1639, translated in J. V. Field and J. J. Gray, *The Geometrical Work of Girard Desargues*, New York, 1987, p.176; René Descartes, *Correspondence*, ed. C. Adam and G. Milhaud, Paris: Vrin, vol.3, 1940, pp.228–9.

50 See D. P. Walker, *Studies in Musical Theory in the Late Renaissance*, London: Warburg Institute, 1978; and Field, 'Kepler's Rejection of Numerology' (full ref. note 48).

Learning the lessons

Galileo may also have learned something from his musician father in the matter of methodology. Music was one of the four mathematical sciences of the *quadrivium*, and as such its theory was taught in universities, but it nevertheless had well-established connections with the craft of the executant musician. Vincenzo Galilei, who was a composer as well as a theoretician, took a lively interest in performance. There was thus nothing particularly startling in his conducting experiments to provide evidence in support of his musical theory, as he describes in his *Dialogo della musica antica e della moderna*. Galileo's use of experiment, albeit in a different context, seems likely to have owed something to the parental example.[51] Moreover, in the mathematical science of music, the reliability of mathematics as a means of obtaining results was not in question. Galileo took this attitude over into areas of natural philosophy where it had not previously been accepted. He naturally appeals to the example of Archimedes, and the mathematics he uses is mainly geometry, but he seems to have been acquainted with some 'practical mathematics', namely commercial arithmetic. Whether he was actually taught this kind of mathematics as a boy is open to conjecture, but in his *Dialogo sopra i due massimi sistemi del mondo*, published in Florence in 1632, he refers to commercial arithmetic for its assumption that the ducats in the real world will behave in exactly the same way as those in the school problem.[52] In context, this becomes a serious assertion, against some unnamed philosophers, of the applicability of mathematics to problems in the real world. Galileo is, of course, rightly famed for his skill in making telling points in a rhetorical fashion – and one may suspect that this verbal dexterity helped to make him enemies – but in the present instance it seems he has merely given characteristically pithy expression to a notion that was of some importance in the learned world.

As we have seen, this sort of disregard for strict philosophical rigour, in favour of trying to find usable answers, had its counterpart in changes within mathematics itself in the sixteenth century. In this respect, mathematics was becoming more like a craft. These changes have connections with the early stages of the Scientific Revolution, not only in the sense that general traces of them can be found in the work of Galileo, but also in the sense that narrower specific effects can be seen in the work of Kepler. They are most apparent in Kepler's work in astronomy, a subject established since ancient times as a branch of mathematics. The occasion for Kepler's departure from the standard mathematical methods used by astronomers arose from his confronting a problem that was in some important respects entirely different from those on which his long line of predecessors had exercised their geometrical skills.

This is not to say that Kepler was in any way lacking in geometrical skills. On the contrary, he seems to have been a very able geometer, and he needed considerable powers of visualization to grasp the three-dimensional relationship of Mars and the Earth as they both moved round the Sun, each at variable speed, along paths whose planes were slightly inclined to one another. His task was to find the orbit of Mars, and to use this to construct tables that would predict future positions of the planet.[53] Other astronomers had not looked at such problems the way Kepler did, first for the obvious reason that all but a handful of

51 See Thomas B. Settle, 'The Pendulum and Galileo, Conjectures and Constructions', in *Galileo's Experimental Research*, Preprint 52, Max-Planck-Institut für Wissenschaftsgeschichte, Berlin, 1996, pp.39–49.

52 G. Galilei, *Dialogo sopra i due massimi sistemi del mondo*, Florence, 1632, p.202 (English translation *Dialogue concerning the Two Chief World Systems*, trans. Stillman Drake, Berkeley, California: University of California Press, 1953, pp.207–8).

53 This series of calculations was mentioned briefly in Chapter 2.

them believed the Earth to be at rest (and the Sun in motion), but also, secondly, because they thought they knew what kind of mathematical models of motion they were seeking. In practice, that meant that a few observations would be used to fix certain parameters and the result was a new improved 'theory' of the motion of the planet in question. For philosophical reasons, Kepler tried to avoid preliminary decisions about the shape of the path of the planet, though he had an immovable conviction that the path must lie in a plane, that this plane must include the Sun, and that the shape of the path must relate to the Sun.

The other important component that made Kepler's starting point different from that of his predecessors was that he had at his disposal an extraordinarily large number of observations of planetary positions, made over a time-span of about twenty years and of unprecedented accuracy. The observations had been accumulated by Tycho Brahe (1546–1601), who apparently regarded them as his private possession, rather as if they were a cabinet of natural curiosities such as other noblemen of the time did indeed amass. Tycho knew what he wanted to use these observations for: to prove his own ideas about the planetary system were correct. However, he seems not to have had clear ideas about exactly how they could be used for this purpose.[54] Kepler, who was at first employed as Tycho's assistant, and was certainly the nearest one could find to a supercomputer in 1600, succeeded Tycho as Imperial Mathematician to the Emperor Rudolph II after Tycho's death in 1601, and in due course inherited the observations. He managed to work out a way of making good use of the huge mass of data, but found himself with a mathematical problem that did not seem to be susceptible of exact solution.

Kepler wanted to find positions for Mars at specified times, so he needed to find not only the shape of the path of the planet but also at what point in the path the planet would be at a specific time.[55] The part of this problem that concerns us here is the second one: finding a way of determining how the position of the planet in its path changed with time. Today, an astronomer would formulate this as finding the speed with which the planet moved, but Kepler did not think in these terms. For the purposes of finding the orbit, to predict positions of the planet, what he needed was something that could be used to measure time. After an attempt to use the sum of distances of the planet from the Sun, what he chose to assume was that a line joining the planet to the Sun would sweep out equal areas in equal times as the planet moved round its path. He justifies this choice by reference to Archimedes' *On the Measurement of a Circle*, and we may reasonably suppose it was also partly based on the fact that the planets further from the Sun (in the Copernican system) move more slowly than those nearer to it.[56] Applying this idea to a single planet, one might expect it to move more slowly when it was further from the Sun. We may note also that if the path of the planet were circular, as in the ideal model envisaged by ancient astronomers (but in Kepler's case with the Sun in the centre), Kepler's rule that the line joining the planet to the Sun sweeps out equal areas in equal times reduces to the planet moving uniformly, as ancient models required.

54 On Tycho see V. E. Thoren, *The Lord of Uraniborg: A Biography of Tycho Brahe*, Cambridge: Cambridge University Press, 1990.

55 That is, he wanted what astronomers now call an 'orbit' for the planet. In normal parlance, the word orbit usually means just the path.

56 For instance, Saturn's distance from the Sun is about ten times that of the Earth, which makes the total distance it travels in a complete circuit ten times that traversed by the Earth so, if the speeds were the same, the time for Saturn to complete its circuit would be ten years, but in fact the period of Saturn is about thirty years.

All the same, Kepler's method of measuring time by measuring area is at first simply a hypothesis. Indeed, the chapter in which he introduces it has the title 'An imperfect method . . . which nonetheless suffices for the theory [that is, model of the motion] of the Sun or the Earth'.[57] The rule turned out to be true, and is now known as Kepler's Area Law. However, his first application of it did not give satisfactory results, which is to say that the orbit gave positions of Mars that did not agree with observed ones found among Tycho's data. Kepler accordingly modified the path. His new model of the path, which allowed him to find distances of the planet from the Sun, used a construction that effectively gave him individual points on the planet's path, and he was not actually concerned with the overall shape of the path as such. The shape of the earlier path had allowed Kepler to find the area swept out in any time (that is, to find it exactly, using geometry). This could not be done for the new path. Instead, he found series of distances such that angles between successive lines joining the planet to the Sun were small (about one degree). He then used the sum of these successive distances as a measure of time.[58]

Although one can see this method of finding time as somewhat resembling the area rule, it differs from the earlier rule in having no solid physical or mathematical rationale behind it. It is simply a calculation device. Its only possible justification would have been that it worked. Unfortunately, it did not. Kepler again found that the planetary positions he obtained from a series of different versions of his construction for finding distances did not agree with Tycho's observations to within the acceptable margin of error. The next modified method of finding distances allowed Kepler to reinstate the area rule and he eventually showed that the path was an ellipse. Which is to say that positions derived from the ellipse and the postulated area rule gave positions that agreed with observed ones to within the margin of error Kepler considered acceptable. However, the lack of success of the earlier non-rigorous method in finding an acceptable path is irrelevant. What is of interest here is the method employed.

Kepler, whose mathematical education at the university of Tübingen had been entirely conventional – apart from introducing him to the Copernican system – was aware that the method he was using was not a rigorous approximation to the area rule.[59] His strategy was to make use of Tycho's observations to check whether the answer, which yielded a position of the planet for a new time, was actually correct (to within his accepted margin). That is, Kepler's non-rigorous method could be employed in practice because he could check the results. This is exactly the mathematical style we have found in algebra. However, we know Kepler had not been taught algebra as a child, and he remained firmly convinced of the superiority of geometry, partly on the grounds that the natural entities he was considering were continuous and it was geometry that dealt with continuous magnitudes, whereas arithmetic – of which he regarded algebra as an offshoot – dealt with discrete quantities. It seems that Kepler was introduced to algebra by Jost Bürgi (1552–1632), who at the time was working as a clockmaker to Rudolph II. Bürgi had also made astronomical instruments,

57 J. Kepler, *Astronomia nova . . .*, Heidelberg, 1609, p.192, title of Chapter 40; KGW, vol.3, p.263.

58 Kepler, *Astronomia nova* (full ref. note 57), pp.215–46, Chapters 45–50; KGW, vol.3, pp.288–322.

59 A well-known example of a rigorous approximation was provided by Archimedes' treatment of the area of the circle in *On the Measurement of a Circle*, in which the area is established as lying between that of a polygon circumscribed around the circle and that of a polygon inscribed within the circle. Archimedes uses a similar method to establish upper and lower limits on the length of the circumference.

Later, Kepler himself and other mathematicians, notably Bonaventura Cavalieri (*c.*1598–1647), were to devise a rigorous version of Kepler's method, which is now known as 'the calculus of indivisibles'. This was an immediate antecedent of the differential and integral calculus of Gottfried Wilhelm Leibniz (1646–1716) and Isaac Newton.

and he was a competent mathematician – he did some useful original work on logarithms – but, unlike Kepler, he was not given to writing his work up for publication. It is a measure of the mutual respect between these two that some of Bürgi's mathematics survives in a draft in Kepler's handwriting, and that Kepler accepted Bürgi's opinion when Bürgi told him that, while his (Kepler's) design for gearing to simulate the motion of planets would in principle yield the right speeds, in practice the train would not run sufficiently freely.

Bürgi had presumably learned algebra as part of his 'practical mathematics' education. As far as I know, Kepler only once made a serious attempt to use algebra for a mathematical problem, and concluded it would not yield the kind of answer he required, essentially because it produced discrete numerical answers rather than continuous magnitudes.[60] This kind of philosophical objection was irrelevant in work on the orbit of Mars, but it is significant that, in the astronomical calculations, Kepler's methods could work only because he had decided how precisely his answers must agree with Tycho's observed positions for the planet. He was, after all, dealing with numbers derived from nature, not ones set by a schoolmaster to test the calculating abilities of pupils. In setting his standard for an answer being accepted as correct, what Kepler introduces into astronomy is the concept of observational error.[61]

Historians are agreed that this is so, but the matter is generally left there. It seems to me that we have here another situation in which, more clearly even than in the interaction between Galileo and his father, practical devices affected theory. Tycho's lavishly illustrated book about his observatory, *Astronomiæ instauratæ mechanica*, published in Wandsbeck in 1598, includes rather detailed accounts of how his exceptionally large and carefully designed astronomical instruments were made and how they were used.[62] In particular, Tycho tells us that each was tested for consistency when the movable parts were in different positions, and sometimes instruments were remade, even up to three times, in order to improve their performance. Kepler never saw Tycho's most important instruments in action, so it must have been from discussion with Tycho that he formed his estimate of their performance. It seems likely that the concept of observational error arose from considering the irreducible margin of inconsistency in the performance of individual instruments. The instruments were large, and scales were sufficiently finely divided for the estimated errors to be detected.[63]

The remainder of this story is given in detail in many histories of astronomy, but it can be summarized very simply. Kepler's work on Mars established the first two laws of planetary motion that bear his name, and these, together with his further work on the other planets, led to his constructing astronomical tables that proved to be accurate over a long period of time. All tables that incorporated recent observations were reliable at first, but their performance generally deteriorated perceptibly within a few decades. For instance, astronomers found that the tables based upon Copernicus' work that were calculated by Erasmus Reinhold (1511–1553) and published in his *Prutenicae tabulae coelestium motuum* in Tübingen in 1551, were noticeably inaccurate by the mid-1570s. The reliability of Kepler's *Rudolphine Tables*, published in Ulm in 1627, did much to establish that Kepler's

60 See J. V. Field, 'The Relation between Geometry and Algebra: Cardano and Kepler on the Regular Heptagon', in *Girolamo Cardano: Philosoph, Naturforscher, Arzt*, ed. E. Keßler (Proceedings of a conference held in the Herzog August Bibliothek, Wolfenbüttel, in October 1989), Wiesbaden, 1994, pp.219–42. Unfortunately, what I have said in this paper about Kepler's calculations of areas in the *Astronomia nova* is not completely correct. I am grateful to Dr A. E. L. Davis for correcting me on this matter.

61 See J. V. Field, 'Tycho Brahe, Johannes Kepler and the Concept of Error', in *Miscellanea Kepleriana. Festschrift für Volker Bialas*, ed. D. Di Liscia et al., Munich: forthcoming.

62 Reprinted Nuremberg, 1602.

63 See J. V. Field, 'What is Scientific about a Scientific Instrument?', *Nuncius* 3.2, 1988, pp.3–26.

laws were correct and, since the laws related the planets to the Sun, was a powerful force for the acceptance of heliocentric astronomy.[64] The notion of observational error seems to have slipped in quietly at much the same time.

Galileo's understanding of terrestrial physics and Kepler's laws of planetary motion were to play a vital part in later developments in natural philosophy. We may thus see the practical tradition as having helped to hand on ideas that proved to be of great significance. Craftsmen continued, of course, to form a social group distinct from that of scholars such as Kepler and Galileo and their counterparts in later generations. Historians of science writing in the last decades of the twentieth century have made much of the connection of seventeenth-century science with the upper classes.[65] If we look at craftsmen's mathematics, we find that from the mid-seventeenth century onwards the practical tradition gradually ceased to be separate from the learned one. Elementary instruction for people who were not expected to progress beyond the elementary became merely a simpler version of elementary instruction in the learned tradition. Thus 'practical mathematics' gradually became an elementary part of learned mathematics. One naturally cannot give a precise date for the completion of the absorption process, but the last major mathematician I know of whose background lay in a craft tradition was Philippe de la Hire (1640–1718), whose first training was as a painter, in the workshop of his father Laurent de la Hyre (1606–1656).[66]

This description of the revival of mathematics from the fifteenth to the seventeenth centuries is not a complete story in itself, and still less a complete account of the origins of the Scientific Revolution. Nor is the sketch of part of a complicated story – a sketch that does no more than trace some strands in a tangled skein – intended to argue for the recognition of a direct debt owed by, say, Galileo to Piero della Francesca. Though Piero's reference to 'the force of lines' in his introduction to the third book of *De prospectiva pingendi* does indeed seem like a pre-echo of Galileo's later advocacy for mathematics, Piero proposed this 'force' as establishing the truth of artificial perspective, that is he was concerned with a matter in which the use of mathematics was entirely accepted.[67] Galileo was extending the use of mathematical methods into areas where they had not been used by his predecessors and were not generally accepted by his contemporaries.

My purpose is not to attempt to establish a direct connection between such later work and Piero, but rather to show what developing story, *pace* Alberti, Piero's work as a whole, both his paintings and his writings, taken together as a contribution to the cultural history of his time, should be seen to belong to. Piero is a good example of the learned craftsman, and the activities of his intellectual kin were to make notable contributions not only to the development of mathematics but also to the emergence of a mathematical natural philosophy in the two centuries following Piero's death.

64 Kepler's laws, in modern form, and with their modern numbering, are:

1 The path of a planet is an ellipse with the Sun in one focus.
2 As a planet moves along its path, a line joining the planet to the Sun sweeps out equal areas in equal times.
3 For any two planets, the square of the ratio of their periods of revolution about the Sun is equal to the cube of the ratio of the mean radii of their paths.

The first two laws were published in *Astronomia nova seu physica cœlestis αἰτιολογητος tradita commentariis de motu planetœ Martis . . .*, Heidelberg, 1609. The third appeared in *Harmonices mundi libri V*, Linz, 1619.

65 For instance, Steven Shapin and Simon Schaffer, *Leviathan and the Air-pump: Hobbes, Boyle, and the Experimental Life*, including a translation of Thomas Hobbes, 'Dialogus physicus de natura aeris' by Simon Schaffer, Princeton and Oxford: Princeton University Press, 1985, reprinted 1989.

66 For further details see Field, *The Invention of Infinity* (full ref. note 7), esp. pp.214–20 and 227–29.

67 See Chapter 5.

Appendix 1

The Method of Double False Position

The method of double false position is a way of solving problems by using arithmetic. It consists of taking two possible values ('positions') for the solution, finding how far each is from satisfying the given numerical relationship, and then using the two incorrect 'positions' and the two 'remainders' to calculate the correct answer. The name 'double false position' is a literal translation from the Latin, and the rather strange use of 'position' (rendered as 'positione', with variable orthography, in Italian vernacular texts), without any spatial connotation, comes from its being a translation of an Arabic word, which in context must have meant something akin to 'substitution'. The method itself is of Islamic origin. Leonardo of Pisa, who describes it in his *Liber abaci* (1202, 1228) says it is called 'elchataieym' in Arabic and that the name is translated into Latin as 'duarum falsarum posicionum regula'.[1]

To see how the method works we shall proceed step by step through one of Piero della Francesca's simplest examples, that of a fish:

> There is a fish that weighs 60 pounds, the head weighs $\frac{3}{5}$ of the body and the tail weighs $\frac{1}{3}$ of the head. I ask what the body weighs.[2]

We may note that this is rather a hefty fish, weighing about 40 Imperial pounds. It probably started its arithmetical career as a sea fish in the Persian Gulf. In any case, the problem is essentially practical, since fish are likely to be bought whole and their heads and tails sell for less per pound than the main body does.

Piero's solution is numerical. To make it clearer what is going on in his arithmetic, we shall call the first 'position' x, the first 'remainder' q, the second 'position' y and the second 'remainder' r. The solution begins

> Do thus; say that the body weighs 30 pounds, $\frac{3}{5}$ of 30 is 18 which is the head, . . .

so we have the weight of the head as $\frac{3}{5}x$. Then we are told:

> the tail weighs $\frac{1}{3}$ of the head which is 6.

This gives us the weight of the tail as $\frac{1}{5}x$. Next we are told:

> Add together 30 and 18 and 6 they make 54; . . .

1 Leonardo of Pisa [Fibonacci], *Liber abaci*, in *Scritti di Leonardo Pisano*, ed. B. Boncompagni, vol.1, Rome, 1857–62, p.38 (English translation in John Fauvel and Jeremy Gray, eds., *The History of Mathematics: A Reader*, London: Macmillan, 1987, p.241).

2 Piero della Francesca, *Trattato d'abaco*: BML MS, p.17 recto; Piero ed. Arrighi, p.64.

That is, adding up the weight of the body (x), the weight of the head ($\frac{3}{5}x$) and the weight of the tail ($\frac{1}{5}x$) gives us $1\frac{4}{5}$ times x. This should be equal to the weight we were given for the whole fish, namely 60, but the remainder, in this case a shortfall, is q, so we got

$$\frac{9x}{5} = 60 - q \qquad (1).$$

In Piero's numerical terms this is:

> you want 60, which is what you have [but] less 6; . . .

and he summarizes the first trial:

> thus for 30 that I put I got 6 too little.

In our symbols, this is 'putting in x we get q too little'. The process is now repeated for the second 'position'.

> Do the next trial and put the body as weighing 25, the head weighs $\frac{3}{5}$ which is 15, the tail weighs a third of the head, $\frac{1}{3}$ of 15 is 5. Add together 25 and 15 and 5, [which] makes 45; and you want 60, which you have [but] less 15; . . .

For Piero the summary is:

> for the 25 I put I get 15 too little.

In our symbols, this is 'putting in y we got r too little', and we have the equation

$$\frac{9y}{5} = 60 - r \qquad (2).$$

For the true answer, which we shall call z, the right side of the equation needs to be 60, so the equation will be of the form

$$\frac{9z}{5} = 60 \qquad (3).$$

To get something that looks like this we need to get rid of the q and r on the right-hand sides of our equations (1) and (2). We can do this by multiplying (1) by r and (2) by q and subtracting. That is, we get from (1)

$$r\frac{9x}{5} = 60r - qr \qquad (4),$$

and from (2)

$$q\frac{9y}{5} = 60q - qr \qquad (5).$$

Subtracting (5) from (4) then gives

$$\frac{9}{5}(rx - qy) = 60(r - q) \qquad (6).$$

So we can divide through by $(r - q)$ and get an equation that looks like (3), with

$$z = \frac{(rx - qy)}{(r - q)} \qquad (7).$$

That is, our correct answer is first position times second remainder minus second position times first remainder divided by the difference second remainder minus first remainder. Or, as Piero puts it:

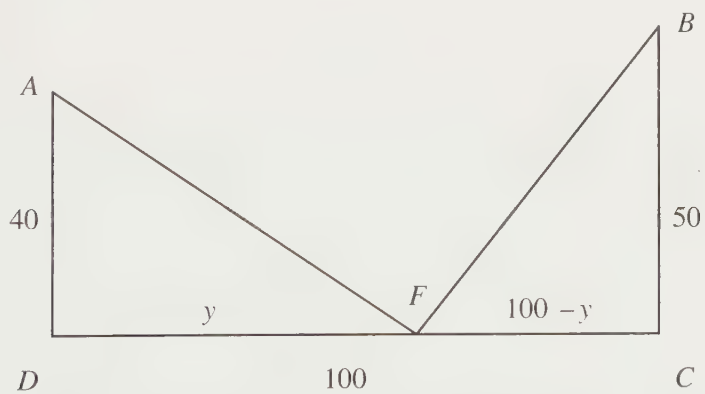
> Now multiply 15 by 30 [which] makes 450, then multiply 6 by 25 [which] makes 150; taking this from 450 there remains 300. Take 6 from 15 there remains 9, which is the divisor. Divide 300 by 9 there results $33\frac{1}{3}$, take $\frac{3}{5}$ of $33\frac{1}{3}$ [which] are 20, and one third of 20 is $6\frac{2}{3}$. Accordingly the body weighs $33\frac{1}{3}$, the head weighs 20, the tail weighs $6\frac{2}{3}$; which added together give 60, as I said the fish weighed.

For a problem as simple as this, the method of double false position is much more cumbersome than using algebra, which would have immediately given us equation (3), which can be solved by a couple of multiplications and divisions by simple numbers. However, the situation is different if we look at the problem of the two towers and the birds flying off to the fountain to drink, which we discussed briefly in Chapter 1. The problem is:

> There is a plain on which there are two towers, one 40 *bracci* high, the other 50 *bracci*, and from one tower to the other it is 100 *bracci*. And on each tower there is a bird, which birds, flying at the same speed, set off at the same moment to go to drink; and they arrive at the same moment at a fountain that is between one tower and the other. I ask how far the fountain is from the tower that is 40 *bracci* [high] and how far from the one that is 50.[3]

For this problem the algebra is not so simple as it was in the problem of the fish. Using today's terms and notation, the algebraic solution would be as follows.

Let the distance of the fountain from the foot of the tower of height 40 *braccia* be y. The set-up will then be as shown in Figure A1.1.



A1.1 Two towers, heights 40 and 50 *braccia*, separated by a distance 100 *braccia*, with a fountain between them. We are required to find the distance of the fountain from the tower of height 40 *braccia*. From Piero della Francesca, *Trattato d'abaco*, BML MS, p.22 recto (Piero ed. Arrighi, p.72), where no diagram is supplied. Diagram for solution by algebraic method. Drawing by JVF.

3 Piero della Francesca, *Trattato d'abaco*: BML MS, p.22 recto; Piero ed. Arrighi, p.72.

We have been told that the birds leave the towers at the same instant, that they fly at the same speed and arrive at the fountain at the same moment. So their flight paths, which we shall assume to be straight, must be of equal length. That is, we have

$$AF = BF.$$

By Pythagoras' theorem in triangle ADF we have

$$AF^2 = AD^2 + DF^2.$$

Substituting known values gives

$$AF^2 = (40)^2 + y^2 \qquad (8).$$

By Pythagoras' theorem in triangle BCF we have

$$BF^2 = BC^2 + CF^2.$$

Substituting known values gives

$$BF^2 = (50)^2 + (100 - y)^2 \qquad (9).$$

Since we know $AF = BF$, the right-hand sides of equations (8) and (9) must be equal, that is

$$(40)^2 + y^2 = (50)^2 + (100 - y)^2.$$

Some arithmetic and expanding the second term on the right using the identity $(a - b)^2 = a^2 - 2ab + b^2$ (proved in Euclid's *Elements*, though not in this form) gives us

$$1600 + y^2 = 2500 + 10000 - 200y + y^2,$$

that is

$$2y = 25 + 100 - 16$$
$$=109$$

and

$$y = 54\tfrac{1}{2}.$$

So Piero's problem really amounted to asking us to solve the equation $2y = 109$, which is in the same form as equation (3) above, and accordingly can be solved by the method of double false position. However, in order to get to the simple form $2y = 109$ we had to do some algebraic manipulation. In modern notation, this is not very difficult, but if everything is written out in ordinary prose, as it would have been in the rhetorical algebra of Piero's time, the manipulation looks much more daunting. Moreover, it involves using the expansion for the square of the difference of two lengths, which is liable to slow down the inexperienced. Piero's solution by means of the method of double false position is fairly long – it is quoted in full in Chapter 1 – but the algebraic method would probably have been at least as long, and certainly more difficult for the pupil to follow. Algebra comes after arithmetic in abacus books because it was generally regarded as harder.

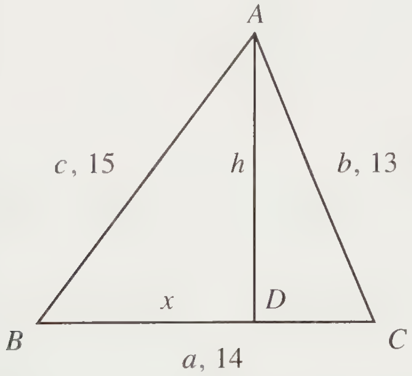
Piero's algebraic examples do include one that is the same as the fish problem (with slightly different numbers, see Chapter 1), but they do not include a version of the towers, birds and fountain problem. His more awkward examples of algebra are all merely numerical.

Appendix 2

Piero della Francesca's Methods of Finding the Height of a Triangle

The two general methods that Piero recommends for finding the height of a triangle in his *Trattato d'abaco* both use the abacists' standard 13, 14, 15 triangle as their example, and take the side of length 14 as the base of the figure. The diagram is lettered as in our Figure A2.1. The first procedure is:

> But when they [the sides] are not equal, as happens when there is a triangle with AB 15, BC 14, AC 13 and the base is 14, multiply it into itself, [which] makes 196, multiply AB which is 15 into itself [which] makes 225, join it with 196 [which] gives 421; multiply AC which is 13 into itself [which] makes 169, take it from 421 there remains 252; divide by double the base BC which is 14 [so doubled] it will be 28, the result is 9: and 9 is [the distance] from B to the point at which the cathetus falls.[1] Multiply 9 into itself [which] makes 81, and multiply AB which is 15 into itself [which] makes 225; subtract 81 there remains 144: its root is the height [*catecto*], which is 12. And this method can be used for all triangles.[2]



A2.1 The 13, 14, 15 triangle, showing height above base 14. Figure for a modern version of the procedure described in Piero della Francesca, *Trattato d'abaco* (BML MS, p.80 recto, Piero ed. Arrighi, p.170). Drawing by JVF.

1 That is, the foot of the perpendicular to the base. 'Cathetus' is the standard word for a perpendicular height.

2 Piero della Francesca, *Trattato d'abaco*: BML MS, p.80 recto; Piero ed. Arrighi, p.170.

Reconstructing the method behind Piero's procedure is hampered by his reliance on numbers. To avoid ambiguities, the following reconstruction will adopt the standard modern usage by which the side opposite vertex A is called a, that opposite B is called b and that opposite C is called c. One length we want to find is BD, so we shall call it x. Let the height of the triangle, above the base BC, be h.

Piero's first instruction is to square the base, BC, which gives us a^2. We then square AB, which gives us c^2. We add them, getting $a^2 + c^2$. We next square AC, which gives b^2, and subtract this from the earlier sum, which gives us $a^2 + c^2 - b^2$. We then divide this by double the base, that is $2a$. Piero says we have now obtained the distance of D, the foot of the perpendicular height, from B, that is the length we have called x. So in algebraic terms we supposedly have

$$x = \frac{a^2 + c^2 - b^2}{2a}.$$

This is indeed correct, as can be seen by looking at the right-angled triangles formed by drawing the height.

By Pythagoras' theorem in triangle ABD, we have $AD^2 = AB^2 - BD^2$.
That is,

$$h^2 = c^2 - x^2 \qquad (1).$$

By Pythagoras' theorem in triangle ADC, we have $AD^2 = CA^2 - CD^2$.
That is,

$$h^2 = b^2 - (a - x)^2.$$

Using the binomial expansion for the second term on the right (this expansion for the square of a sum is found in Euclid's *Elements*), then gives us

$$h^2 = b^2 - a^2 - x^2 + 2ax \qquad (2).$$

Both equations (1) and (2) give us expressions for h^2, so their right-hand sides must be equal; that is, we have

$$c^2 - x^2 = b^2 - a^2 - x^2 + 2ax.$$

This can be tidied up to give us

$$2ax = a^2 + c^2 - b^2$$

or

$$x = \frac{a^2 + c^2 - b^2}{2a} \qquad (3).$$

This is the same as the result of the calculation in Piero's first procedure. So the procedure is correct.

Piero's second procedure is:

> Again one can find the height by another method; that is add up the two sides on either side of the base, that is 15 and 13, [which] makes 28, divide by 2 [which] gives 14, and dividing by 2 again gives 7; add the divisor, which is 2, [which] makes 9; and 9 is [the

length of the line from] B to D where the height falls. Multiply it into itself [which] makes 81 and 15 into itself makes 225, subtract 81 from it there remains 144; its root is the height, which is 12.

(Ancora se po' trovare il catecto per questo altro modo; cioè giogni i doi lati a canto la basa, ch' è 13 e 15, fa 28, parti per 2 ne vene 14, et parti pure per 2 ne vene 7; giognici il partitore ch' è 2 fa 9; et 9 è da B ad D dove cade il catecto. Moltiplicando in sè fa 81 e 15 in sè fa 225, tranne 81 resta 144: la sua radici è il catecto, ch' è 12.)[3]

The answer Piero obtains, 12, is correct, but his method appears not to be. As we shall see, like many a mathematician, child and adult, before and after him, Piero has got the right answer by luck.

Piero's first instruction is to add the two sides on either side of the base, that is *AB*, *CA*. So we are looking for a formula that will contain a factor $c + b$. As before, one length we want to find is *BD*, so we shall call it x. And we may again let the height of the triangle, above the base *BC*, be h. So we can again use the figure shown in Fig. A2.1, and carry out calculations exactly as before, to obtain equation (3). Since Piero's procedure involves finding the sum $c + b$ and a divisor, equation (3) will be rearranged, using the identity $c^2 - b^2 = (c + b)(c - b)$, to give

$$x = \frac{(c+b)(c-b)}{2a} + \frac{a}{2} \qquad (4).$$

The expression on the right now includes the sum $c + b$ and a divisor, $2a$, a fraction of which is added on after the division has been carried out.

The numerical coincidences that give Piero his correct answer can be seen if we put the appropriate numbers for the 13, 14, 15 triangle into equation (4). It gives us

$$\begin{aligned} x &= \frac{(15+13)(15-13)}{2\times 14} + \frac{14}{2} \\ &= \frac{28\times 2}{2\times 14} + 7 \\ &= 2 + 7. \end{aligned}$$

So we are indeed going to get 9 as our answer, and get it by adding up 2 and 7, but contrary to Piero's procedure it is the term involving $b + c$ that has given us the 2 and the term involving the 'divisor', $2a$, that has given us the 7. The remainder of Piero's procedure for finding h is perfectly correct. It uses Pythagoras' theorem in triangle *ABD*, that is our equation (1) above.

Getting the known right answer is a most effective damper for one's instinct to check any calculation. The question still remains, however, as to how the text came by the nonsense it contains. Mere inattention in writing or copying out something 'well known' is the most likely explanation, if the manuscript of the *Trattato d'abaco* in the Laurentian Library in Florence is indeed autograph, which it certainly seems to be. It is hard to see how a straight-

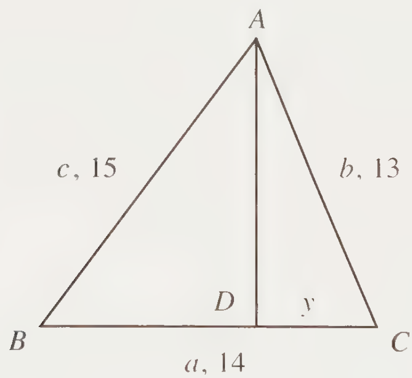
3 Piero della Francesca, *Trattato d'abaco*: BML MS, p.80 recto; Piero ed. Arrighi, p.170.

forward scribal error by a copyist – skipping some words, or even some consecutive lines of text – could introduce this amount of disorder. The Tuscan text has been supplied here so that readers may, if they wish, exercise their own ingenuity in the matter.

A great deal of the geometry in Piero's *Trattato d'abaco* also appears in his *Libellus de quinque corporibus regularibus*, which is believed to be a later work.[4] Methods of finding heights of triangles come at the very beginning of the *Libellus*. The number of methods is only two: one for the equilateral triangle and one for the 13, 14, 15 triangle. The latter is a slight variant on the first of the two general methods given in the *Trattato*. It is:

> Let the sides of the triangle ABC be such that AB is 15, BC 14, and AC 13; and let BC be the base, which is 14: which number [14] multiplied into itself gives 196, and AC, which is 13, also multiplied into itself makes 169, which adding 196[5] will be 365. Now make the same multiplication [into itself] of AB, which is 15, the result is 225, which number subtracted from 365 will leave 140, which always has to be divided by twice the base, which is 28, you will have 5 and you are to say that 5 is [the measure of the distance] from C to the point in which the cathetus meets the base, which is the smaller part, which multiplied into itself makes 25. Then you bring back to the [process of] multiplication into itself the smaller side, which is 13, you will have 169, from which, after subtracting 25 there will remain 144. And the root of 144, which is 12, is the cathetus falling on BC.[6]

The figure required for this method (Fig. A2.2) is slightly different from that for the one in the *Trattato* because Piero is finding the distance from C to the foot of the perpendicular,



A2.2 The 13, 14, 15 triangle, showing height above base 14. Figure for a modern version of the procedure described in Piero della Francesca, *Libellus de quinque corporibus regularibus* (Tractatus 1, casus 1; Piero ed. Mancini, p.490). Drawing by JVF.

4 The dating and content of the *Libellus* are discussed in Chapter 4. The work is printed, in Tuscan, as the final part of Luca Pacioli, *De divina proportione*, Venice, 1509, under the title 'Libellus in tres partes divisus . . .' and without any mention of Piero's name.

5 At this point, the text in G. Mancini's edition (Piero ed. Mancini) gives the number as 169, which is clearly incorrect since it is incompatible with the remainder of the calculation. With the exception of this single emendation, I have followed Mancini's text, which includes a few small corrections to readings found in the manuscript. All the errors he corrects are of the usual trivial kind that can readily be ascribed to a copyist, that is repetition of words, and suchlike.

6 Piero ed. Mancini, p. 490. The sense of this is exactly the same as that of the Tuscan text of the corresponding passage given in Pacioli, *De divina proportione* (full ref. note 4), 'Libellus in tres partes divisus . . .', [Piero della Francesca, *Libellus de quinque corporibus regularibus*], separately paginated, p.1 recto.

whereas he previously found the distance from *B*. In modern algebraic terms, Piero's procedure is as follows:

We first square the base, *BC*, which gives us a^2, and square *AC*, which gives us b^2. We add these terms, which gives us $a^2 + b^2$. We also square *AB*, which gives us c^2, and subtracting this from the previous sum gives us $a^2 + b^2 - c^2$. We then divide by twice the base, that is $2a$. Piero says this gives us the distance of *D* from C; that is, the distance marked y in Figure A2.2. So we have

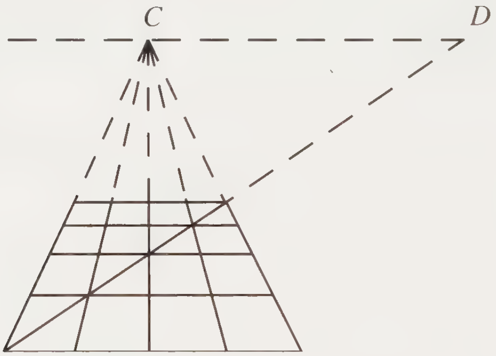
$$y = \frac{a^2 + b^2 - c^2}{2a}.$$

It is left as an exercise to the reader to show that this is equivalent to equation (3) above. (Note that $x + y = a$, and bear in mind that Piero regards numbers merely as measures of lengths, and treats the line *BC* as the same as the line *CB*, since both measure the distance between the points *B* and C. It is thus not significant that the modern algebraic treatment yields some apparently negative lengths.)

Appendix 3

The Distance Point Method of Perspective Construction

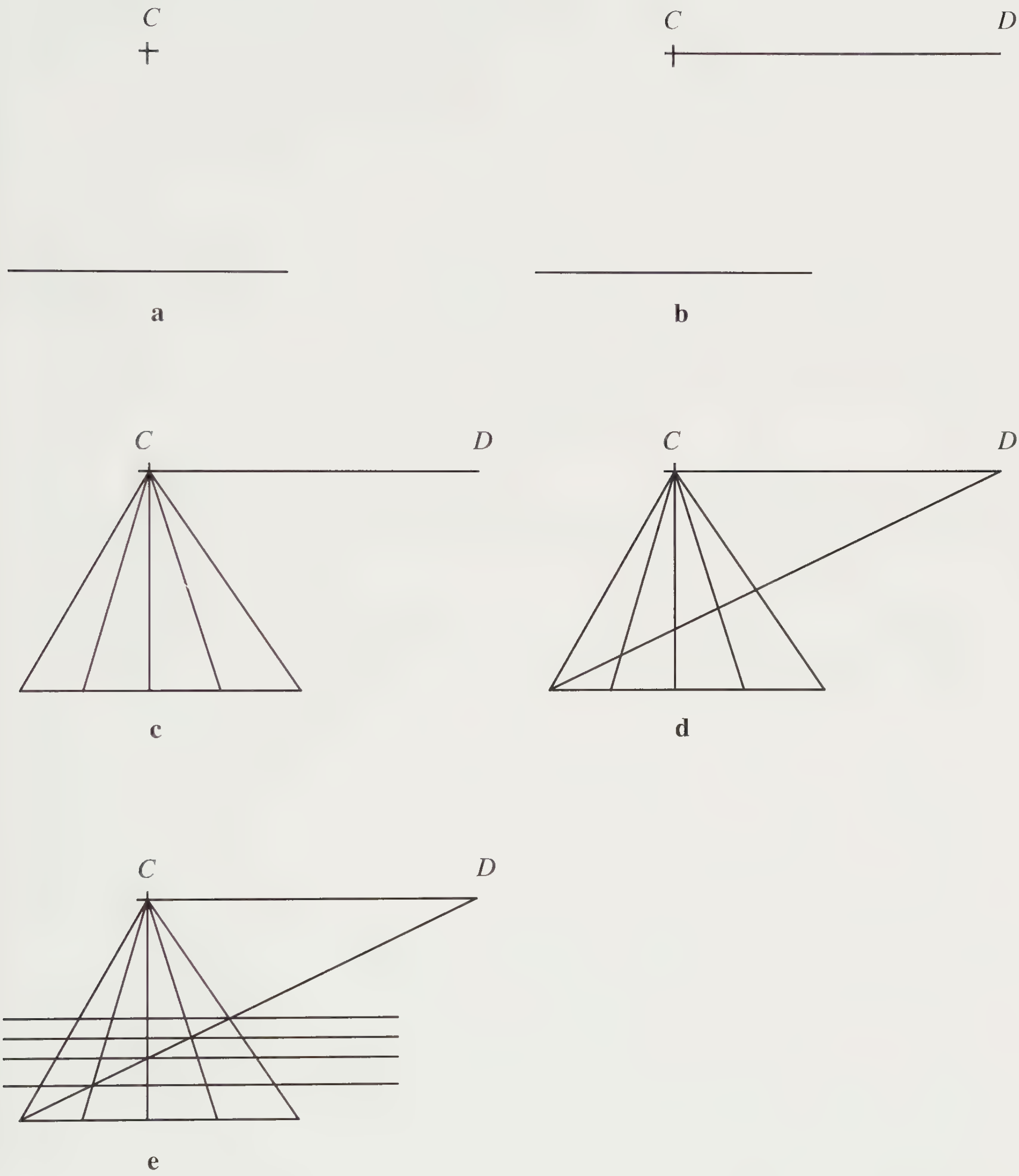
The history of the distance point method is obscure. This appendix is concerned with its mathematics, which surfaces in a more or less explicit form only in the perspective treatises of the sixteenth century. It usually appears in the form of a diagram rather like that shown in Figure A3.1.



A3.1 The distance point construction for a square-tiled pavement. The centric point is *C*, the line *CD* is horizontal and the distance of the eye from the picture is *CD*. Drawing by JVF.

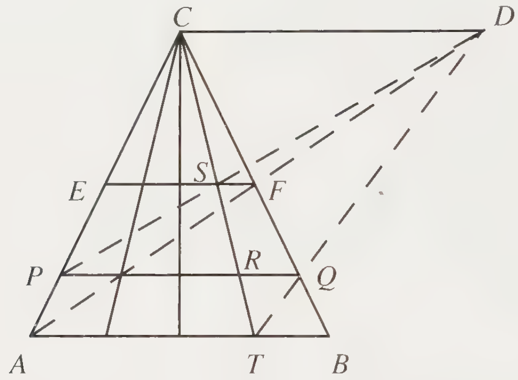
The method of drawing the square-tiled pavement is as shown in Figures A3.2 a to e. The steps are:

a. Draw the front edge of the pavement, and choose the position of the centric point, *C*.
b. Draw a horizontal line (that is, one parallel to the front edge of the pavement) through *C*, and extend it to a point *D* such that the distance *CD* is the distance of the eye from the picture.
c. Divide the front line into the required number of parts, and join the points of division to *C*.
d. Join *D* to the further end of the line representing the front of the pavement.
e. Where this line through *D* cuts each of the lines through *C* put in a line parallel to the front edge.



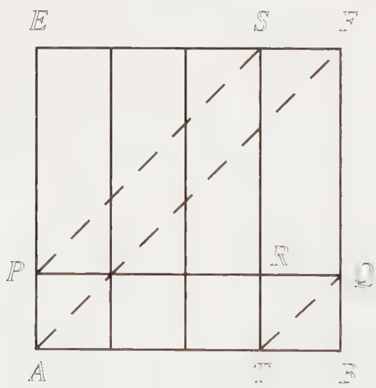
A3.2 Drawings by JVF.

Let some of the points in the preceding set of figures be lettered as in Figure A3.3. The line we drew through *D* in Figure A3.2 was the diagonal of the square *ABFE*. There is, however, nothing special about *ABFE*, except that one of its sides is parallel to the horizon line *CD*. In fact, the diagonal of any other similarly oriented square would also pass through *D*. That is, lines such as *PS* and *TQ* can be extended so that they also pass through *D*.



A3.3 The distance point construction used to draw the image of a square *ABFE*, with centric point *C* and distance point *D*. Diagonals of all squares with one side parallel to *AB* converge to the distance point. Drawing by JVF.

Figure A3.4 shows which are the corresponding lines in the plan of the pavement. Corresponding points have been given letters in outline typeface.



A3.4 The lines in the original square corresponding to those shown in its perspective image in Figure A3.3. Drawing by JVF.

Appendix 4

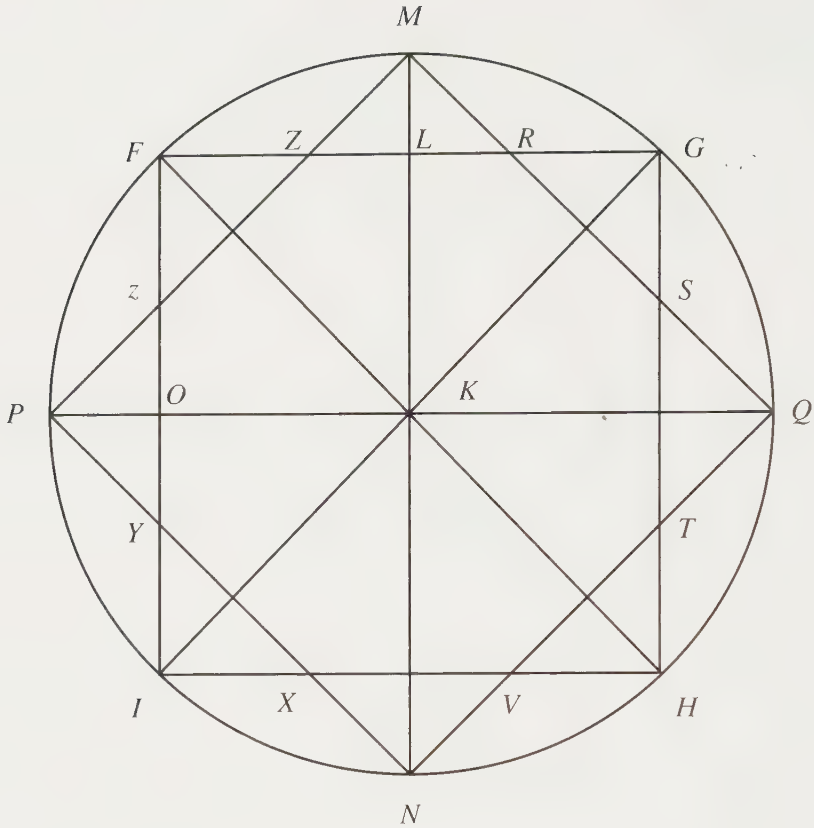
Making a Square into an Octagon

Piero cuts the corners off a square to turn it into a regular octagon as a preliminary to constructing the perspective image of an octagon in *De prospectiva pingendi*, Book 1, section 26. The problem in Book 1, section 16 required one to construct an octagon from a square, but there Piero was concerned simply with the perspective construction, and did not explain how to obtain the vertices of the octagon. The construction was presumably a standard problem. In section 26 Piero does not supply a diagram. The passage in question is Proposition 26, 'On the degraded plane to draw the given octagon':

> Here is the degraded plane .BCDE., on which it is desired to put the given equilateral octagon. I shall first make the square .BCDE. in its proper form, in which I shall draw [*descrivero*] the given surface with eight edges [*facce*], first drawing [*descrivendo*] in the said plane a quadrilateral [i.e., a square], let this be .FGHI., in which I shall construct [*menero*] the diagonals .FH. and .GI., which will cut one another in the point .K., and on the point .K. I shall place the fixed foot of the compasses [*sexto*], and the other, movable, foot I shall rotate [with a separation] the size of .KF. making a circle, touching .FGHI.; then I shall divide .FG. into equal parts at the point .L., and I shall draw [*tirero*] the line parallel to .FI., passing through .L. and through .K., meeting [*contignente*] the circle in the point .M. and the point .N.; then I shall divide .F[I]. in two equal parts at the point .O. and I shall draw from .O. a parallel to .FG. passing through .K. and meeting the circle in the point .P. and the point .Q., and I shall construct [*menero*] .MQ., which will intersect [*segnera*] .FG. in the point .R. and .GH. in the point .S., and I shall draw .QN., which will intersect .GH. in the point .T. and .HI. in the point .V.; then I shall construct [*menero*] .NP. which will divide .HI. in the point .X. and .IF. in the point .Y., and I shall make the line [*linero*] .PM., which will intersect .IF. in the point .z. and .FG. in the point .Z.; then I shall construct [*menero*] .RS. .TV. .XY. .zZ. [By our standards he has already drawn them, but they occur as segments; here he draws them as lines in their own right – perhaps inking them in?], and the octagon in its proper form will be complete.[1]

As Piero constructs all regular polygons in a circle, it is obvious that the above method can be extended to cut the corners off any regular polygon, though for ones with odd numbers of sides the centre of the circumcircle cannot be found as the point of intersection of diagonals but will need to be constructed as the point of intersection of perpendicular bisectors of the sides (that is, lines corresponding to the lines *MN* and *QP* in Fig. A4.1).

1 Piero della Francesca, *De prospectiva pingendi*, Book 1, section 26: Parma MS, p.12 recto; BL MS, p.13 recto; Piero ed. Nicco Fasola, p.89.



A4.1 Diagram for Piero della Francesca, De prospectiva pingendi, Book 1, section 26. There is no corresponding Figure in the Parma or British Library manuscripts. Drawing by JVF.

Appendix 5

Constructing an Octagon from a Square and Drawing the Patterned Pavement in Piero della Francesca's *Flagellation of Christ*

Appendix 4 describes a construction used to cut corners off a square in such a way as to turn it into a regular octagon. It seems possible that a development from the diagram for the construction, extending some lines, and adding others in a symmetrical way – the sort of thing we see Leonardo da Vinci doing in many series of drawings of overlapping or interconnected polygons – may have led Piero della Francesca to the complicated pattern of black and white tiles that fills three of the squares in the paving shown in his *Flagellation of Christ* (Galleria Nazionale delle Marche, Urbino, Fig. 5.28). However, this mathematical doodling process does not provide a good way of preparing to make a perspective image of the pattern. As a demonstration of the method Piero describes in his perspective treatise, we shall show how he could have used it to draw this piece of patterned pavement.

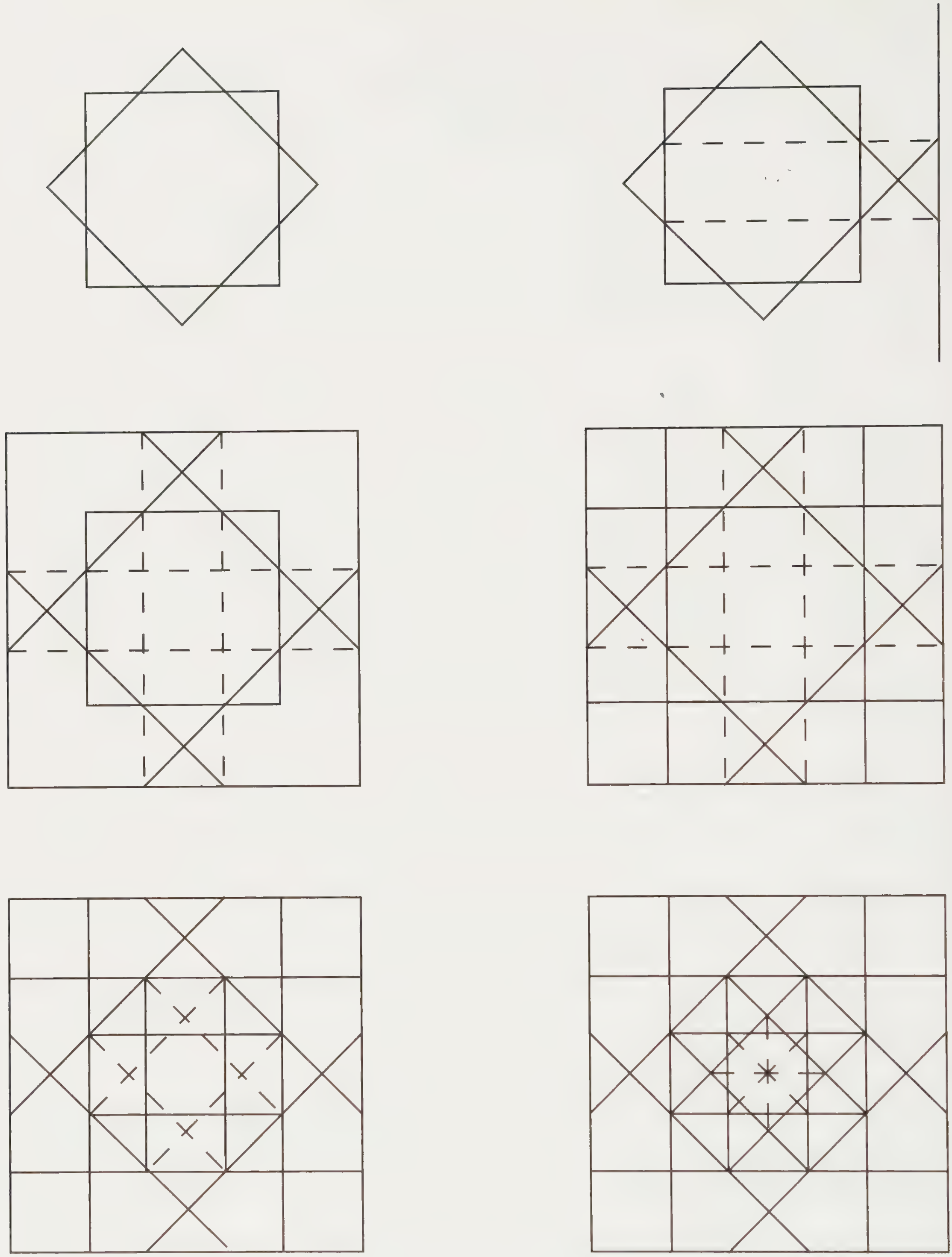
Square to octagon extended

The stages of the possible extension of the pattern are shown in Figure A5.1. The steps of the doodling might be expressed more formally as follows, starting at top left:

a. Initial figure (compare Fig. A4.1). An initial square, with vertical and horizontal sides, is converted into a regular octagon by having corners cut off by constructing a new square whose diagonals are vertical and horizontal.
b. and c. Draw lines through pairs of points of intersection of sides of squares to meet extensions of sides of the new square, and join the points obtained. The line through pairs of these points will be an edge of the outermost square, whose sides are parallel to those of the first square.
d. Extend the sides of the first square to meet those of the outermost one.
e. Tidy up by removing parts of construction lines that will not appear in the final drawing.
f. Draw diagonals of the two innermost squares to get lines that will divide the rhombic tiles of the eight-pointed star.

Drawing the pattern in perspective, using the method of *De prospectiva pingendi*, Book 1

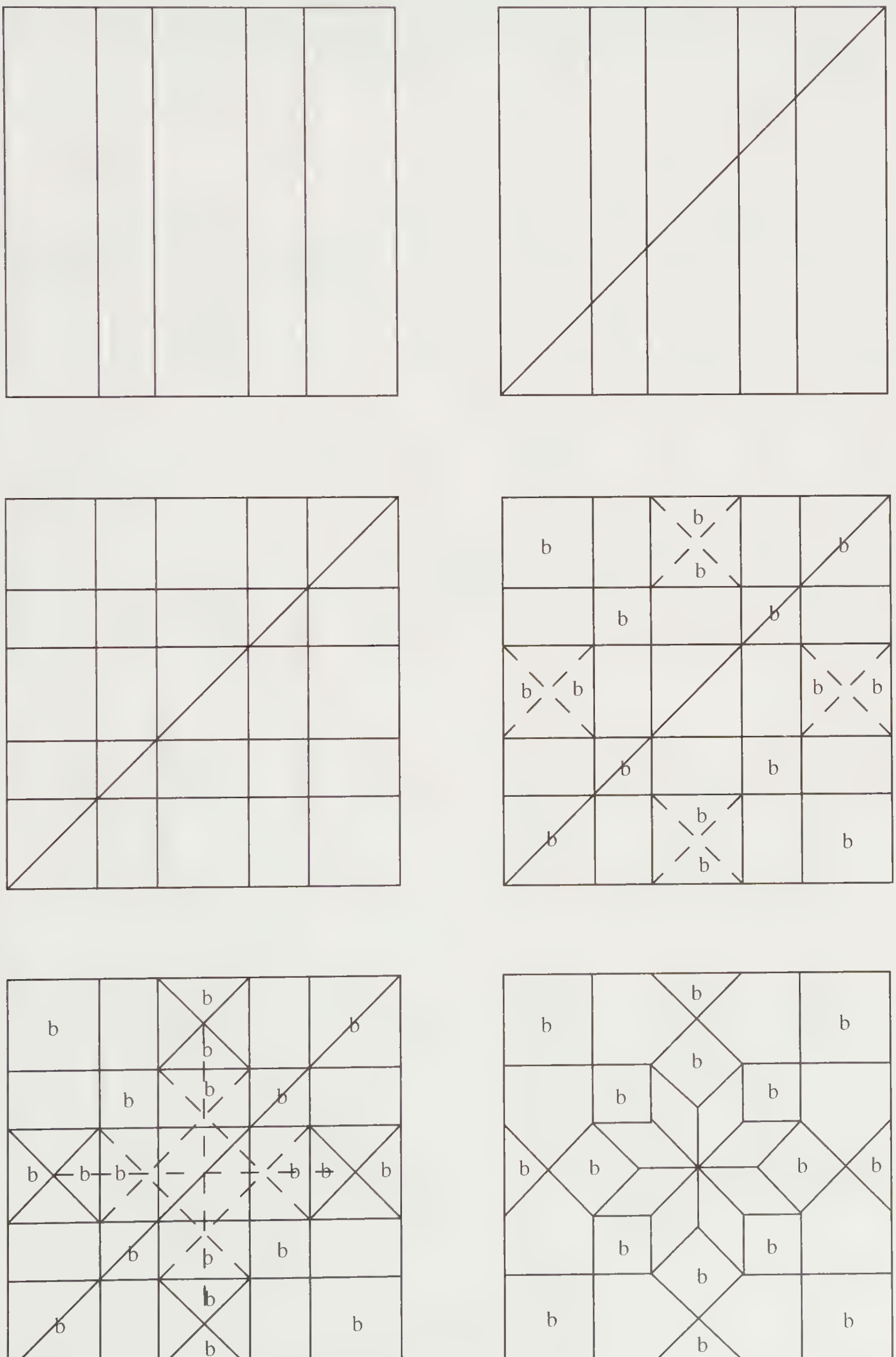
The above is a possible doodling method for inventing the pattern in the first place. To draw it in plan, as a preparation for a perspective rendering, one would start with the outermost



A5.1 (*above*) Stages for extending the diagram for cutting the corners off a square to make a regular octagon into the pattern of tiles found in the Judgement Hall in Piero della Francesca's *Flagellation of Christ*.

A5.2 (*facing page*) Drawing the 'perfect' form of the pattern on the pavement in Piero della Francesca's *Flagellation of Christ*, using the methods of *De prospectiva pingendi*, Book 1. The letter b has been used to indicate than an are will be black. Drawings by JVF.

square, divide its front edge in the ratio $\sqrt{2}:1:\sqrt{2}:1:\sqrt{2}$ (which is quite easily done since $\sqrt{2}$ is easy to construct, as a diagonal of a unit square); then put in orthogonals through these points of division; then draw a diagonal; through its points of intersection with the orthogonals put in parallels to the front line (which will be transversals); then colour in bits as required. The steps of the construction of the pattern are shown in Figure A5.2.



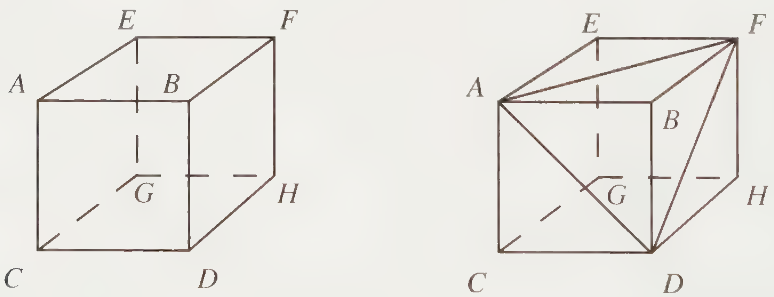
Appendix 6

Some Examples of Three-Dimensional Geometry from Piero della Francesca's *Trattato d'abaco* and *Libellus de quinque corporibus regularibus*

The edition used for the *Trattato* is that of Arrighi: Piero della Francesca, *Trattato d'abaco: Dal Codice Ashburnhamiano 280 (359*.291*) della Biblioteca Medicea Laurenziana di Firenze*, ed. G. Arrighi, Pisa, 1970; and for the *Libellus* that of Mancini: Piero della Francesca, 'L'Opera "De corporibus regularibus" di Pietro dei Franceschi detto della Francesca, usurpata da Fra' Luca Pacioli', ed. G. Mancini, *Memorie della R. Accademia dei Lincei*, series 5, 14.8B, Rome, 1916, pp.441–580.

1. The greatest tetrahedron that can be fitted inside a cube (*Trattato d'abaco*, BML MS, p.106 verso; Piero ed. Arrighi, p.227).

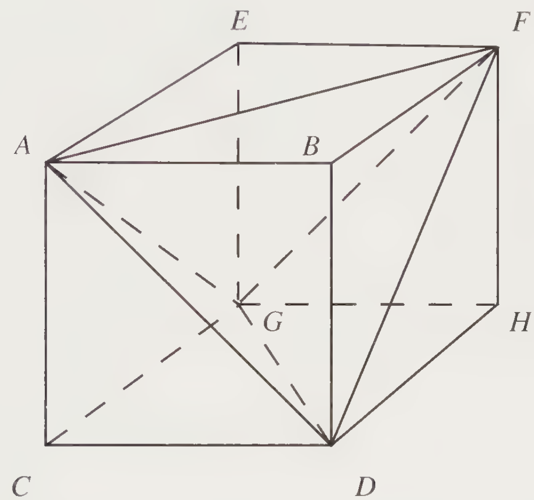
> There is a cube, which is 4 *bracci* along each side; I want to put inside it the greatest solid equilateral triangle figure that can be put [there]. I ask for its sides.
>
> You have the given cube ABCD.EFGH,[1] first draw a line from A to D, then AF and DF and cut off the [solid] angle B [see Fig. A6.1.1], then join AG and DG taking away the [solid] angle C: there will remain ADF and ADG which are two sides of the required



A6.1.1 Cube lettered as in Piero's diagrams, first complete and then showing the lines drawn to cut off the solid angle at *B*. Drawings by JVF.

1 Experience suggests, and the present example confirms, that Piero's normal order of lettering for points is to go across the diagram in rows, as shown in Figure A6.1.1.

triangle. Then join FG and AF and AG and DF and DG, taking away the [solid] angle E and the [solid] angle H, and there will remain AFG and DFG two more sides of the figure with four triangular faces [see Fig. A6.1.2]. And because AB is 4 and BD 4 and they make a right angle opposite AD, so the power of AD is the sum of the powers of the two, and each is 4 so its power is 16, which is AB, and 16 is the power of BD. Added up they make 32; and the root of 32 is the power of AD which is a side of the said figure with four triangular faces. So I say that the greatest solid triangle which can be constructed in the cube is ABCD.EFGH has each side root 32; which was what was required.

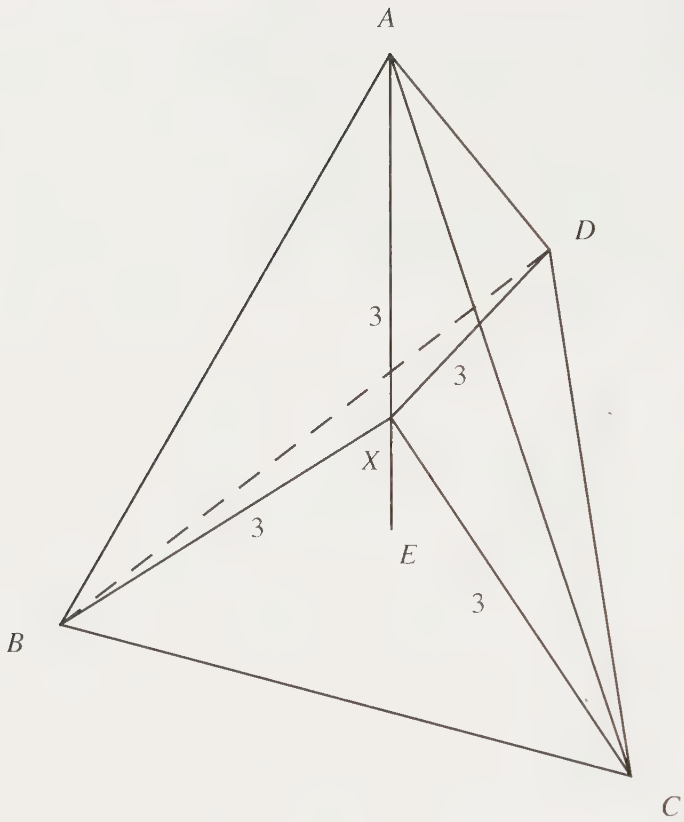


A6.1.2 Cube with inscribed tetrahedron. Drawing by JVF.

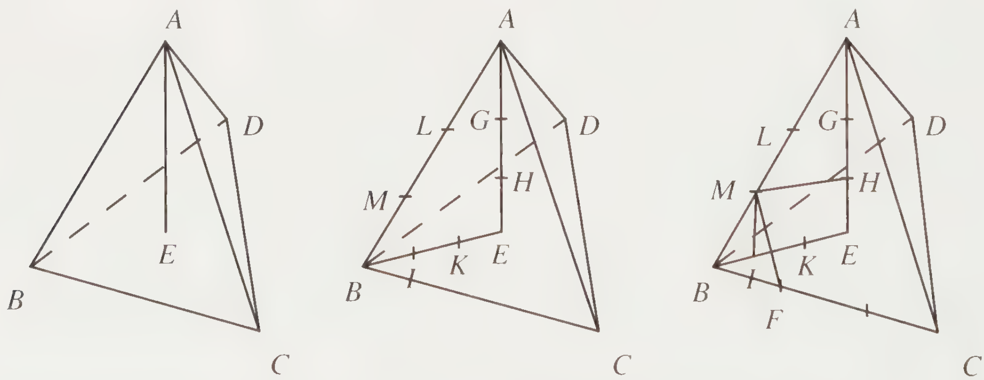
2. 'A body with eight faces', modern name: the truncated tetrahedron (*Trattato d'abaco*, BML MS, p.107 recto; Piero ed. Arrighi, p.230).

There is a spherical body whose diameter is 6; I want to put in it a body with 8 faces, 4 triangular and 4 hexagons. I ask what its edge is.

You have, by the fifth [proposition] on bodies a 4-faced triangular figure which is ABCD and from the centre [of the sphere] to each of its corners is 3; and because each of its corners touches the circumference of the sphere, 3 will be the semidiameter and the complete diameter will be 6, for the sphere which contains this body with 4 triangular faces; and the axis [i.e., height] is 4 which is AE, and its sides are each the root of 24 [see Fig. A6.2.1]. And, because this body has four faces, and we wish to make them 8, it is necessary to cut off its four angles and leave it such that it still has equal sides. So make each of its sides into three equal parts, that is divide AB, which is root 24, into three equal parts, so make 3 into a root it is 9, divide 24 by 9 the result is $2\frac{2}{3}$, and the root of $2\frac{2}{3}$ will be [the length of] each part, that is AL, LM, MB [see Fig. A6.2.2]. Now make three parts of the axis AE, which is 4, which will be AG, GH, HE, and each will be $1\frac{1}{3}$. Divide BE, which is $\frac{2}{3}$ of the diameter BO [*sic*: no corresponding line is marked in any of Piero's diagrams], which is the root of 8, into three equal parts in the points I and K, which will

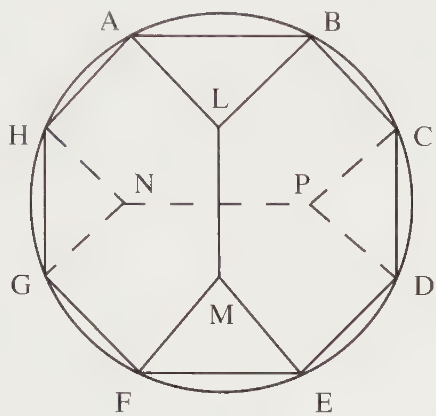


A6.2.1 Regular tetrahedron. Piero has proved above that if the height, *AE*, is 4 then the centre of the circumsphere of the solid, called *X* in our diagram, which by symmetry must lie on *AE*, is at a distance 3 from each vertex. This gives him a diameter 6 for the circumsphere. The side of the tetrahedron is $\sqrt{24}$. Drawing by JVF.



A6.2.2 Regular tetrahedron, showing the steps towards cutting off its corners. As elsewhere in his work, Piero here uses some letters more than once. So direct comparisons between these intermediate diagrams, not supplied in the Florence manuscript, and the final ones that do appear in the manuscript, Fig. 4.18 and copy in Fig. A6.2.3, requires caution. Drawings by JVF.

each be root of $\frac{8}{9}$ And. IE, which is $\frac{2}{3}$, is equal to MH and $\frac{2}{3}$ of the root of 8 are the root of $3\frac{5}{9}$; therefore MH is the root of $3\frac{5}{9}$, which multiplied into itself makes $3\frac{5}{9}$, and FH is $\frac{1}{3}$ which multiplied into itself makes $\frac{1}{9}$;[2] added to $3\frac{5}{9}$ makes $3\frac{2}{3}$. Therefore MF is the root of $3\frac{2}{3}$; so you say thus: if the root of $3\frac{3}{5}$ gives me the root of $2\frac{2}{3}$, what will give me the root of 9, which is the semidiameter of the sphere, the root of $3\frac{2}{3}$ of the body with 8 faces and side $2\frac{2}{3}$?[3] So multiply the root of $2\frac{2}{3}$ by the root of 9, it gives root 24, divide by the root of $3\frac{2}{3}$, the result is $6\frac{6}{11}$. And the root of $6\frac{6}{11}$ will be the length of each side of the body with 8 faces, 4 triangular and 4 hexagons, contained in the spherical body whose diameter is 6 *bracci*.



A6.2.3 Copy of diagram supplied in the Florence manuscript at the end of Piero della Francesca's propositions on the figure with eight faces (truncated tetrahedron), see Fig. 4.18. Drawing by JVF.

3. 'A body with fourteen faces', modern name: the cuboctahedron (*Trattato d'abaco*, BML MS, p.108 recto and verso; Piero ed. Arrighi, pp.231–2).

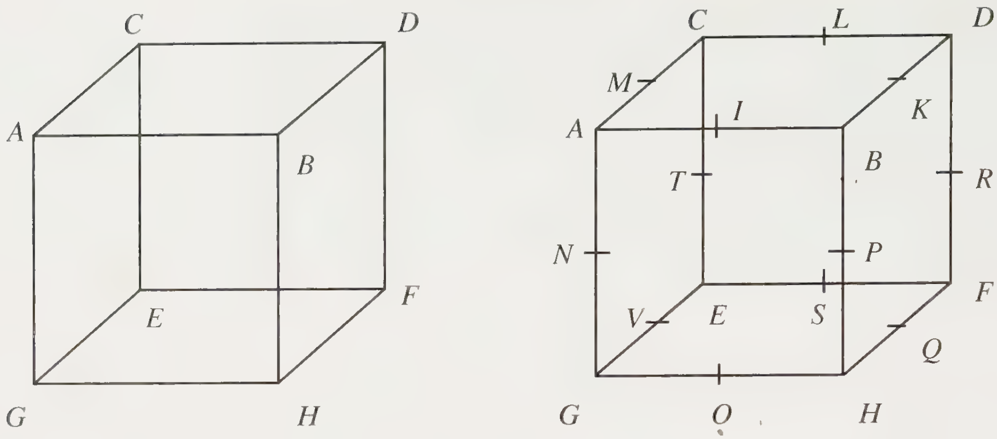
There is a spherical body, whose diameter is 6 *bracci*; I want to put in it a figure with fourteen faces, 6 square and 8 triangular, with equal edges. I ask what each edge will be.

Such a figure as this is cut out from the cube, because it [the cube] has 6 faces and 8 corners; which, cutting off its 8 corners, makes 14 faces, that is thus. You have the cube ABCD.EFGH, divide each side in half: AB in the point I, and CD in the point L, BD in the point K, AC in the point M, and AG in the point N, GH in the point O, HB in the point O [*sic*, means P], HF in the point Q, FD in the point R, FE in the point S, EC in the point T, EG in the point V [see Fig. A6.3.1]. Draw a line from T to P passing through the centre K,[4] which [line] is in the power equal to [the sum of] the powers of the two

2 The point *F* has not been mentioned before, but it can be found in Piero's diagrams. It is one of the points of trisection of an edge of the original tetrahedron, as shown in Figure A6.2.3 (copied from Piero's diagram, Fig. 4.18) and in Figure A6.2.2 (my own diagram).

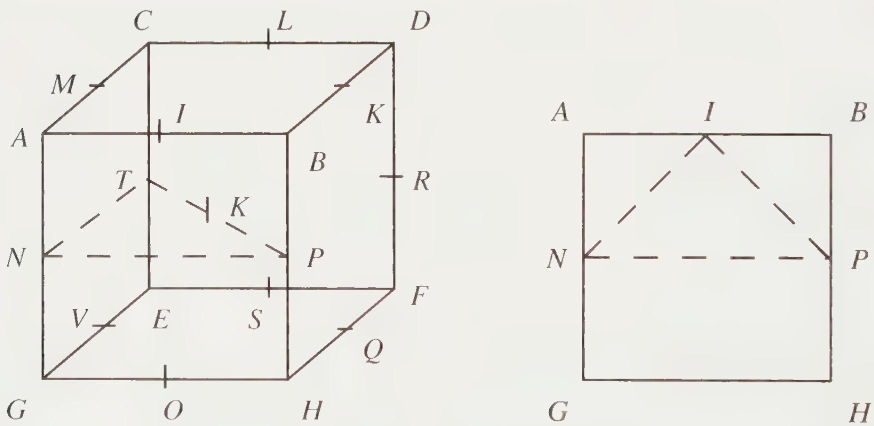
3 The application of the rule of three that follows is required to scale all the lengths so that they apply to a truncated tetrahedron inscribed in a sphere of semidiameter 6, as required in this problem. The lengths Piero has obtained refer to the truncated tetrahedron formed by cutting corners off the tetrahedron from an earlier problem, in which the solid was inscribed in a sphere of diameter 6.

4 This is Piero's second point called *K*. He has already used the letter *K* to designate one of the points of bisection of edges of the cube (see Fig. A6.3.1).



A6.3.1 Cube lettered in Piero's manner, and then showing points obtained by bisecting each edge. Drawings by JVF.

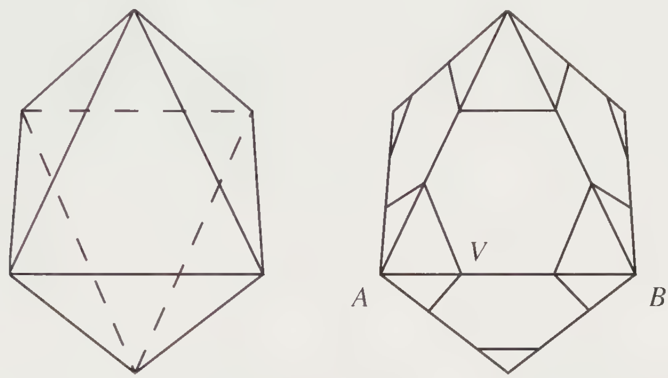
lines TN and NP, because N is a right angle opposite the line TP [see Fig. A6.3.2]; and the angle P and the angle T touch the circumference of the sphere, and so do the angles M, K, I, L, O, Q, N, S, V, R. So TP is a diameter of the sphere which is 6, and its power is 36 which is the power of the two lines NP and NT [that is, the sum of the powers of the two lines], so each is the root of 18. And if NP is the root of 18, whose power is equal to those of the two lines NI and IP [see Fig. A6.3.2], then NI is the root of 9; and so is IP [both of] which are sides of the figure with 14 faces contained by the sphere which has diameter 6. So that I say that the side of the figure with 14 faces, 6 square and 8 triangular, is the root of 9, which is 3.



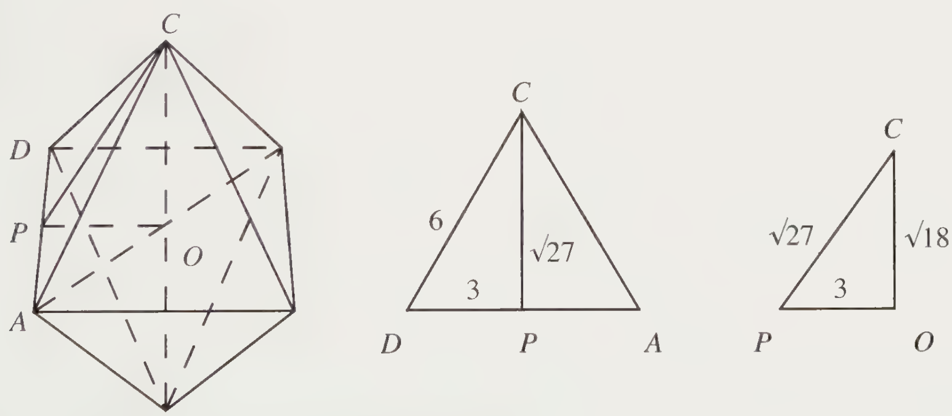
A6.3.2 The figure on the left shows the method of constructing *PT*, a diameter of the circumsphere of the new solid, whose vertices are the mid points of the edges of the original cube. That on the right shows the isosceles right-angled triangle *NIP* drawn on one face of the original cube in the course of finding the length of the edge of the new solid. Drawings by JVF.

4. 'A body with 14 faces, 6 square and 8 hexagons', modern name: the truncated octahedron (*Libellus de quinque corporibus regularibus*, Tractatus 4, casus 4; Piero ed. Mancini, p.562).[5]

> Given a body with 14 faces, 6 square and 8 hexagonal, with the side of each face 2, to find its surface area, and its volume, and the diameter of the sphere that encloses the body [i.e., the circumsphere].
>
> This body is formed from the body with 8 triangular faces [the regular octahedron] by cutting off its six solid angles, and dividing each of its sides [edges] into three equal parts.



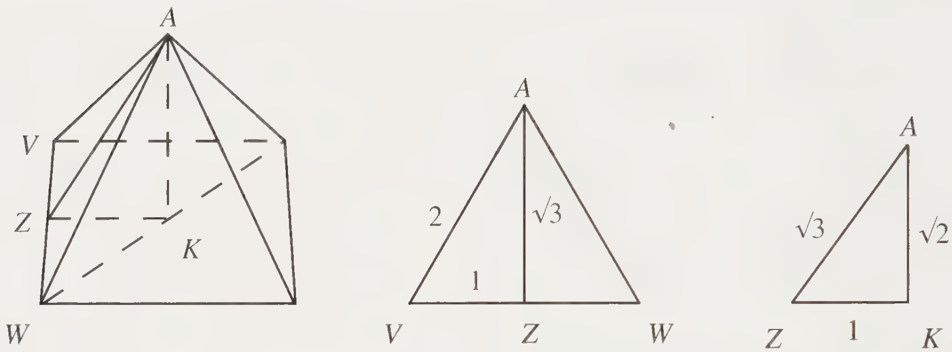
A6.4.1 a. Regular octahedron; b. Regular octahedron with corners cut off to make each triangular face into a regular hexagon. Drawings by JVF.



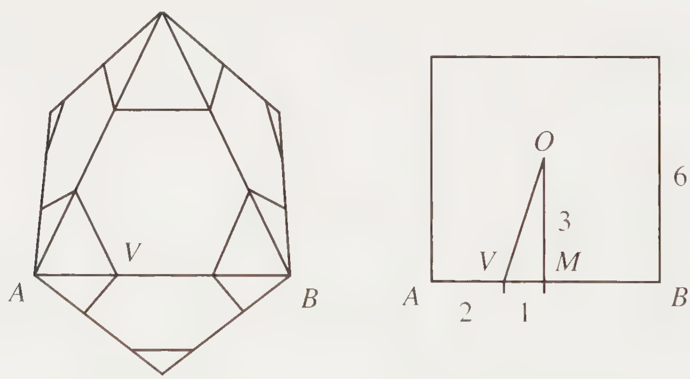
A6.4.2 Finding the height of the regular octahedron: a. Centre, *O*, found by the intersection of two diagonals. *P* is the foot of the perpendicular from *O* to *AD*. By symmetry, *P* is the mid point of *AD*; b. Triangle *ADC*, to find *CP*; c. Triangle *CPO* to find *CO*. The height of the regular octahedron is 2*CO*. Drawings by JVF.

5 No relevant diagram is supplied for this passage in Pacioli's printed version of the *Libellus*, or in the Latin manuscript in the Vatican Library. The diagrams provided here adopt today's drawing conventions, since they are intended merely as an aid to following Piero's mathematical reasoning.

And because we want each of its sides to be 2, each side of the body with 8 faces [regular octahedron] must be 6. So, if each of the sides of the triangular faces of the 8-faced figure is 6, its height will be ℞ 72,[6] which multiplied by 36 and made a root gives root 93312. You will divide by 9 there remains the root of 10368. And thus the root of 10368 is the volume of the body with eight triangular faces, whose 6 angles you will cut off, they will be 6 square pyramids, all with each of their sides [of length] 2, and the area of the base of each will be 4, and the axis [height] of each will be root 2. So you will take $\frac{1}{3}$ of the



A6.4.3 Diagrams for finding the height of a square pyramid, that is one of those removed from the octahedron. *A* is a vertex of the original octahedron, *V* is a vertex of the new solid. Compare Fig.s 6.4.1, 6.4.2 above. Drawings by JVF.



A6.4.4 left: Regular octahedron with corners removed, making each triangular face into a regular hexagon. *V* is a vertex of the solid formed by this truncation; right: Plane of section running symmetrically through the regular octahedron, passing through four vertices of the solid and its centre, *O*. *M* is the mid point of the edge *AB*. Drawings by JVF.

6 The symbol ℞, which is found in manuscript and printed texts of the fifteenth and sixteenth centuries, is an abbreviation for the Latin 'radix' and indicates a square root. Piero is in the habit of taking everything inside the root sign, so that where we now write $3\sqrt{2}$, he has $\sqrt{18}$, and twice this gives him $\sqrt{72}$. This habit presumably arises from the fact that if one had to evaluate the term one would work with the larger number rather than extracting the root and then having to do a more complicated multiplication.

surface [area] of any of the 6 bases, which is 8. Multiply into itself, it makes 64, which multiplied by 2 gives 128, and this you will subtract from 10368 as root, there remains 8192, and ℞ 8192. And this is the volume of the body with 14 faces as was proposed. Now for the surface [area]. You have that 6 faces are square, and the side of each is 2, and squared is 4. Then 6 is multiplied times four, it makes 24; that is the surface [area] of the 6 square faces. And each of the 8 hexagonal faces is divided into 6 equilateral triangles, each of whose sides is 2, and the height is root 3. You will take half the eight faces, in which there are 48 triangles; half is 24 faces, and each is 2, which is 48: multiply into itself, it gives 2304: multiply this by the height which is 3, it gives 6912, and root 6912 is [the area of] the hexagonal faces, which joined with the 6 square faces, which are 24, the surface [area] of the body will be 24 plus root 6912. It now remains only to find the diameter of the sphere that contains the said body. You have that from the centre of the said body to the mid point of the side of the eight-faced body is 3, which turned into roots makes 9, joined with the power of half the side of the hexagon, which is 1, it makes 10, and ℞ 10 is the semidiameter of such a body. The complete [diameter] is ℞ 40, and the surface [area] is 24 added to root 6912. And the volume is root 8192.

Appendix 7

The Prefatory Letter to the Latin Text of Piero della Francesca's *Libellus de quinque corporibus regularibus*

The manuscript of the *Libellus de quinque corporibus regularibus* in the Vatican Library has been printed as Piero della Francesca, 'L'Opera "De corporibus regularibus" di Pietro dei Franceschi detto della Francesca, usurpata da Fra' Luca Pacioli', ed. G. Mancini, *Memorie della R. Accademia dei Lincei*, series 5, 14.8B, Rome, 1916, pp.441–580. The prefatory letter that precedes the *Libellus* in the manuscript naturally does not appear in Luca Pacioli's vernacular version of Piero's work, printed as the final part of *De divina proportione* (Venice, 1509), where there is a separate dedication that implies the work is in Pacioli's own gift, and Piero's name is not mentioned.

In Pacioli's volume, the four-line paragraph before the beginning of the main text says: 'Short book divided into three treatises, as parts, in which the regular bodies and those derived from them are carefully examined. Dedicated specially to Lord Petro Soderino perpetual ruler of the Florentine people by Luca Paciolo of Borgo, Minorite, with good wishes. It begins.'

In this particular case Giorgio Vasari's accusation that Pacioli took credit for Piero's mathematics seems well founded, though the notion of plagiarism is certainly anachronistic. In fact, a fair proportion of Piero's work is taken from earlier sources. In the abacus tradition, good coverage of the relevant material was much more important than originality, a state of affairs that still obtains in elementary textbooks.

To his most illustrious highness the prince
Guido Ubaldo
Duke of Urbino, Peter of Borgo, painter, his Preface

Among ancient painters and sculptors, O Guido, distinguished prince, Policretus, Phidias, Miro, Praxiteles, Apelles, Lisippus, and others who sought nobility through art, are not more revered for other things, and among themselves more praised, and indeed through the memory of posterity also [given] more lasting fame, [than] has been allotted to Aristomenes, Thasius, Polides, Chionis, Pharaxes, Bœdas, and others who in their art were no less by their study and skill [*ingenium*], diligence [*sollertia*] and hard work, unless this [that is, the greater fame of the former set] was because they made works either for great states [*civitates*], or kings, or princes known for their virtue. The former indeed spending their time with humbler people, their lack of standing and the slenderness of their

> fortune stood in the way and obscured their virtues.[1] Nor was it unimportant for Virgil, Flaccus, and other poets who flourished in that age, that the glory of Octavianus Augustus and Maecenas is transmitted to eternity. Now since my works and pictures have taken all they have of lustre from the highest and most glittering star and the greatest luminary of our time, that of your father, the best of men, it was not considered inappropriate to dedicate to your genius [*numini tuo*] the little work in this last mathematical exercise [*calculo*] of my old age [*aetate meo*], which I published [*edidi*], lest the mind should become torpid by inaction, concerning the mathematics of the five regular bodies, so that it too should be brought out of[2] obscurity by being illuminated by your radiance.[3] Nor will your highness be dishonoured by accepting from this little plot of land, now out of use and almost consumed with age, from which also your most illustrious father received more abundant ones, these slender and hollow fruits, and placing the book itself among the numberless volumes of your and your father's most ample library, close by [*penes*] another little work of ours a Perspective, which we published [*edidimus*] in earlier years, as a footman and lowly servant of others, or in a corner. For it is not the custom not to admit upon a most opulent and luxurious table, uncultivated things [*agrestia*] and fruits gathered on a humble farm. For, despite its novelty, it may not displease. And indeed it deals with things noted by Euclid and other geometers, in this work however newly expressed in arithmetical terms. And it will serve you as a monument and reminder of me, and an uncultivated shoot [from the trunk of] my old affection and perpetual service to you.

The opening passage of Piero's text – presumably as translated by Maestro Matteo – is:

> Inter antiquos pictores et statuarios, Guido princeps insignis, Policretum, Phidiam Mironem, Praxiteles, Apellem, Lisippum, ceterosque qui nobilitatem ex arte sunt consecuti, non ab aliud digniores fuisse, et apud suos maiorem gratiam, apud vero posteritatem memoriam et famam diuturniorem, Aristomene, Thasio, Polido, Chione, Pharaxe, Boeda, ceterisque, qui non minori artis studio, ingenio, solertia, et industria fuerunt, habuisse perhibent, nisi quod ii aut civitatibus magnis, aut regibus, aut principibus virtutis experimentatae opera fecerunt. Illis vero inter humiliores versantibus eorum dignitate exiguitas, imbellicitasque fortunae obstituit, et virtutes obscuravit.

This seems to be a condensed paraphrase (85 words as against 145) of Vitruvius' *De architectura*, Book 3, preface, para 2 (modern text):

> Maxime autem id animadvertere possumus ab antiquis statuariis et pictoribus, quod ex his qui dignitate notas et commendationis gratiam habuerunt, aeterna memoria ad posteritatem sunt permanentes, uti Myron, Polycletus, Phidias, Lysippus ceterisque, qui nobilitatem ex arte sunt consecuti. Namque ut civitatibus magnis aut regibus aut civibus nobilibus opera fecerunt, ita id sunt adepti. At qui non minori studio et industria sollertiaque fuerunt nobilibus et humili fortuna civibus non minus egregie perfecta fecerunt opera, nullam memoriam sunt adsecuti, quod hi non ab industria neque artis sollertia sed a Felicitate fuerunt decepti, ut Hegias Atheniensis, Chion Corinthius, Myagrus Phocaeus.

1 The above passage is discussed further below.

2 Possibly 'saved from'.

3 The fact that Piero mentions only works written for Guidobaldo's father suggests that the *Libellus* is the first work written for Guidobaldo himself.

> Pharax Ephesius, Boedas Byzantius et alii plures. Non minus item pictores, uti Aristomenes Thasius, Polycles et Androcydes <Cynzice>ni, Theo Magnes ceterisque, quos neque industria neque artis studium neque sollertia defecit sed aut rei familiaris exiguitas aut imbecillitas fortunae seu in ambitione certationis contrariorum superatis obstitit eorum dignitati.

Some of the phrasing is so close that, at the least, Matteo must have known Piero wanted to remind the reader of Vitruvius' text.

Appendix 8

Some Theorems and Problems in Piero della Francesca's Construction of the Perspective Image of a Square-Tiled Pavement

In the first book of *De prospectiva pingendi*, Piero gives mathematically rigorous proofs of a number of preliminary theorems before setting out his construction of the perspective image of a square in Book 1, Proposition 13. He proves this construction is mathematically correct, and then proceeds to use the results proved in some of the earlier theorems to divide the square into an array of smaller squares (Propositions 14 and 15).

Some of the most important propositions of Book 1 are translated here. They are:

1. The proof of the convergence of images of orthogonals ('Piero's theorem', Book 1, Proposition 8).
2. The proof that a diagonal can be used to transfer ratios from one side of a rectangle to one perpendicular to it (Book 1, Proposition 9).
3. The corollary for dividing up a square (Book 1, Proposition 10).
4. The construction of the perspective image of any surface (Book 1, Proposition 12).
5. The construction of the perspective image of a square (Book 1, Proposition 13).
6. The division of the perspective image of a square by the images of orthogonals (Book 1, Proposition 14).
7. The subdivision of the perspective image of a square, divided as in section 14, into small squares (Book 1, Proposition 15).

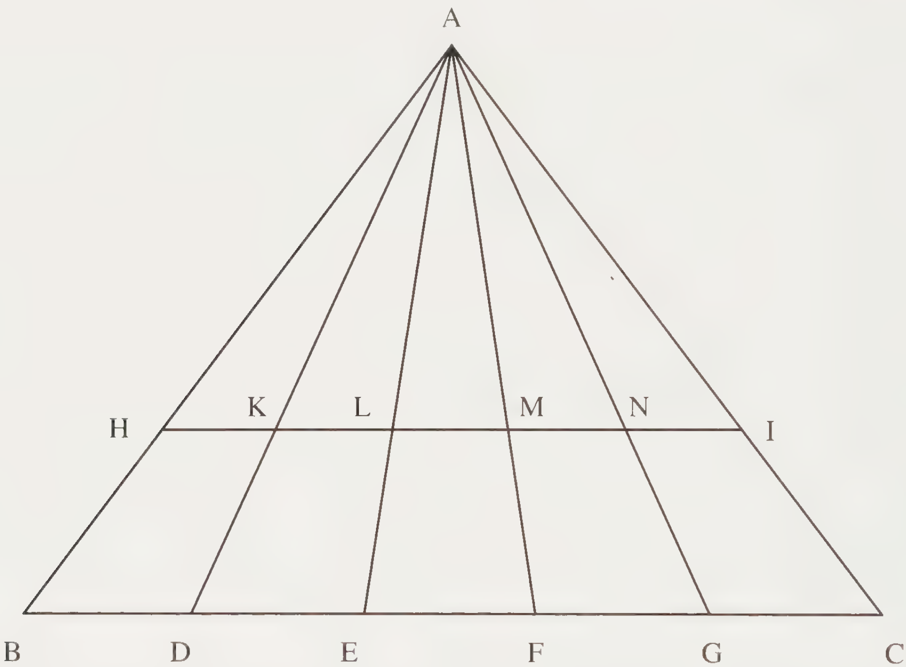
1. Book 1, Proposition 8 ('Piero's theorem')

Parma MS, p.4 recto; BL MS, p.4 recto; Piero ed. Nicco Fasola, p.70.

> [Proposition 8] If above a given straight line divided into several parts another line be constructed parallel to it and from the points dividing the first line there be drawn lines which end at one point [that is, they are concurrent], they will divide the parallel line in the same proportion as the given line.
>
> Given the line .BC. which is divided in .D.E.F.G., and let another line be drawn parallel to it, and let this be .HI. And[1] from the point .A. let there be drawn .AB. .AD. .AE.

1 Here, and in several other places in the proposition, the Latin text in the BL MS starts a new sentence where the vernacular text in the Parma MS merely has a comma. The choppy style of the Latin text, which I have followed in the translation for reasons of clarity, is characteristic of learned mathematical works, such as Euclid's *Elements*, and its use in the Latin version of Piero's work is indicative of the way that the text is not simply a translation from the vernacular but also tends to 'naturalize' the work into the learned tradition.

.AG. .AC.; which divide the line .HI. in the points .K.L.M.N. I say it is divided in the same proportion as the given line .BC.[2] Because .BD. is to .DE. as .HK. is to .KL., and .EF. to .FG. And because [this ratio is the same as] .LM. to .MN., and .FG. to .GC. is as .MN. to .NI. And the triangle .ABC. is similar to the triangle .AKI..[3] And the triangle .ABD. is similar to the triangle .AHK. And [so] is .ADE. to the triangle .AKL. And .AEF. is similar to the triangle .ALM. So that they are proportional and the proportion that obtains between .AB. and .BC. is the same as that of .AH. to .HI. And since the greater sides were proportional,[4] the smaller sides are also proportional. And the angles of the triangle .ABD. are similar to [that is, are the same as] the angles of the triangle .AHK., so [the triangles] are proportional, as is shown by the 21st of the sixth of Euclid;[5] and similarly for the others [that is, the other triangles],[6] which is what is proposed.



A8.1.1 Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 8 – 'Piero's theorem'. Drawing by JVF.

2 The Latin text in the BL MS does not name .BC. here.

3 This sentence does not occur in the vernacular text in the Parma MS, but is found in the Latin in the BL MS.

4 G. Nicco Fasola's edition of the vernacular text has 'es, endo', which appears to be a misreading for 'essendo', associated with a line break, fol.4 recto. The Latin text in the BL MS makes the meaning clear.

5 *Elements*, Book 6, Proposition 21, as given by Campanus, quoted in Euclid, *Elementa*, trans. Bartolommeo Zamberti with a preface by Philipp Melanchthon (Philipp Schwartzerd, 1497–1560), Basel, 1537, p.156, is: 'If there be a number of proportional lines, and on any two there be constructed similar areas, these areas will also be proportional. If indeed similar surfaces constructed on lines are proportional then the lines themselves must be proportional.' ('Si fuerint quotlibet lineae proportionales, atque super binas & binas similes superficies designentur, ipsae quoque superficies erunt proportionales. Si vero super binas & binas similes superficies constitutae fuerint proportionales: ipsas quoque lineas proportionales esse necesse est.') The text of the recension of the *Elements* by Theon of Alexandria (active A.D. 364), found in Zamberti's edition (Venice, 1505, and cited on p.155 of the 1537 Euclid) is different.

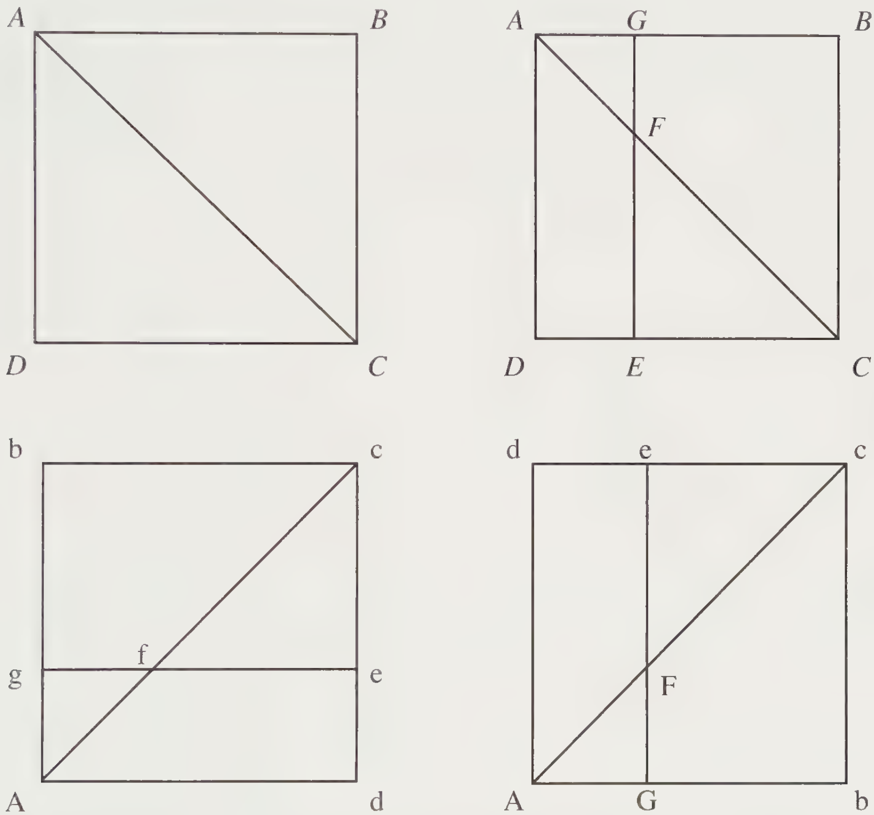
6 This phrase is omitted in the Latin text in the BL MS.

2. Book 1, Proposition 9

Parma MS, p.4 verso; BL MS, p.4 verso; Piero ed. Nicco Fasola, p.70.

[Proposition 9] If in a quadrilateral surface the diagonal is drawn, it will divide the surface into two equal parts, and if other lines are drawn parallel to the sides, the diagonal will divide them and the sides in the same proportion.[7]

For example [*Exemplo*]: let the quadrilateral surface be .ABCD. and the diagonal be .AC.[8] I say that .AC. divides the surface .ABCD. into two equal parts, because the quadri-



A8.2.1 Figures for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 9. The upper pair of diagrams results from the stages of my following the drawing instructions given by Piero. The diagram at lower left is a copy of the one in the BL MS, and the one at lower right is from the Parma MS, which is autograph. Drawings by JVF.

7 The quadrilateral has to be a parallelogram for this to be true. The diagrams in both the Parma and BL manuscripts show what seems to be a square. Later, Piero does use 'quadrilatero' to mean a square, though here it is possible that his 'exemplo' should be construed as his version of today's 'without loss of generality'.

8 This is a departure from Piero's normal method of lettering: from left to right in rows as necessary. The draughtsman in the Parma MS, who is almost certainly Piero himself, seems to have had to alter his first attempt at lettering. The upper pair of diagrams are by JVF, and use cyclic lettering (in accordance with today's convention). The Parma and BL manuscripts have slightly different diagrams, copied in the lower pair here.

lateral .ABCD. has its sides and angles equal[9] and the diagonal .AC. dividing it through the middle [*per lo mezzo*] from angle to angle, and the angle .D. and the angle .B. are equidistant from the line .AC. as is proved by the 34th of the first of Euclid.[10] The first [part of the present proposition] is clear. Let there be constructed a line parallel to .AD., which divides .DC. in the point .E. and the diagonal .AC. in the point .F. and .AB. in the point .G.; I say that it [the parallel] divides these lines and itself in the same proportion, because the proportion of .AG. to .GB. is the same as that of .DE. to .EC.,[11] and the rectangle enclosed by .AF. and .FE. [*et quello che si fa da .AF. in .FE.*] is equal to that enclosed by .FG. and .FC.,[12] and that enclosed by .FE. and .AC. is equal to that of .FC. and .AD.,[13] and the rectangle of .AG. and .AC. is equal to that of .FA. and .AD.,[14] and [.GF. to .FE. is as .AG. to .GB.],[15] so that they are proportional. Otherwise, by the sixth[16] of this [book] the line .EG. is equal to the line .BC. and is parallel [to it] and divides .AB. and .AC. of the triangle .ABC. in the same proportion, as was proved by that [proposition], which is what is proposed.

9 That is, it has been made a square.

10 *Elements*, Book 1, Proposition 34: 'In parallelogrammic areas the opposite sides and angles are equal to one another and the diameter bisects the areas.'

11 This follows from the fact that, since GE is parallel to AD and BC, we have $AG = DE$ and $GB = EC$, therefore the proportion of AG to GB is the same as that of DE to EC. In modern quasi-algebraic notation this is

$$\frac{AG}{GB} = \frac{DE}{EC} \quad (1).$$

Thus the line GE divides the sides of the square in the same ratio, which is part of what Piero wants to prove.

12 In the Latin text of the BL MS the last line is incorrectly identified as .fe. In modern quasi-algebraic notation the relationship given in Piero's last phrase would be written

$$AF.FE = FG.FC \quad (2).$$

It follows from the fact that triangles AFG, CFE are equiangular and therefore similar, giving

$$\frac{AF}{FC} = \frac{DF}{FE}.$$

What we have is that the ratio in which F divides the diagonal (shown on the left of our last equation) is equal to the ratio in which it divides the line GE (shown on the right).

13 In modern notation, we have

$$FE.AC = FC.AD \quad (3).$$

This follows from the fact that the triangles CFE, CAD are equiangular and therefore similar, giving

$$\frac{FE}{AD} = \frac{FC}{AC}.$$

That is, the ratio between the segment FC and the whole diagonal is equal to the ratio between FE and GE (since $GE = AD$).

Here, as elsewhere, Piero expresses his results in terms of areas, in the manner of Euclid. This kind of formulation continued to be normal for at least a further two centuries.

14 In modern notation, we have

$$AG.AC = FA.AD \quad (4).$$

This follows from the fact that the triangles AGF, ADC are equiangular and therefore similar, giving

$$\frac{AG}{AD} = \frac{FA}{AC}.$$

That is, the ratio between FA and the whole diagonal is equal to the ratio between AG and AB (since $AD = AB$). Either this relationship or the one described in the previous note establishes that the point F divides the diagonal in the same ratio as the points G and E divide the sides.

15 Nicco Fasola notes that in the Parma MS this bit is erroneously given as saying that the rectangle of GF and FE is equal to that of AG and GB. She points out that the error was corrected by Winterberg in his edition of the vernacular text (C. Winterberg, ed., *Petrus pictor burgensis de prospectiva pingendi*, Strasbourg, 1899), but adds that the same error occurs in the Latin manuscript in the Ambrosiana Library in Milan (MS C307 inf). The Latin text of the BL MS is correct.In modern notation we have

$$\frac{GF}{FE} = \frac{AG}{GB} \quad (5).$$

That is, the point F divides GE in the same ratio as the point G divides AB. This is established by putting together the result from the equation we have called (2) with that from (3) or that from (4).

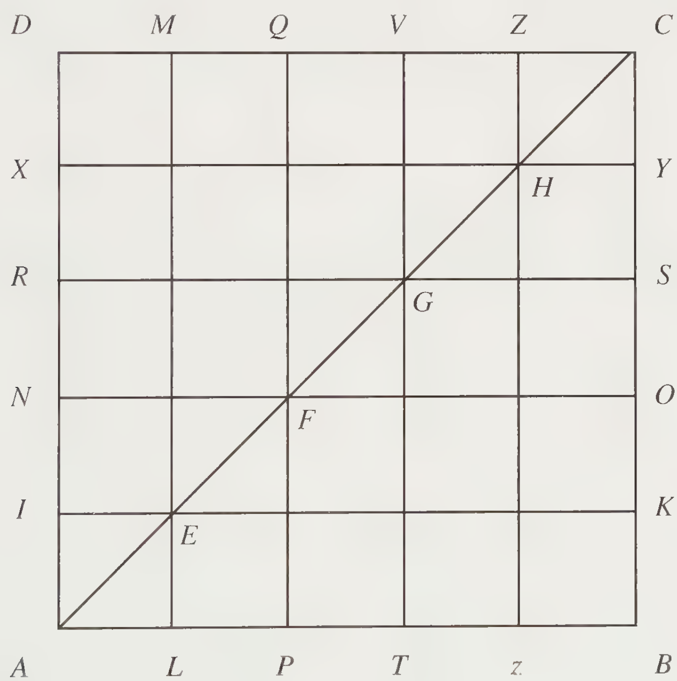
16 The Parma MS refers to the fifth proposition, which is incorrect. The BL MS has 'sextam'. Piero does actually mean to refer to the sixth proposition, and this correction may have been made in the process of revision carried out in connection with the Latin translation of his work.

3. Book 1, Proposition 10

Parma MS, p.4 verso; BL MS, p.4 verso; Piero ed. Nicco Fasola, p.71.

[Proposition 10] If in a square surface with equal sides and angles[17] the diagonal is constructed, and it is divided into several equal parts, and from [the points dividing] these parts there are drawn lines parallel to the four sides of the surface, it will be divided into similar surfaces [that is, into squares].[18] Let the surface be .ABCD. and let the diagonal line be .AC. divided into several equal parts, so that it is .EFGH.. I say that if there is drawn [through] .E. a parallel to .AB. and another [line] parallel to .AD., and there is drawn [through] .F. a parallel to .AB. and another [line] parallel to .AD., and there is drawn [through] .G. a parallel to .AB. and another [line] parallel to .AD., and there is drawn [through] .H. a parallel to .AB. and another [line] parallel to .AD., which lines will make .25. surfaces similar to the surface .ABCD. and [they have] both similar sides and similar angles.

Let the surfaces be made as was said and the diagonal be divided into .EFGH., as above, and let there be constructed [through] .E. a parallel to .AB., which will cut .AD. in the point .I. and .BC. in the point .K., and let there be constructed another line through .E. parallel to .AD., which will cut .AB. in the point .L. and .DC. in the point .M., and let



A8.3.1 Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 10. Drawing by JVF.

17 The text in the Parma MS has 'superficie quadrata'. The same words, but in the inverse order, are found in the Latin text of the BL MS. Clearly, Piero's word 'quadrata' does not quite correspond to our 'square'.

18 This result would hold for any parallelogram, though Piero is considering a square.

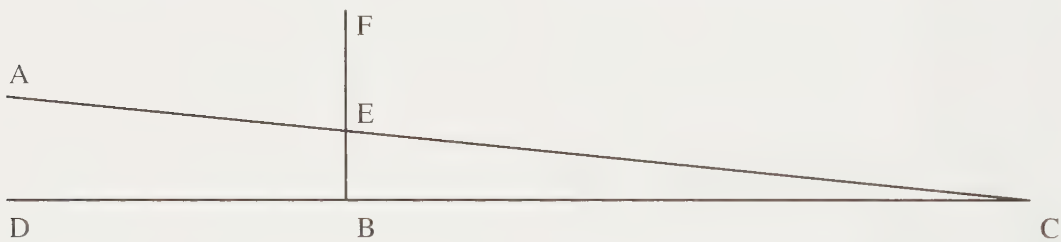
there be drawn [through] .F. a parallel to .AB., which will cut .AD. in the point .N. and .BC. in the point .O.; and let there be drawn another line through .F. parallel to .AD., which will cut .AB. in the point .P. and .DC. in the point .Q.; and let there be constructed [through] .G. a parallel to .AB., which will cut .AD. in the point .R. and .BC. in the point .S.; and let there be drawn another line through .G. parallel to .AD., which will cut .AB. in the point .T. and .DC. in the point .V., and let there be constructed [through] .H. a parallel to .AB., which will cut .AD. in the point .X. and .BC. in the point .Y.; and let there be drawn another line through .H. parallel to .AD., which will cut .AB. in the point .z. and .DC. in the point .Z.. I say that all these squares contained in the surface .ABCD. are similar to it and to one another,[19] because they are composed of lines that are parallel [to the sides of the original square] and have similar angles; therefore the sides are in the proportion of the sides of the surface .ABCD., as is demonstrated by the [forty-]third [proposition] of the first book of Euclid.[20]

4. Book 1, Proposition 12

Parma MS, p.6 recto; BL MS, p.6 recto; Piero ed. Nicco Fasola, p.75.

[Proposition 12] From the given eye [position] and on the determined limit[21] to degrade the assigned surface.

Let there be given the eye .A. above the line .DC. perpendicularly over .D., and let the line be divided in the point .B., which is to be the determined limit, and above .B. let



A8.4.1 Copy of figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 12. Drawing by JVF.

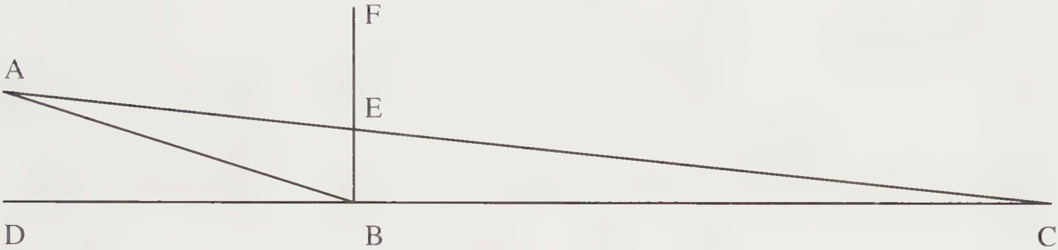
19 Nicco Fasola reads 'et infra loro', and I agree with this reading of the text of the Parma MS, but suspect a scribal error for 'fra loro'. The Latin text of the BL MS, fol.5 recto, l.4 makes the sense clear.

20 The Parma MS refers to 'the 30th of Euclid'. Nicco Fasola identifies this as a reference to *Elements*, Book 6, Proposition 30, but in the standard edition of the Greek text of the *Elements* (Euclid, *Opera Omnia*, ed. J. L. Heiberg and H. Menge, 8 vols., Leipzig: Teubner, 1883–1916; English translation of the *Elements*: Euclid, *The Thirteen Books of Euclid's Elements*, ed. and trans. T. L. Heath, Cambridge: Cambridge University Press, 1908) this proposition is to divide a line in extreme and mean ratio, and in Campanus (cited in the 1537 Euclid (full ref. note 5), p.164) it is to do with triangles on bases. An earlier proposition, Heiberg/Heath, *Elememts*, Book 6, Proposition 24, is 'In any parallelogram the parallelograms about the diagonal are similar both to the whole and to one another.' This may be relevant.

Nicco Fasola also notes that the Ambrosiana Latin manuscript (C307 inf) makes this a reference to Euclid, Book 1, Proposition 3. This agrees with the reading in the BL MS: 'Veluti per tertiam primi Euclidis demonstratur', which is the text we have translated here. However, *Elements*, Book 1, Proposition 3 merely concerns a line. As the confusion is in words not numerals, it seems likely a word has been omitted before 'tertiam'. A check was accordingly made on the propositions that Campanus gives as *Elements* 1, Propositions 13, 23, 33 and 43. It turns out that we need *Elements*, Book 1, Proposition 43 (p.34 in the 1537 Euclid (full ref. note 5), which concerns parallelograms constructed about the diameter. The proposition has the same number in Theon's recension, which was the basis for some Renaissance editions of the *Elements*.

21 'nel termine posto', that is, on the chosen picture plane, as is made clear below. See note 26.

there be drawn[22] the perpendicular line .FB., and .BC. will be the assigned plane, which is to be degraded. Let there be drawn[23] from the point .A. a line to the point .C., which is the edge of the assigned plane. And this line will divide .BF. in the point .E.. I say that .BE. is the degraded plane, that is [the degraded version of] .BC. For .BE. will also present itself to the eye equal to .BC. at the determined limit. To prove this. Let there be drawn .AB., making a triangle, which will be .ABC., and [we have] the bases [which] are .BC. and .BE. opposite the same angle, so that they present themselves to the eye as equal, as was proved by the second [proposition] of this [book]. We shall say[24] that .BE. is the assigned plane degraded. To show this another way we take a different course: because this is the first degradation, and must be well understood so that we may understand the others much more easily.



A8.4.2 Copy of figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 12. The line *AD* is shown in the BL MS diagram and in the Parma MS. Drawing by JVF.

I said[25] that the eye was given, it was meant that you had positioned yourself in the place where you wished to be to see the assigned plane; the assigned plane is meant to be of whatever length you wish to make the plane. The determined limit [*termine posto*] is the place where the said plane is to be degraded, that is [at] the distance from the eye to the wall or panel [*taula*] or other thing where it is wished to put the degraded things, placing the eye high or low, close to or far away, according to what the work requires.[26] Let us take the assigned plane .BC. to be .20. *braccia* [*sic*], and let .DB., which is [the distance from] the limit as far as the eye, be .10. *braccia* [*sic*], and let the eye be raised above .D. by .3. *braccia* [*sic*],[27] which [the position of the eye] I put as .A.. Let there be

22 The Parma MS has 'I shall draw' ('linearò'), whereas the BL MS has the more formal wording translated here. There are numerous examples of this kind of difference between the vernacular and Latin texts.

23 The Parma MS has 'I shall draw' ('tirarò'). The BL MS has the more formal wording translated here.

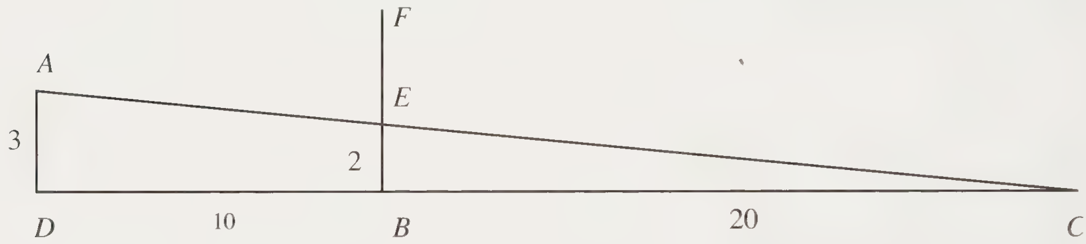
24 The Parma MS has the present singular 'I say' ('Dico').

25 Neither the Parma nor the BL manuscript starts a new paragraph here. We have done so for clarity.

26 This definition of the meaning of 'termine posto' makes it unambiguously clear that Piero means the words to refer to what would now be called 'the picture plane'. Technical vocabulary for writing about perspective remained fluid for a considerable time. Even in 1636, Girard Desargues found it advisable to define his terms before starting on his work.

27 Three *braccia* is the traditional, and the Albertian, height for the ideal man, but note that Piero is here dealing with eye height so the total height of the man has been made more than 3 *braccia*. As can be seen in connection with the last proposition of his first book, Piero tends to avoid explicit discussion of the height of the eye (see J. V. Field, 'Piero della Francesca's Treatment of Edge Distortion', *Journal of the Warburg and Courtauld Institutes* 49, 1986, pp.66–99 and plate 21c, and Chapter 5). Piero's usual form for the plural of *braccio* is *bracci*.

drawn the line .AC. which will divide .BF. in the point .E., as was said above: I say that .C. is higher than .B. at the limit, [by] the quantity .BE., because .A. is higher than .BC.; [this is] proved by the 10th of Euclid's *De aspectuum diversitate*. Therefore I shall say .BE. will be .2., which is two thirds of the height at which I put the eye above the plane, [which was] three *braccia* [*sic*], two-thirds [of this] are two *braccia* [*sic*]; because the line which starts from the point .A. divides the parallel lines in proportion, so that the proportion of .BC. to .DC. is the same as that of .BE. to .DA.; and .DA. is .3. and .BE. is .2., and .DC. is .30. and .BC. .20.; the proportion of .20. to .30. is the same as that of .2. to .3., so I shall say .BE. will be .BC. degraded, which I said was to be degraded.



A8.4.3 Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 12. Drawing by JVF.

5. Book 1, Proposition 13

Parma MS, p.6 verso; BL MS, p.6 verso; Piero ed. Nicco Fasola, p.76.

[Proposition 13] To make the degraded surface a square. Or proofs of the preceding [proposition].[28]

As in the preceding [proposition] let .DC. be a line divided at the point .B. [see Fig. A8.5.1a][29] and let .BF. be drawn perpendicularly [to it] [see Fig. A8.5.1b] and .A. in its position above .D. [see Fig. A8.5.1c.][30] and let a perpendicular line be drawn up from .C. equal to .BC. which is to be .CG. [see Fig. A8.5.1d] [;] and from the point .G. let there

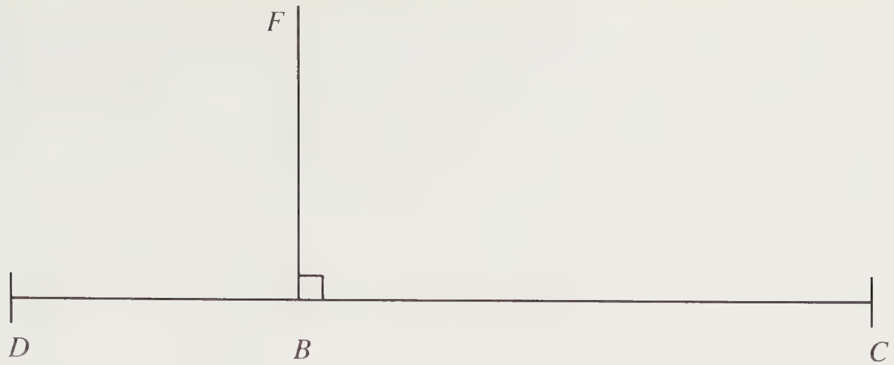
A8.5.1a Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 13. Drawing by JVF.

28 The second sentence is not present in the Parma MS, which gives the statement as 'Il piano degradato in quadro reducere'. The Latin text of the BL MS has 'Planitiem degradatam in quadratam formam reducere. Veluti in ea quae processit demonstrationes.'

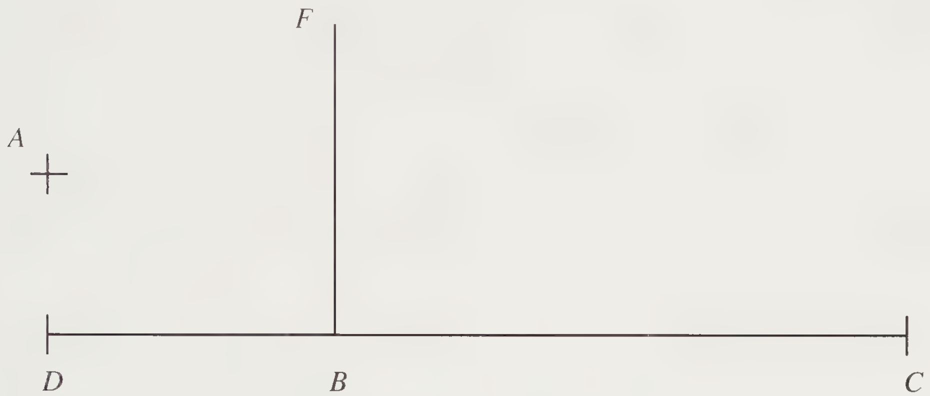
The emphasis on proof in the Latin is further evidence of the way Piero's vernacular original was not simply translated but was also revised to make it into something more like a learned mathematical text.

29 In the previous section (Book 1, Proposition 12) Piero suggests $BC = 20$ *braccia*, $DB = 10$ *braccia*.

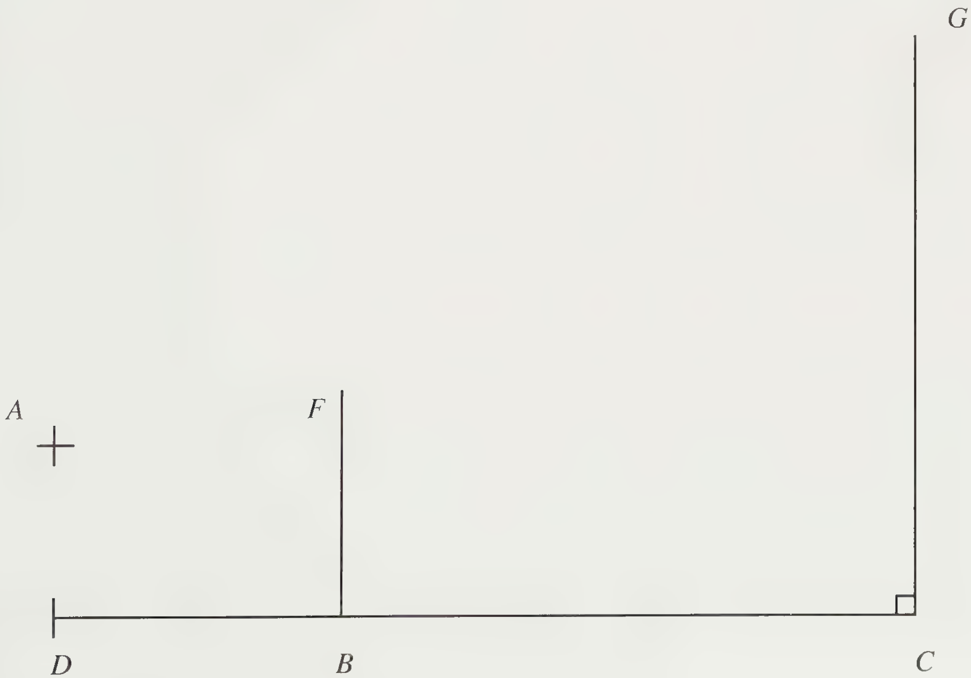
30 In the previous section (Book 1, Proposition 12) Piero suggests $AD = 3$ *braccia*. Here, this has been roughly doubled, for clarity.



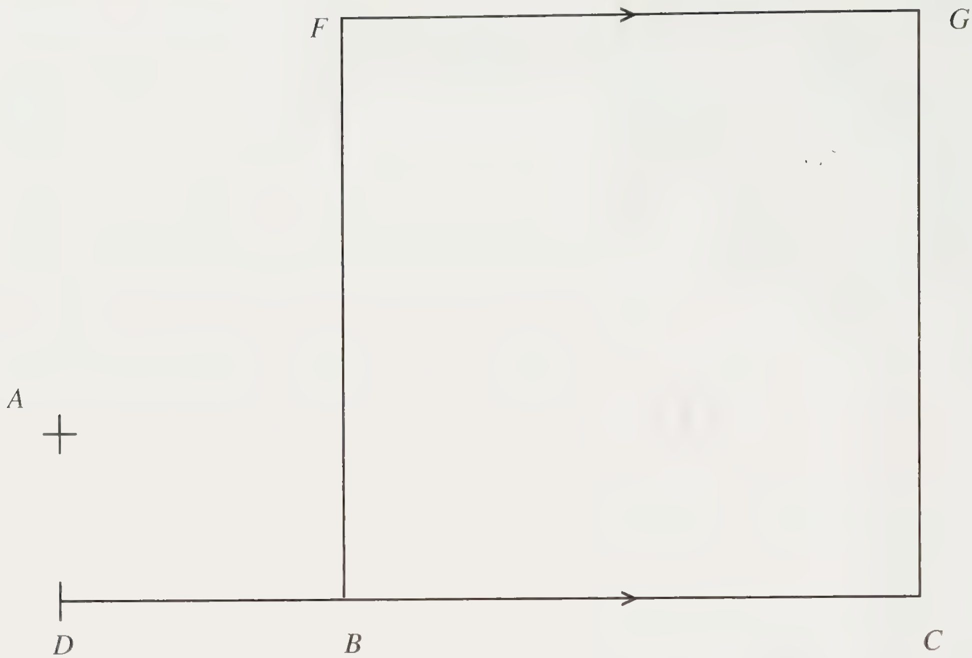
A8.5.1b Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 13. Drawing by JVF.



A8.5.1c Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 13. Drawing by JVF.



A8.5.1d Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 13. Drawing by JVF.



A8.5.1e Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 13. Drawing by JVF.

be drawn a line parallel to .BC. which is to be .GF. [see Fig. A8.5.1e] such I say will be a square with equal sides .BC. .CG. .GF. .FB..[31] Now draw from the point .A. the line[s] .AC. and .AG., which will divide .BF. in two points: .AC. will divide .BF. in the point .E. and .AG. will divide .BF. in the point .H. [see Fig. A8.5.1f]. I say that .E. will appear [when seen] from .A.[32] as higher than .B. because .A. is higher than .B. and .H. will appear [when seen] from .A. as lower than .F. because .A. is lower than .F., as is proved in the 10th and 11th propositions of Euclid's *De aspectuum diversitate*.[33] I say that .BE. will appear [when seen] from the said place [that is, .A.] equal to .BC., and .EH. will appear [when seen] from the said place [that is, .A.] equal to .CG., and .HF. appears equal to .GF..[34] Draw .AF. and .AB.; we have three triangles, each with two bases [see Fig. A8.5.1g], the triangle .ABC. has two bases .BC. and .BE., and the triangle .ACG. has two bases .CG. and .HE., and the triangle .AGF. has two bases .FG. and .FH.;[35] whence, by the second [proposition] of this [book] the base .BE. appears [as seen from .A.] equal to

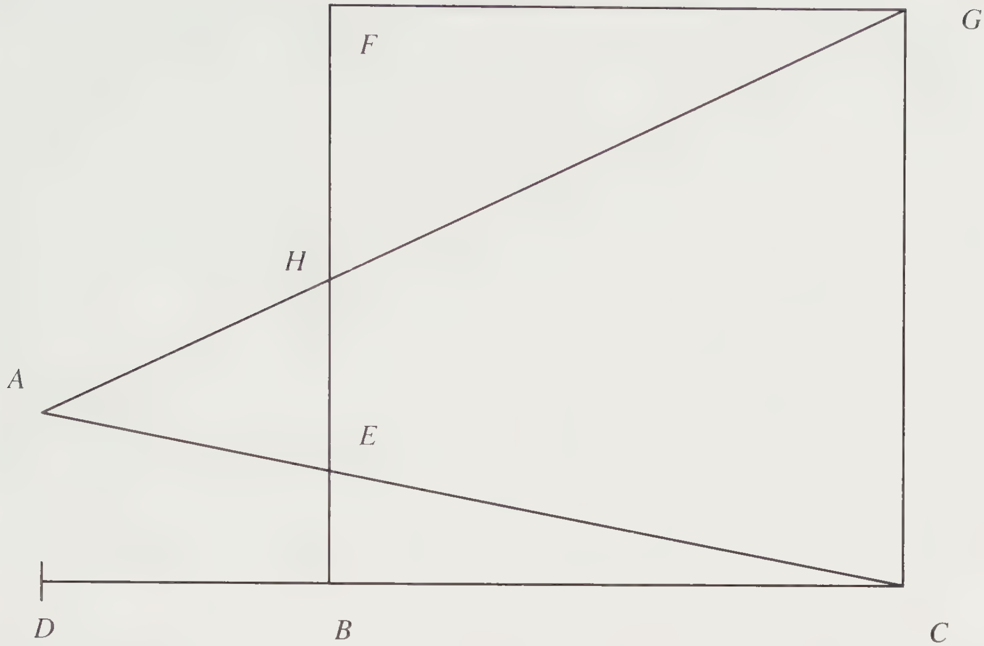
31 Piero's 'such' presumably means the resultant diagram; his style is not very clear.

32 Literally '.E. will represent [or 'present'] itself at .A. as being'.

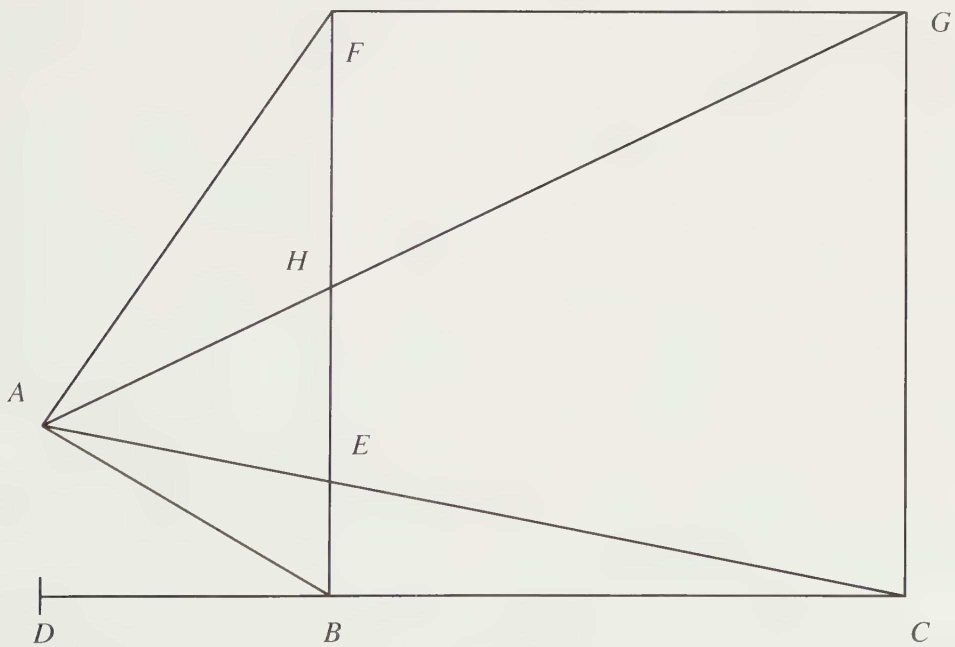
33 The reference could be to Euclid's own text or to the recension by Theon of Alexandria (active A.D. 364). Translated in Ver Eecke, *Euclide, L'Optique et la Catoptrique*, Paris and Bruges, 1938, pp.8–9, 62–63. It is, of course, possible that Piero knew the text through a secondary source.

34 Note that Piero has made this correct point by point: *EH* is *CG*, that is, *E* is *C* and *H* is *G*. A diagram of the three-dimensional set-up is needed to show Piero's assumptions here, see Chapter 5 (esp. Fig. 5.10). See also J. V. Field, 'Piero della Francesca as Practical Mathematician: the Painter as Teacher', in *Piero della Francesca tra arte e scienza. (Atti del convegno internazionale di studi, Arezzo, 8–11 ottobre 1992, Sansepolcro, 12 ottobre 1992)*, ed. Marisa Dalai Emiliani and Valter Curzi, Venice: Marsilio, 1996, pp.331–54; and J. V. Field, *The Invention of Infinity: Mathematics and Art in the Renaissance*, Oxford: Oxford University Press, 1997.

35 That is, triangles are thought of as having an apex, at *A*, joined to 'bases' by lines from *A*, which is the abacus-book rather than the *Elements* way of looking at triangles.



A8.5.1f Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 13. Drawing by JVF.



A8.5.1g Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 13. Drawing by JVF.

the base .BC. because they both subtend the same angle [at] .A., and the base .EH. is equal to .CG. in appearance, because they subtend the same angle, and the base .HF. appears equal to .FG. For they are contained by one [and the same] angle. And the proportion of .AE. to .AC. is [the same as] the proportion of .DB. to .DC.[36] And the same proportion of .EH. to .CG. is [given by] .AE. to .AC.[37]. And the proportion there is between .BE. and .FH. [added] together and .CG. is [the same as] the proportion of .HG. to .AG.[38] And when the distances and the things are in the same proportion as the height of the eye and the thing as degraded, it is clear that the degradation is correct.[39] So I shall say that [the rectangle] .EH. .CG. is the surface of [that is, it has the same area as] .BE. made into a square.[40]

Now let there be drawn[41] from the point .A. a line parallel to .BC., without an end point [see Fig. A8.5.1h] then divide the line .BC. into equal parts at the point .I. and construct a perpendicular from .I. and where it cuts the line from the point .A. parallel to .BC. make the point .A. [*sic*] [see Fig. A8.5.1i] then draw [through] .E. a line parallel to .BC. to cut .CG. in the point .K. [see Fig. A8.5.1j] then draw from the point .A. [(bis)] to the point .B. [a line] to divide .EK. in the point .D. [(bis)], then join the point .A. [(bis)] to the point .C., [with a line] to divide .EK. in the point .E. [(bis)] [see Fig. A8.5.1k]; I say I have made a square [in the] degraded plane, [the square] which is .BCDE. The proof

36 Because triangles *ADC*, *EBC* are similar, we have

$$\frac{AE}{AC} = \frac{DB}{DC}.$$

The Latin text in the BL MS has 'And the proportion of .AD. to .BE. is [the same as] the proportion of .DC. to .BC'.. The relation follows from the same similar triangles as those used in the vernacular text, and we have

$$\frac{AD}{BE} = \frac{DC}{BC}.$$

37 That is,

$$\frac{EH}{CG} = \frac{AE}{AC} \qquad \text{(g1)},$$

because triangles *AEH*, *ACG* are similar.

38 This follows from the fact that we have similar triangles *AHE*, *AGC*, but requires a little manipulation. I shall use modern notation.

We wish to prove

$$\frac{BE + FH}{CG} = \frac{HG}{AG}.$$

The right hand side can be expanded. Since $HG = AG - AH$, We get

$$\frac{HG}{AG} = \frac{AG - AH}{AG} = 1 - \frac{AH}{AG}$$

Now,

$$\frac{AH}{AG} = \frac{HE}{GC},$$

since triangles AHE, AGC are similar. So we have

$$\frac{HG}{AG} = 1 - \frac{HE}{GC} = \frac{GC - HE}{GC} \qquad \text{(g2)}.$$

Now, $GC = FB$, by construction (a construction Piero gave in laborious detail) and

$$FB - HE = BE + FH,$$

by inspection of the diagram, so equation (g2) simplifies to

$$\frac{HG}{AG} = \frac{BE + FH}{GC},$$

which is what Piero requires.

Piero would probably have got his result by manipulating ratios, for which there are rules that seem to have been committed to memory at abacus school. The Latin text in the BL MS gives the final ratio, the one of the left of our first equation, as that of .AH. to .HG. The mathematics involved is essentially similar.

39 That is, the perspective construction has given an optically correct result. The word 'distance' is a bit vague, but the preceding results show it must mean lines such as *DC*, that is, the distance to the furthest point of the object if it is horizontal, making *BC* correctly degraded as *EB* if we have

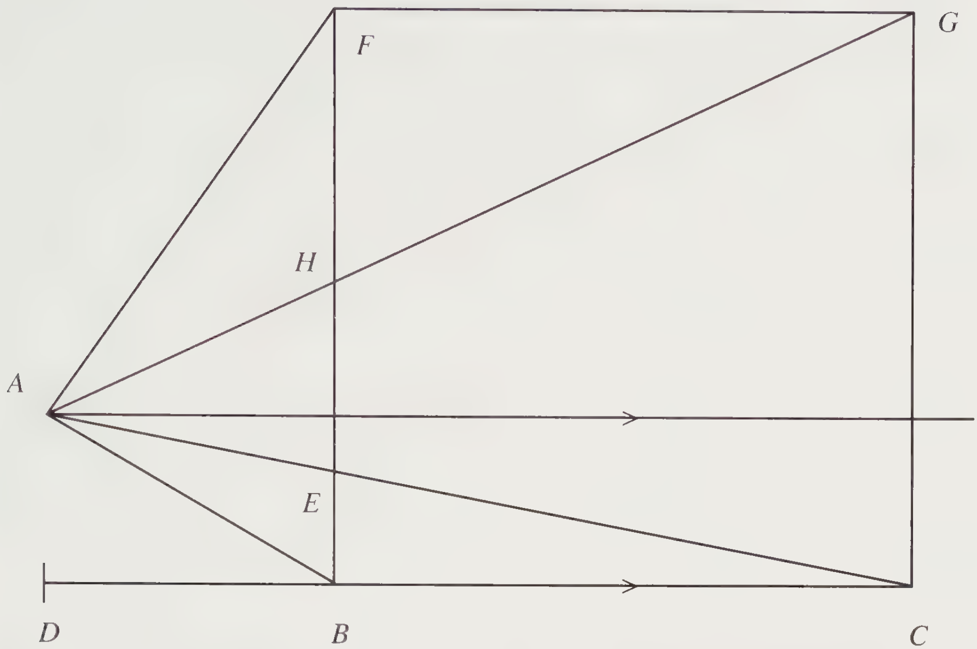
$$\frac{DC}{BC} = \frac{AD}{EB}.$$

This proportionality does in fact hold, because triangles *ADC*, *EBC* are similar.

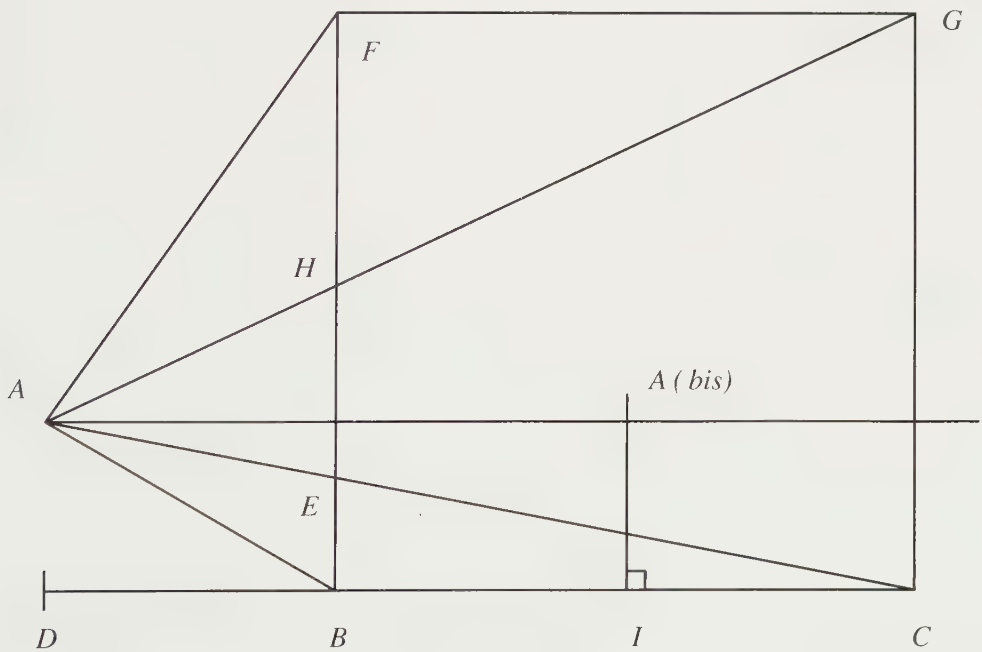
40 That is, in modern quasi-algebraic terms

$$EH.CG = BE^2.$$

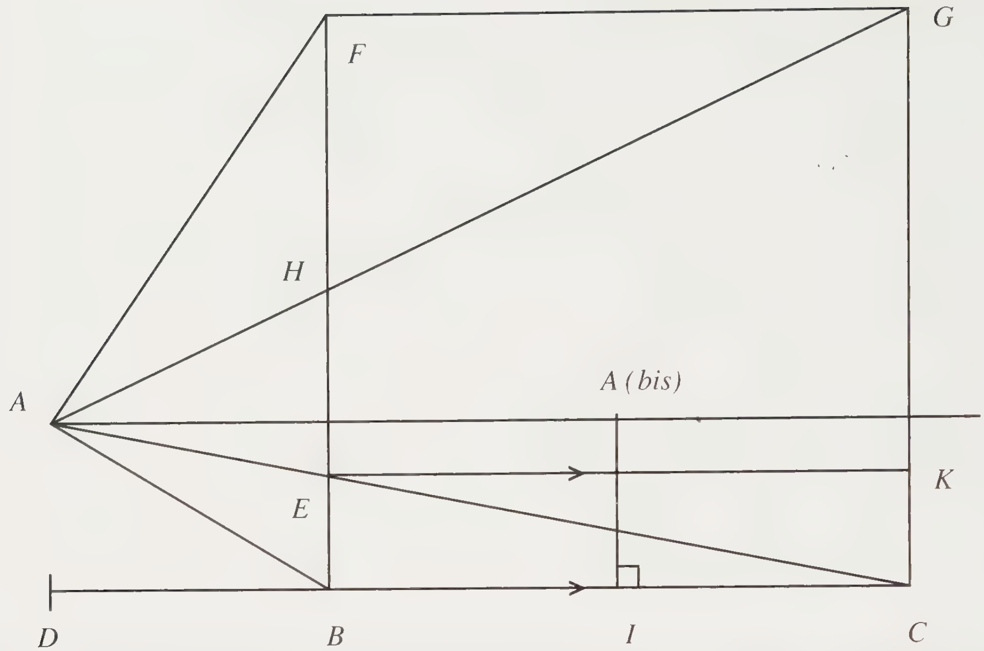
41 Paragraphing, not found in either the Parma or the BL manuscript, has been introduced for clarity. The Parma text has the imperative 'Now draw'.



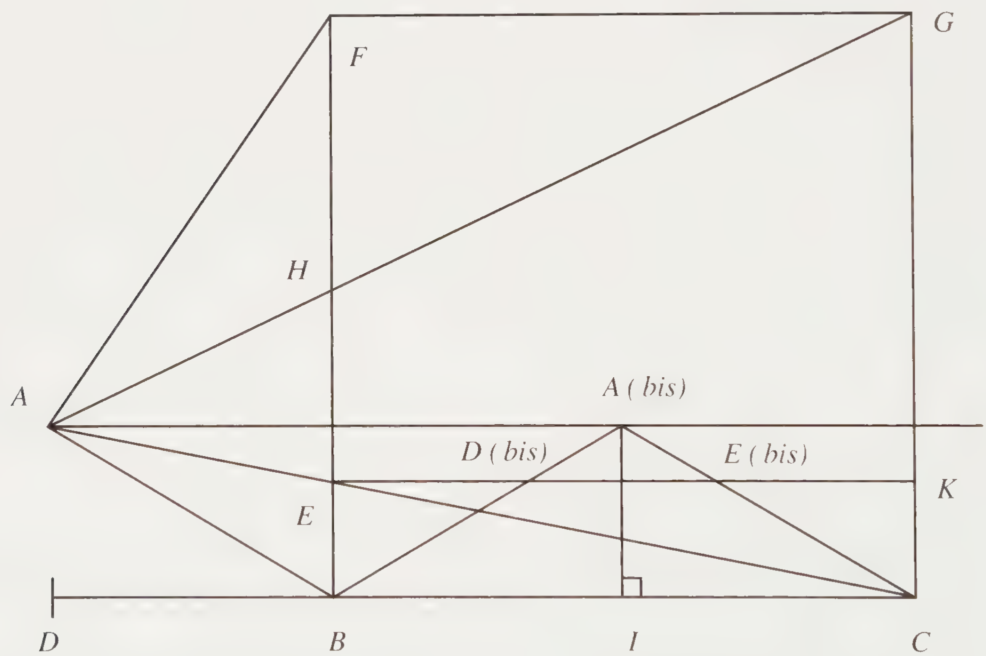
A8.5.1h Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 13. Drawing by JVF.



A8.5.1i Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 13. Drawing by JVF.



A8.5.1j Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 13. Drawing by JVF.



A8.5.1k Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 13. The BL MS has this diagram twice, on p.6 verso and p.7 recto. Both show line *AD*; the former omits *AI* and *AB*; the latter omits *AI*. However, these are the fullest forms of the diagram that I know. Also, on p.7 recto the text creeps round the diagram as if the diagram were drawn first or at least planned carefully. This diagram is larger than the previous one. Drawing by JVF.

[*prova*]: it is clear that .DE. is equal to .EH., which I put [*misi*] as the appearance of the quantity .CG.,[42] as is proved above.[43] I say it is equal or similar because the proportion of .AB. to .AD. is [the same as] that of .AC. to .AE.[44] And there is the same proportion between .DE. and .BC. as between .EH. and .CG..[45] So since they are[46] proportional they are either equal or similar, but they must be equal, because we put .BC. of the one equal to .BC. [*sic*] of the other, from which it is clear that the proposition holds.[47] But if you were to say: why do you put the eye in the middle?[48] Because it seems to me more convenient for seeing the work; all the same, it can be put wherever one wants, provided you do not go beyond the limits that will be shown in the final figure [of this book],[49] and, wherever you put it, it will see in the same proportion.[50]

6. Book 1, Proposition 14

Parma MS, p.7 recto; BL MS, p.7 verso; Piero ed. Nicco Fasola, p.77.

[Proposition 14] To divide the degraded square into several equal parts.

For example [*Verbi gratia*] let the square be .BCDE. and let the eye be .A., as was said in the preceding [propositions], which I have placed above the degraded square surface, which, as has been demonstrated, is the same [as putting it] in the place it was put before,[51] so I shall follow this [procedure], because it has the same result and is shorter.[52] As was said, let .BCDE. be a square and let the eye be .A.. Divide .BC. into as many parts as you like it divided, [say] in .FGHI. equally, then join .F. to the point .A. and .G. and .H. and

42 That is, the horizontal D(bis)E(bis) is equal to EH.

43 It seems possible that at this stage Piero expects one to test the truth of this assertion with dividers in the diagram.

44 Let us call A(bis) a, D(bis) d, and E(bis) e. What Piero has said is ambiguous, but we can take it he means

$$\frac{aB}{ad} = \frac{aC}{ae} \qquad \text{(k1)},$$

which follows from the fact that we have similar triangles aBC, ade (and conversely). But we also have similar triangles CEe, CAa, from which we obtain

$$\frac{aC}{ae} = \frac{AC}{AE} \qquad \text{(k2)}.$$

Now, from equation (g1) above we have

$$\frac{EH}{CG} = \frac{AE}{AC} \qquad \text{(g1)}$$

(this result follows from the similarity of triangles AEH, ACG). So equation (k2) gives us

$$\frac{aC}{ae} = \frac{CG}{EH} \qquad \text{(k3)}.$$

45 Again, this is ambiguous. The proportion de to BC suggests we look in the similar triangles ade, aBC, from which we obtain

$$\frac{de}{BC} = \frac{ae}{aC} \qquad \text{(k4)}.$$

The right side of (k4) is the inverse of the left side of (k3), so we clearly have

$$\frac{de}{BC} = \frac{EH}{CG} \qquad \text{(k5)},$$

which is as Piero said. Thus his last clause is a conclusion deduced from previous results, not a new independent assertion.

46 The Latin text in the BL MS, translated here, has 'Cum itaque sint proportionales', while the vernacular text in the Parma MS has the less formal 'being proportional' ('essendo proportionali').

47 The undefined 'they' (in the masculine plural) must mean the proportions because what we have is

$$\frac{de}{BC} = \frac{EH}{CG} \qquad \text{(k5)},$$

and

$$CG = BC,$$

by construction in Figure A8.5.1d. So the denominators of the two fractions in (k5) are equal. Therefore the numerators must be. So

$$de = EH.$$

And we are home and dry.

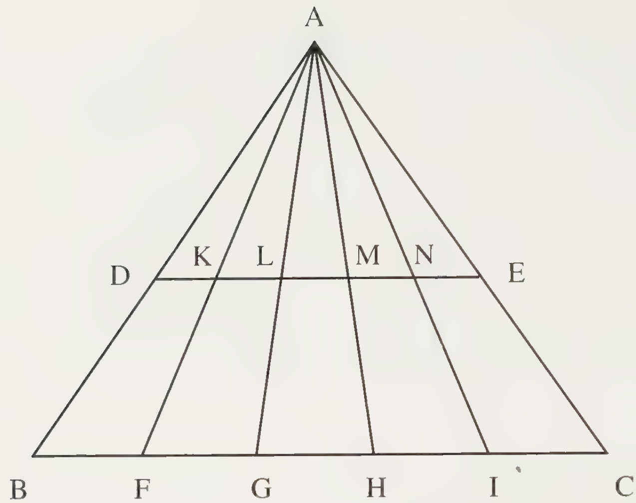
48 That is, directly above I.

49 That is, Book 1, Proposition 30, where Piero does not, however, discuss non-central a, nor AD.

50 'verrà in quella medesima proportione', presumably meaning that moving a, while keeping DB the same, does not affect $BCde$. The Latin text of the BL MS has 'it will have the same system (?) of proportion' ('eidem proportionis modus habebit'), which seems to mean the same thing.

51 That is, outside the square.

52 The vernacular text in the Parma MS is not very clear, but help is provided by the Latin in the BL MS.



A8.6.1 Copy of figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 14, BL MS, p.7 verso. In the BL MS, the text creeps round the diagram. – Compare this diagram with that for Book 1, Proposition 13 (Fig. A8.5.1k). The orthogonals from *F*, *G*, *H*, *I*, to *A* are not shown above the line *DE*, though the lines shown do go over it a little, suggesting they were drawn with a ruler placed to take them through *A*. Drawing by JVF.

.I. to the point .A., which will divide .DE. in the points .KLMN.. I say that .DE. is divided in the same proportion as that in which .BC. is divided, because the proportion of .BF. to .DK. is the same as that of .BC. to .DE., and that of .FG. to .KL. is equal to that of .GH. to .LM., and that of .HI. to .MN. is equal to that of .BC. to .DE., so that they are in proportion. Otherwise, because .BC. and .DE. subtend the same angle, so .BF. and .DK. subtend the same angle, and .FG. and .KL. subtend the same angle, and .GH. and .LM. subtend the same angle, so .HI. and .MN. subtend the same angle, and .IC. and .NE. subtend another and are parallel bases; it follows that they are in the same proportion. As is proved by the sixth[53] of this [book], which is what is proposed.

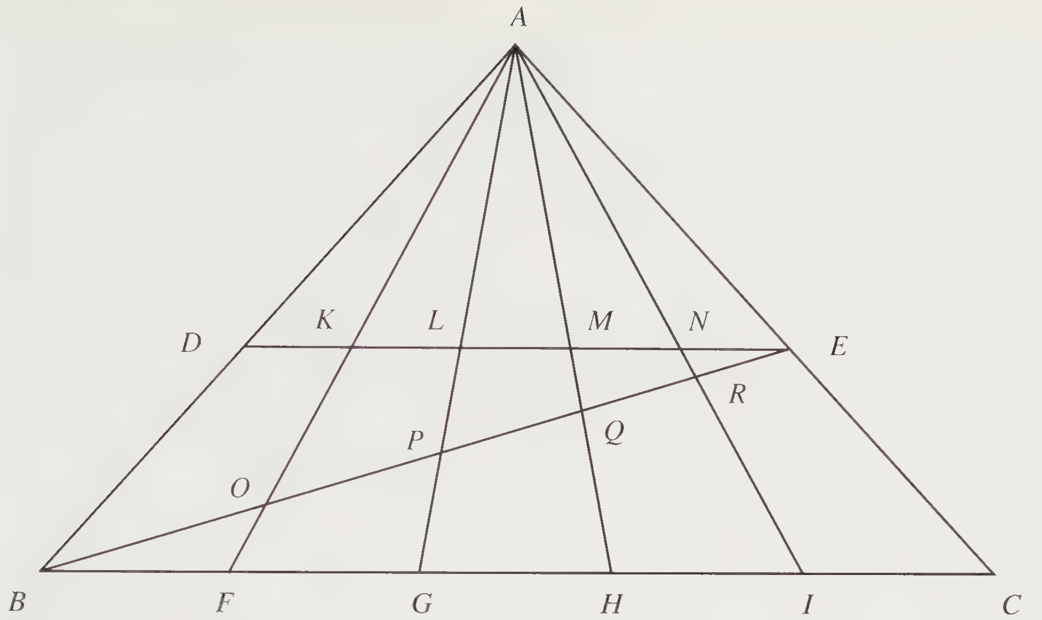
7. Book 1, Proposition 15

Parma MS, p.7 recto; BL MS, p.7 verso; Piero ed. Nicco Fasola, p.78.

[Proposition 15] When the diminished [that is degraded] square surface is divided into several equal parts, to make those divisions squares.

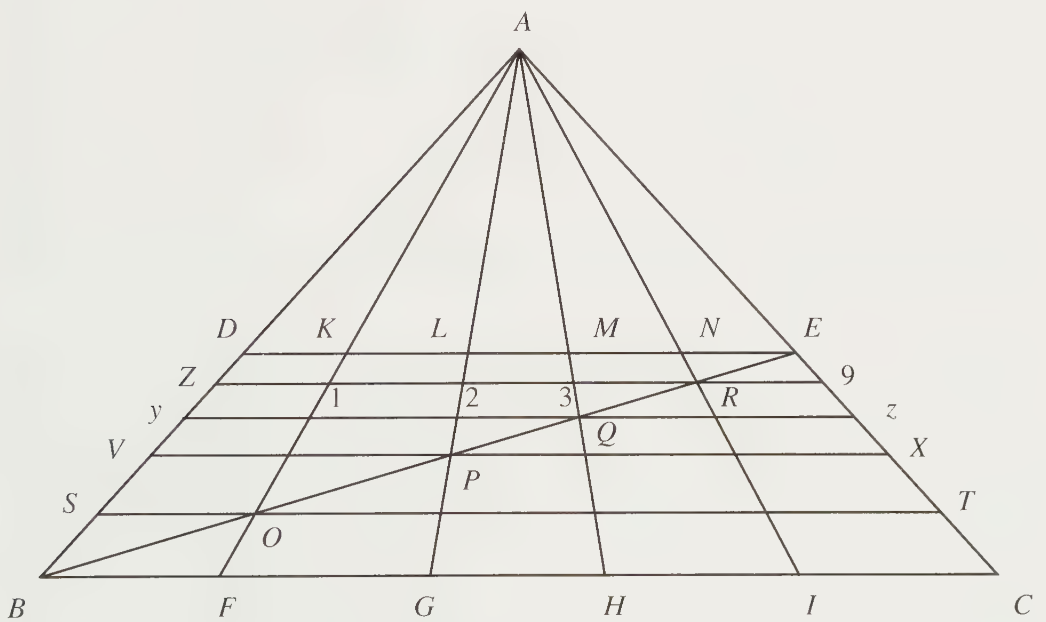
We have the degraded square surface .BCDE. divided into several equal parts, as .BC. in the points .FGHI. and .DE. in the points .KLMN.. Let them be joined to the eye .A. as before, and let there be drawn from the angle .B. to the angle .E. the diagonal .BE.; which will divide .FK. in the point .O. and .GL. in the point .P., and .HM. in the point .Q., and .IN. in the point .R. Draw through .O. a parallel to .BC., which will cut .BD. in the point .S. and .CE. in the point .T.; draw through .P. a parallel to .BC., which will cut .BD. in the point .V. and .CE. in the point .X.; draw through .Q. a parallel to .BC., which will cut .BD. in the point .y. and .CE. in the point .z.; draw through .R.. a paral-

53 The Latin text in the BL MS has 'sextam', where the Parma MS has '5a'. Actually, he means 'eighth', the result we have called 'Piero's theorem'. This error may be evidence for the text having been revised, and specifically for some extra propositions having been introduced before what is now section 8.



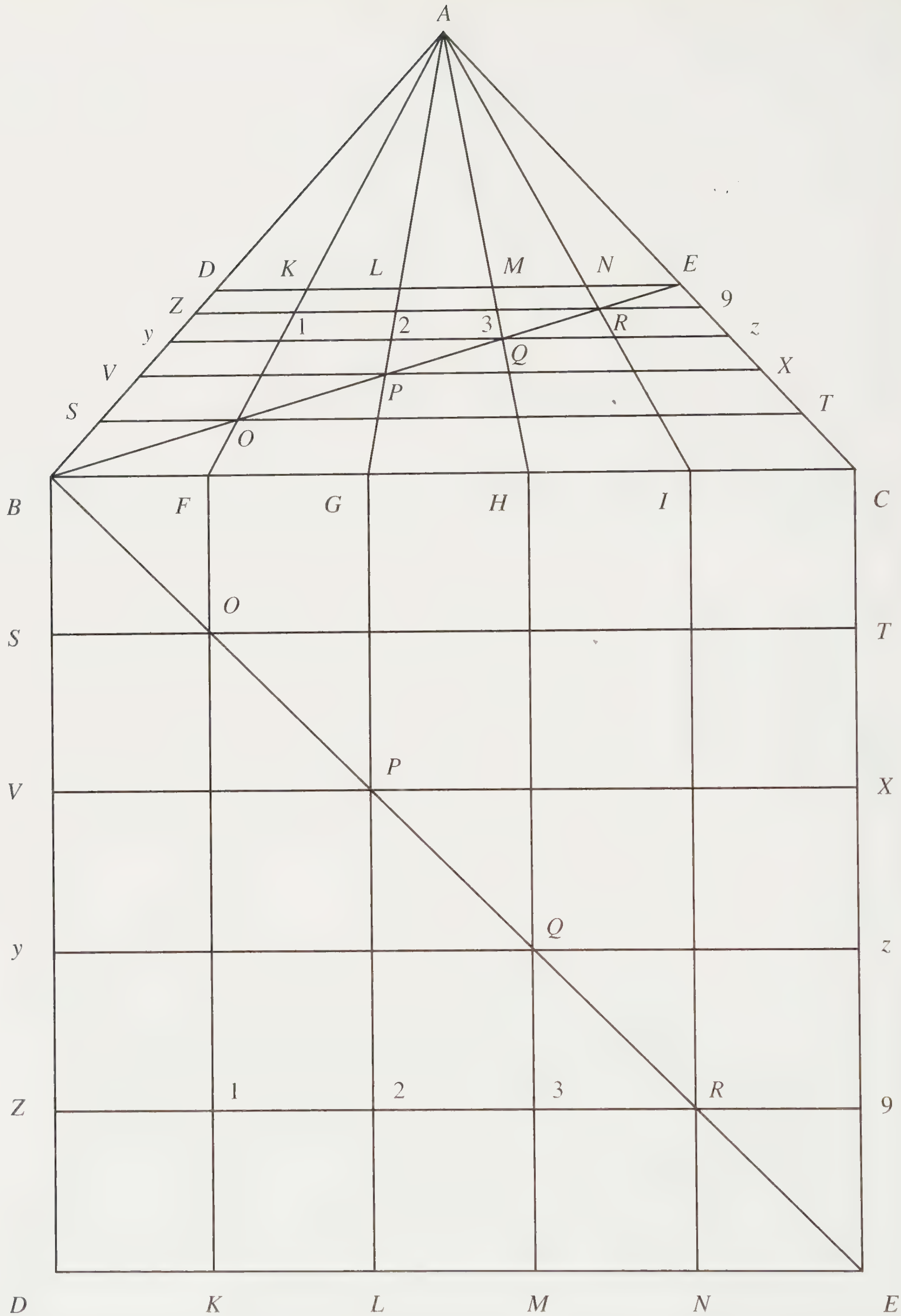
A8.7.1 Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 15. Drawing by JVF.

lel to .BC., which will cut .BD. in the point .Z. and .CE. in the point .9., and it will cut .FA. in the point one[54] and .GA. in the point .2. and .HA. in the point .3.. I say that these divisions will be made squares, as we said was to be done.

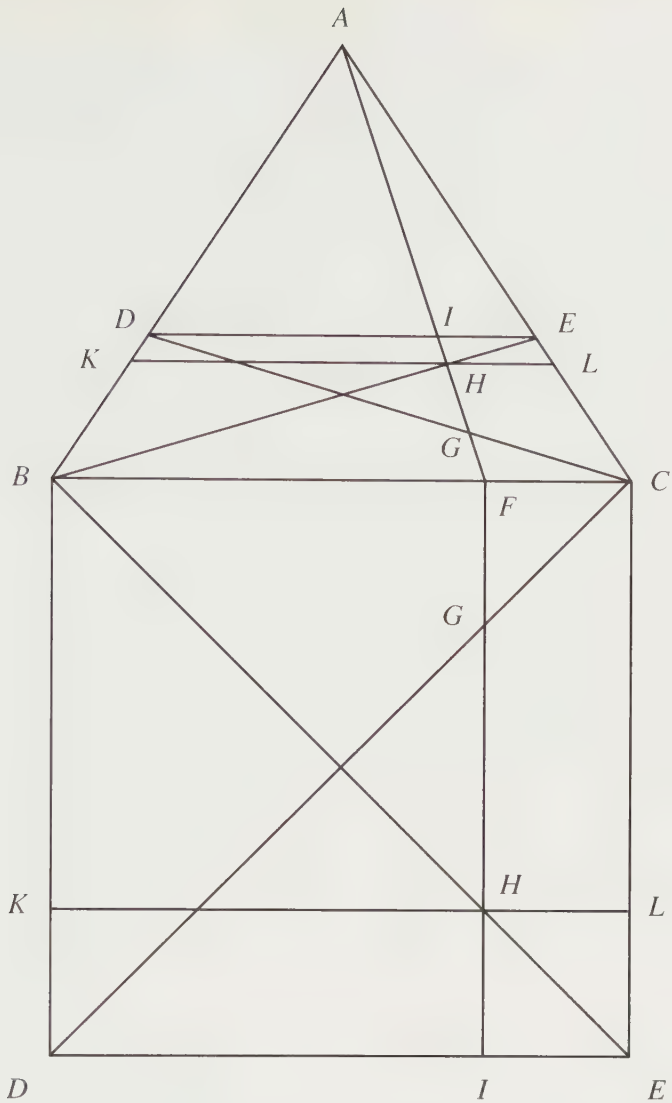


A8.7.2 Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 15. Drawing by JVF.

54 .1. in the BL MS Latin text.



A8.7.3 Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 15. The diagram in the BL MS has *DC* joined also in the 'proper' square below (p.8 recto). Drawing by JVF.



A8.7.4 Figure for Piero della Francesca, *De prospectiva pingendi*, Book 1, Proposition 15. Drawing by JVF.

It is proved thus:[55] make a square in its proper form, which will be below the line .BC., the same size as .BCDE. as above, and divide it into the same parts as .BC. is [divided], so that there will be squares with equal sides, and let the diagonal .BE. be drawn; I say that these lines will divide it in the same points as the degraded square is divided by the diagonal. Therefore if the diagonal in the square in proper form divides the parts in proportion, I shall say that it divides the degraded square surface in the same way in degraded

55 Paragraphing introduced by Nicco Fasola. There is no break in the Parma MS, or in the BL MS. In the latter, this clause was part of the previous sentence.

proportion, that is, the proportion of .AD. to .AB. is the same as that of .DK. to .BF., and that of .KL. to .FG., and that of .LM. to .GH., and that of .MN. to .HI., and that of .NE. to .IC., and the proportion of .AZ. to .AB. which is that of .Z1. to .BF., and that of .1.2. to .FG., and that of .2.3. to .GH., that of .3.R. to .HL.,[56] that of .R[.]9. to .IC., thus it follows that they are in proportion, so that the divisions produced are square, as I said I should show.

And[57] when the said square is not divided into equal parts, the diagonal divides it in proportion, as is shown in the second figure [for this proposition].[58] Let .BCDE. be a square in its proper form, and let there be drawn the diagonals .BE. .CD. which divide the surface into four equal parts, and any line drawn parallel to the sides will divide them [the diagonals] in proportion. For example: there is the figure .BCDE., as was said, in its proper form, in which I wish to insert three-quarters of the said surface. I shall take .BF., which will be three-quarters of .BC., and I shall draw the line [through] .F. parallel to .BD., which will divide the diagonals in two points, .DC. in the point .G. and .BE. in the point .H., and .DE. in the point .I., which divisions are proportional, because .BF. to .BC. is the same as .BH. to .BE. and .DG. to .DC. is the same as .BF. to .BC., and of .DI. to .DE.; and if a line is drawn parallel to .DE., [and] passing through .H., it will divide .BD. in the point .K. and .CE. in the point .L..

I say that .BK. is equal to .BF., because .FH. to HI. is the same as .BF. to .FC., and the diagonal line divides .FI. and .KL. in the same point, namely .H., and finally the diagonal of the quadrilateral produces quadrilaterals, so that .BF. .FH. .HK. .KB. are necessarily equal; and I say I want to insert into the square .BCDE. [an area equal to] three-quarters [of it], so I say the line .KL. is three-quarters inside.[59] Let the point .F. be joined to the point .A., [this line] will divide the degraded diagonals, .CD. in the point .G. and .BE. in the point .H.; draw [through] .H. a line parallel to .DE., which will cut .BD. in the point .K. and .CE. in the point .L. and this line is inside the degraded square .BCDE. in the same way as it is inside the square in its proper form.[60]

56 The BL MS gives this line as .hi. (all points except *A* are named in lower case in this MS).

57 Paragraphing introduced by Nicco Fasola. No break in Parma MS or BL MS.

58 Figures are supplied on BL MS, p.8 recto, unhelpfully labelled 15|1 and 16|2. Numbers of the corresponding sections of text are usually given with figures, though the sections of text are not actually numbered.

59 That is, three-quarters of the way up the quadrilateral.

60 That is, it has the same proportions in regard to the square in each case. What Piero has proved in this second part is that using an orthogonal to set up a proportion along the diagonal gives us a way of constructing any required proportion in the degraded figure.

Appendix 9

The Ground Plan and Lighting of the *Montefeltro Altarpiece*

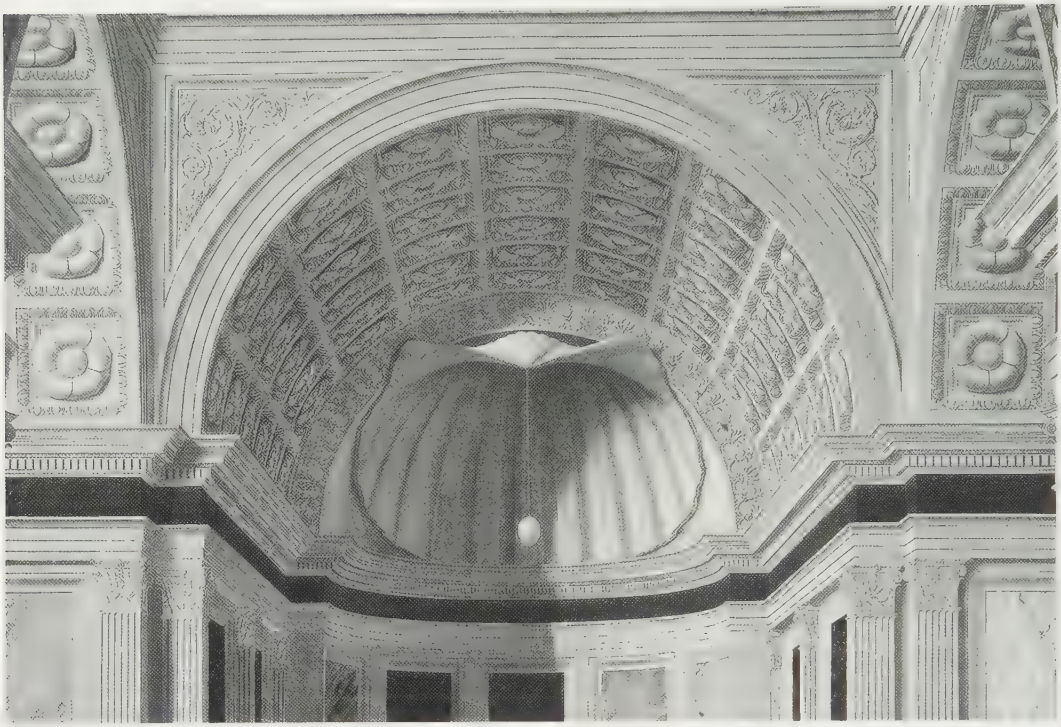
The barrel vault

The area covered by the barrel vault shown in the background of the *Montefeltro Altarpiece* (Galleria di Brera, Milan) must be nearly square. Proving this has nothing to do with perspective but simply involves ordinary geometry, together with the assumption that the coffers are square – an assumption that is in accord with what we know of classicizing architecture of this date. The overall shape of the vault is part of a cylinder (see Fig. A9.1), so the transverse section is clearly half a circle. The length of the curved edge of the transverse section is approximately the width of nine coffers, including the ribs, of which there may be nine or ten, depending on how we imagine the vault meets the moulding on which it rests. Using the fact that the circumference of a circle is π times its diameter, and taking $\pi = \frac{22}{7}$, gives the diameter of the vault as about $18 \times \frac{7}{22}$ coffer widths. This gives us $\frac{63}{11}$, which is not much under 6. (We could probably get exactly 6 by tinkering with the way ribs meet mouldings.) Piero has shown the barrel vault as six coffers deep, so he must have intended it to be approximately square.[1] Thus, given what we know about the Renaissance style in architecture, it seems reasonable to assume he meant the area under the vault to be exactly square.

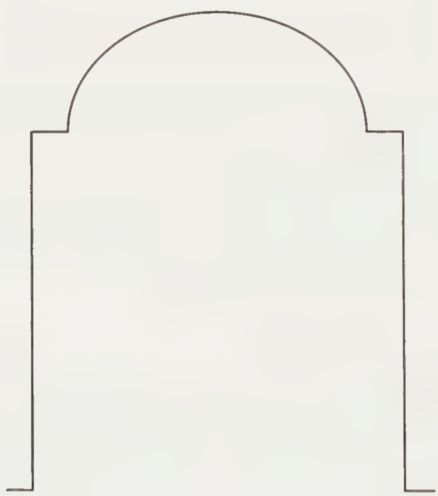
From our understanding of the Renaissance style it might seem that we should presume the plan of the apse is semicircular. However, close inspection of the lines at its base shows that it is in fact a little flattened, as are the apses in Giovanni Bellini's San Giobbe and San Zaccaria altarpieces. Unfortunately there seems to be no way of estimating the exact shape, but we may assume that, omitting mouldings, the plan of the barrel-vaulted area, and the apse beyond it, looks more or less as in Figure A9.2. As we shall see, the exact shape of this part of the structure is not crucial in what follows.[2]

1 A similar line of reasoning, including the assumption that the coffers are square, can be used to show that the area under the barrel vault in Masaccio's *Trinity* fresco (Santa Maria Novella, Florence) is not square. It has eight coffers running round the curved edge, and is seven coffers deep. See also J. V. Field, R. Lunardi and T. B. Settle, 'The Perspective Scheme of Masaccio's *Trinity* Fresco', *Nuncius* 4.2, 1988, pp.31–118.

2 Huge assumptions regarding the simplicity and symmetry of forms to be expected in Renaissance architecture are made in most attempts to reconstruct the ground plan of the building shown by Piero, for instance in Millard Meiss, '*Ovum struthionis*. Symbol and Allusion in Piero della Francesca's *Montefeltro Altarpiece*', in *Studies in Art and Literature for Belle da Costa Green*, ed. D. E. Miner, Princeton, 1954, pp.92–101. For a refreshingly sceptical view, and a full bibliography, see Matteo Ceriana, 'Sull' architettura dipinta della pala', in *La pala di San Bernardino di Piero della Francesca. Nuovi Studi oltre il restauro*, ed. Emanuela Daffra and Filippo Trevisani (Quaderni di Brera 9), Florence: Centro Di, 1997, pp.115–66.



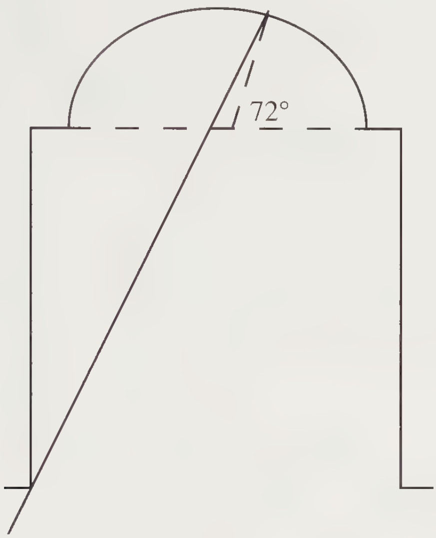
A9.1 Piero della Francesca (*c.*1412–1492), *Madonna and Child Enthroned with Saints and Angels* (*The Montefeltro Altarpiece*), detail of the upper part, showing the architectural setting, Galleria di Brera, Milan. For the complete picture, see Fig. 6.35 above.



A9.2 Plan of part of the architectural setting shown in the *Montefeltro Altarpiece*: the area under the barrel vault and the apse beyond it, omitting mouldings. The exact shape of the plan of the apse cannot be established, though the shape of the line in which the end wall meets the base of the vault indicates the plan is a little flattened compared with a semicircle. Here the plan has been shown semi-elliptical. Drawing by JVF.

The light falling into the apse

Light falls into the apse from above and from the left, casting a shadow of the edge of the left wall onto the panelling at the back of the apse. The panelling divides the apse wall into five parts, presumably equal, so the direction of the light, in plan, must be more or less as shown in Figure A9.3. The lighted part of the wall is the width of two panels and, as there are five across the semicircle, each panel must be 36°, making the lit part about 72° (or a little more, since a white vertical moulding is also illuminated). There is clearly no problem about the uppermost part of the stone scallop shell being lit, since it protrudes into the volume under the barrel vault, and this likewise ensures that the egg dangling from the chain attached to the cusp of the shell is also illuminated.

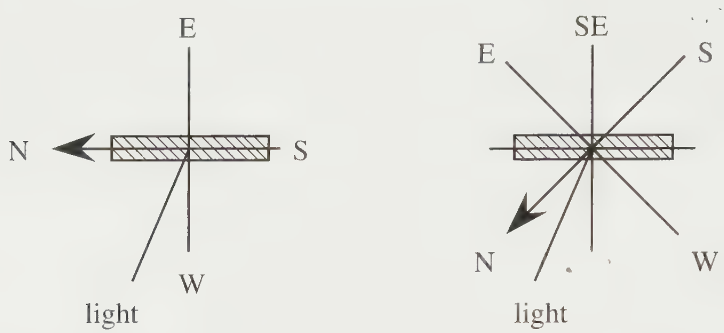


A9.3 Plan of part of the architecture in the *Montefeltro Altarpiece*: area under the barrel vault and the apse beyond it, showing the direction of the incident light. Drawing by JVF.

There is, however, a severe problem with this lighting if we assume the picture was intended for the main altar of a church oriented east. If straight ahead is east, then the light, which is coming from behind our left shoulder, is from somewhere to the west of north. Moreover, the Sun is rather high, since the top of the cusp of the scallop shell is illuminated. This is not realistic in the northern hemisphere. However, Italian churches are not on the whole notable for strict eastward orientation. In the case of San Bernardino, Urbino, the church with which this picture is most commonly associated, the high altar faces almost exactly south-east.[3] If the painting was for the high altar, then it puts the Sun between north

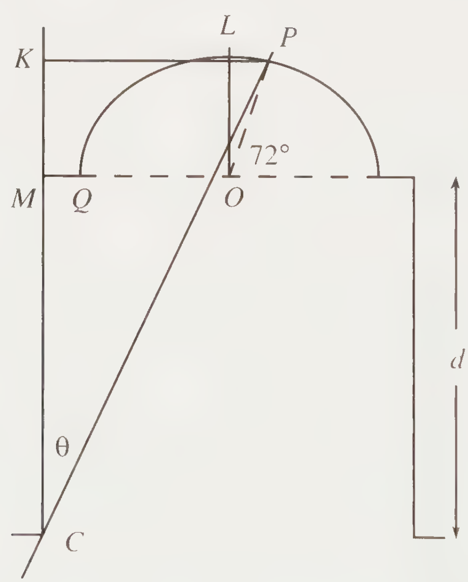
3 The plan of the church is so simple that guidebooks do not include it, but a plan with compass indications is given in Howard Burns, 'San Bernardino a Urbino', in *Francesco di Giorgio architetto*, ed. Francesco Paolo Fiore and Manfredo Tafuri, Milan: Electa, 1993, pp.230–8, p.230 for the plan. Burns demonstrates that the dating of Francesco di Giorgio's work argues strongly against Piero's altarpiece having been designed to go in it.

and north-west. Thus neither assuming the church was taken to be facing east nor the actual orientation of San Bernardino's main altar will make the light Piero shows come from a naturalistic direction. The two set-ups are shown in Figure A9.4.



A9.4 The direction of the lighting shown in the *Montefeltro Altarpiece*. Left, if the picture were above an altar facing east; right, if the picture were above the main altar of San Bernardino, Urbino, which faces approximately south-east. Drawings by JVF.

Since the situation looks far from straightforward, it will be as well to calculate more exactly the angle that the incident light makes with the long axis of the vault and apse. The plan is as shown in Figure A9.5, in which the length and width of the area under the barrel



A9.5 Plan of the area under the barrel vault and the apse in the *Montefeltro Altarpiece*, showing the incident light making an angle θ with the axis of the architecture. Drawing by JVF.

vault is *d*, and the angle that the incident light makes with the axis, ∠PCK, is θ. We shall find θ from the triangle *CPK*, since tan θ = *KP/KC*. In order to find *KP* and *CK* we need to know the radius of the apse. Measurement of the picture indicates that the diameter of the apse is about four fifths of the width of the vaulted area. So if the width of the vaulted area is *d*, and letting the radius of the apse be *r*, we have $r = 0.4d$.

To find the length *KP*, we use the fact that

$$\begin{aligned} KP &= KL + LP \\ &= OM + r.\cos 72° \\ &= (0.5 + 0.4\cos 72°)d \\ &= 0.623607d \text{ (to 6 dec. pl.)} \end{aligned}$$

To find the length *CK*, we use the fact that

$$\begin{aligned} CK &= CM + MK \\ &= CM + OL \\ &= CM + r\sin 72° \\ &= (1 + 0.4\sin 72°)d \\ &= 1.380423d \text{ (to 6 dec. pl.).} \end{aligned}$$

Now,

$$\tan\theta = \frac{KP}{CK},$$

therefore we have

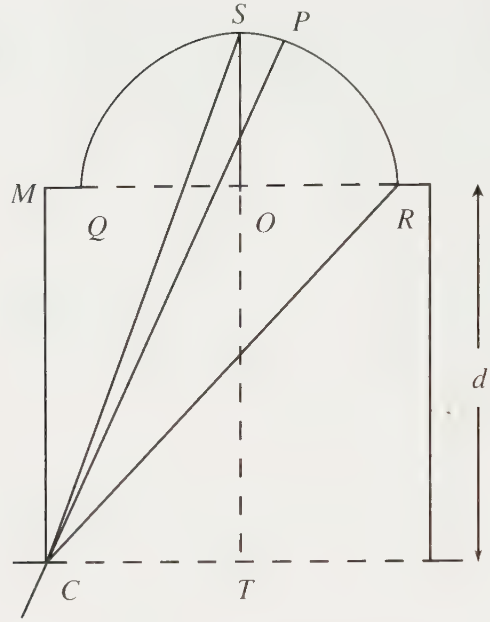
$$\begin{aligned} \tan\theta &= \frac{0.623607}{1.380423} \\ &= 0.451751 \text{ (to 6 dec. pl.).} \end{aligned}$$

Therefore

$$\theta = 24°\ 19' \text{ (to the nearest minute).}$$

This precision is, however, spurious, since we have made a number of rough estimates in the preliminaries to our calculation. That is we have made simplifying assumptions to permit us to carry out a calculation. The first was the assumption that the plan of the apse should be taken as semicircular. The rather sharp curves Piero has shown in the moulding near its outer edges suggest that the apse is shallower than this, perhaps having a plan that is more or less half an oval (or half an ellipse, as shown in my Figures). On the other hand, we may note similarly sharp curves in the moulding round the apse in Giovanni Bellini's *San Zaccaria Altarpiece* (1505), in which other visual evidence suggests the apse does have a semicircular plan. Flattened curves seem to be used as outlines of the apses in some of Francesco di Giorgio's designs for San Bernardino.[4]

4 See Burns, 'San Bernardino a Urbino' (full ref. note 3).



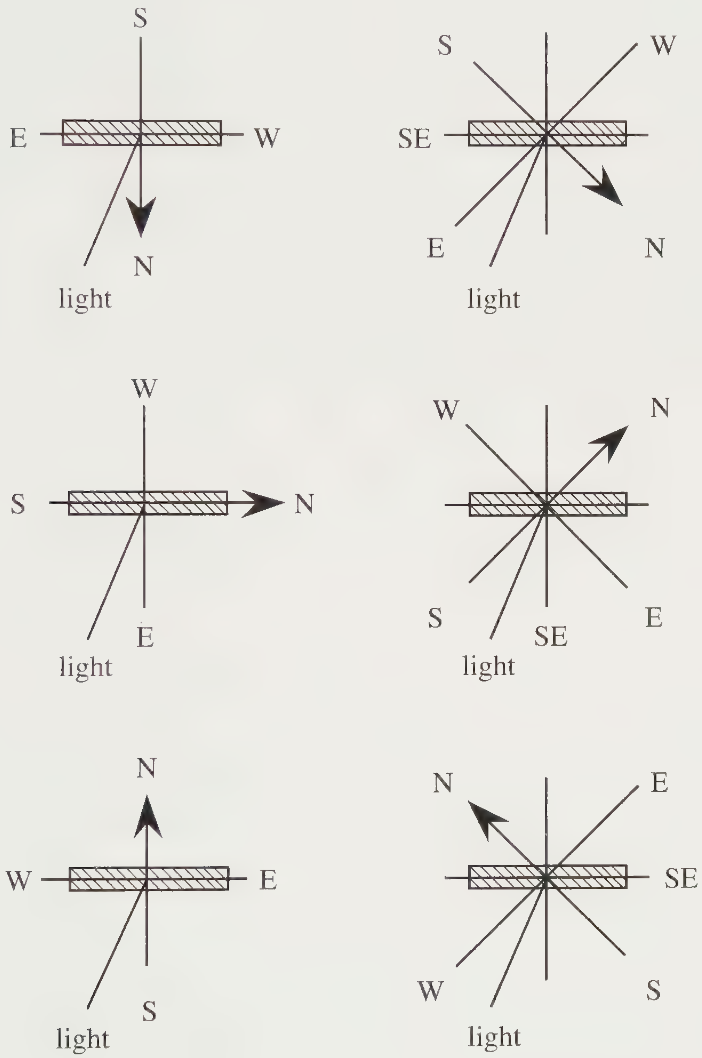
A9.6 Plan of the area under the barrel vault and apse in the *Montefeltro Altarpiece*, taking the plan of the apse to be semicircular, thus showing the limits on the angle made by the incident light. Drawing by JVF.

However, we can see from Figure A9.6 that the exact shape of the plan of the apse does not alter θ in any important way. Points have been given the same letters in this diagram as in Figure A9.5 above. It is clear from Piero's painting that the point *P* must lie to the right (as we see it) of the line *OS*. Simple geometry ordains that *CP* will lie to the left of *CR*, which lies to the left of the diagonal of the square under the barrel vault, so θ, that is, ∠*PCM*, must be less than 45°. To find the least possible value of θ, we want to see how far the line *CS* can move to the left. This depends on *S* moving upwards, and the extreme (reasonable) case is as shown in Figure A9.6, where the plan of the apse has been made semicircular. The diameter of the apse is less than the diameter of the barrel vault, so we have $ST < \frac{3}{2}d$. $CT = \frac{1}{2}d$, so this means that θ must always be greater than the angle whose tangent, given by *CT*/*ST*, is $\frac{1}{3}$. The angle whose tangent is $\frac{1}{3}$ is 18° 26′ (to the nearest minute). So we have established that

$$18°26' < \theta < 45°.$$

If we look back to Figure A9.4, we see that this means that, if Piero's painting is against a wall facing east, the light is coming from a direction between north-west and W18°26′N; and if the painting is against the altar wall of San Bernardino, which faces south-east, the light is coming from between north and N26°34′W (since 45° – 18°26′ = 26°34′). Since the light is also from fairly high in the sky, neither of these ranges of direction is realistic in the northern hemisphere. It is accordingly clear that no amount of playing about with the shape of the apse or the position of the shadow on its surface is going to get us out of this particular problem. Effectively, the direction of the light shows that if Piero meant it to follow the natural lighting in the actual building in which the picture was placed, then the picture was not designed either for a generic east-facing chapel or for the high altar of San

Bernardino – assuming, as is not unreasonable, that at the time of painting Piero had foreknowledge of the direction in which the church would face. Merely for the sake of completeness, the directions of the light implied by other orientations of the picture are shown in Figure A9.7. From these diagrams, it would seem that if Piero intended the light to be naturalistic, then he expected the picture to be placed against a wall that faced in some direction between north-west and north-east. Alternatively, he may not have intended the direction of the light to be naturalistic.



A9.7 Supplement to Fig. A9.4. The directions of the lighting shown in the *Montefeltro Altarpiece*, assuming a set-up like that shown in Fig. A9.5. The left column shows directions if the picture were placed parallel to the wall of a church oriented east (or N, S, W); for a wall (not necessarily the altar wall) facing east, see Fig. A9.4. The right column shows the directions if the picture is placed parallel to one of the walls of San Bernardino, Urbino, whose main altar faces south-east; (for a picture on the altar, see Fig. A9.4). Drawings by JVF.

There is, of course, no reason to suppose that Piero ever carried out a calculation of the kind performed here. If he wanted to find the direction of the light he could simply have drawn a plan of the architecture and drawn the relevant lines. His requirement was that the picture should look convincing. Ours was inevitably different: to draw conclusions about the limitations on possibilities of interpreting the picture.

The architecture itself

The light provides evidence for another limitation. This does not apply either to the plan of the building Piero has shown, or to its orientation, but concerns its architectural structure. Since the light we see in the apse is coming from behind us and from above left, reconstructions showing structures to left and right that are copies of the vaulted part we see in front of us are clearly impossible.[5] The light would need to flow through any barrel vault on the left. So there cannot be a barrel vault on the left. An opening to a loggia or courtyard in this position on the left might explain the intensity of the lighting. Many scholars have noted that near the shoulder of Federigo's armour, we have what seems to be the reflection of a round-headed window, so there must presumably be a wall containing the window lying to the spectator's left and on our side of the opening to a loggia or courtyard just noted. The implication of the existence of this wall is that we are in a building of some kind. It is conceivable that Piero intended to show a church – as is sometimes done in Netherlandish pictures of roughly this date – but if this was his intention it seems likely that, in view of its being such an unusual choice in Italian art, he would have made it absolutely clear that this was what he had chosen to do. As it is, we merely have indications that can partly be construed as suggesting a cruciform structure, but certainly do not compel us to infer one. In the circumstances, it seems much more probable that Piero is showing the Court of Heaven, and that his classical architecture is a reference to the ancient Roman basilica, a secular building that became the model for many Christian churches.[6]

Towards a ground plan with guesswork and arithmetic

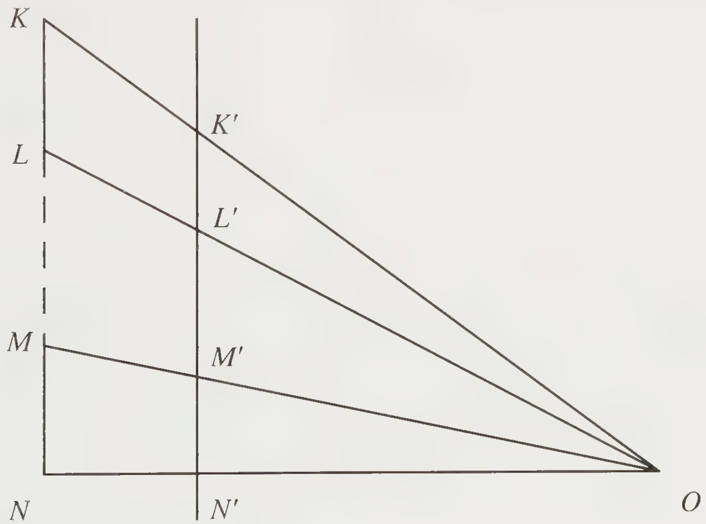
As we have seen, the area under the barrel vault shown in the background of the *Montefeltro Altarpiece* is square. So too is the area under the 'crossing' lying between the arches we see to left and right. In fact, the nearer square is the better defined, since we can see appropriate corners of mouldings, and it can be used to find a viewing distance for the picture (see Chapter 6). This viewing distance turns out to be twice the width of the panel, that is 340 cm. Unfortunately, knowing the correct viewing distance for the picture does not, by itself, allow us to find the true size or distance of an object we see in the picture. What it tells us is the angle that object is meant to subtend at the eye. Thus the thing can be twice as big and twice as far away, or half as big and half as far away. What we need is to be able to establish the true size of some object, which we can then use as a measure – hence

5 Such a reconstruction was shown in John Shearman, 'The Logic and Realism of Piero della Francesca', in *Festschrift für Ulrich Mitteldorf*, ed. Antije Kosegarten and Peter Tigler, Berlin: De Gruyter, 1968, pp.180–6, and appears to have been widely accepted among later scholars. It seems to be based on some simple calculations in Meiss, '*Ovum struthionis* ' (full ref. note 2).

6 This suggestion is made in R. Lightbown, *Piero della Francesca*, London: Abbeville Press, 1992, pp.251–2.

art historians' sometimes excessive concern with painters' possible use of a 'module'. In the case of Piero's *Flagellation of Christ* (Galleria Nazionale delle Marche, Urbino), the paving comes right up to the picture plane, so we can measure its true size there, and then use that measurement to supply a scale for everything else. Matters are not so simple in the *Montefeltro Altarpiece*. The tiny piece of side wall visible at the far left establishes that the architecture we see is entirely behind the figures. There is accordingly no architectural element that we can measure directly. We have a choice between despair and using some guesswork. What follows is the result of opting for guesswork. The use of guesswork is as minimal as the demands of the mathematical set-up will allow. Since there will be an explanation of the processes of calculation, readers will be at liberty to substitute their own guesses if the present ones seem implausible.

Most of the calculation will depend upon the use of similar triangles, each with an apex at the eye and a base consisting of the thing seen (the object or its image). This is where Piero begins the first book of *De prospectiva pingendi* (see Chapter 5 and Appendix 8). He assumes, however, that one is looking straight at the object, that is, that one of the lines joining its ends to the eye makes a right angle with the object. This simplification does not affect the mathematics, as can be seen from considering the properties of some of the similar triangles shown in Figure A9.8. Let O be the position of the eye, and let it look at an object KL, and let the image of KL on the picture plane ($K'L'M'N'$) be $K'L'$. The line $KLMN$ is parallel to the picture plane and ON is at right angles to it and to $K'L'M'N'$. What we are going to be interested in is the ratio of the length KL to the length $K'L'$.



A9.8 Finding the distances of things behind the picture plane. The eye is at O and it is looking at magnitudes KL, MN, whose images on the picture plane are $K'L'$ and $M'N'$. Drawing by JVF.

Triangles KLO and $K'L'O$ are similar, since KL and $K'L'$ are parallel. Therefore,

$$\frac{KL}{K'L'} = \frac{KO}{K'O}.$$

Triangles KNO and $K'N'O$ are similar, so

$$\frac{KO}{K'O} = \frac{NO}{N'O}.$$

Triangles MNO and $M'N'O$ are similar, so

$$\frac{NO}{N'O} = \frac{MN}{M'N'}.$$

Since the right-hand side of the first equation is the same as the left-hand side of the second, and the right-hand side of the second is the same as the left-hand side of the third, we can put these three equations together to get

$$\frac{KL}{K'L'} = \frac{MN}{M'N'}.$$

So, whatever the true height of the eye compared to the thing it is looking at, we can always simplify the diagram into a form like that of the lower part of Figure A9.8. Accordingly, rather than adopting the style of the abacus book and drawing a new diagram for each example we consider, we shall adopt today's style of taking each example as a special case of the general one that is shown in the pair of similar triangles OMN, $OM'N'$ in Figure A9.8. Since we know that the distance of the eye from the picture plane is 340 cm, we shall always have $ON' = 340$. If we want to know the distance of something behind the picture plane, that is, NN', we shall use the relation

$$\frac{MN}{M'N'} = \frac{NO}{N'O},$$

which follows from the fact that triangles MNO, $M'N'O$ are similar, and then find $N'N$ by using

$$N'N = NO - N'O.$$

Since architecture can be any size, we may, if we wish, decide on the size of the background architecture and find its distance. There is, however, one constraint: as already noted, the small piece of wall visible on the left establishes that the near edges of the arches seen at the sides are some way behind the figures. Calculating the distances of the figures depends upon our guessing their real size, shown as MN in Figure A9.8. The obvious guesses are that their true heights are either that of an average European man in the fifteenth century, about 165 cm, or the 3 *braccia* – that is, 175 cm – of the 'perfect' man.

For St John the Baptist, the height of the image, $M'N'$, is about 116.8 cm. (The height was measured on a photograph of about A4 size and then scaled up, by a factor of about 10.) Assuming $MN = 165$ gives us

$$NO = \frac{165}{116.8} \times 340$$

That is,

$$NO = 480.3 \text{ (to 4 sig. fig.)}.$$

So if St John is 'average' size, his distance from the eye is 480 cm (to the nearest centimetre) and his distance behind the picture plane is 140 cm. If his true height is 175 cm we have

$$NO = \frac{175}{116.8} \times 340.$$

That is,

$$NO = 509.4 \text{ (to 4 sig. fig.)}.$$

This makes his distance from the eye 509 cm (to the nearest centimetre) and his distance behind the picture plane becomes 159 cm.

For St Jerome, the height of the image, $M'N'$, is about 109.2 cm. (Again a measurement was made on a photograph of about A4 size and scaled up.) Assuming $MN = 165$ gives us

$$NO = \frac{165}{109.2} \times 340$$

That is,

$$NO = 513.7 \text{ (to 4 sig. fig.)}.$$

So if St Jerome is 'average' size his distance from the eye is 513 cm (to the nearest centimetre) and his distance behind the picture plane is 173 cm. If his true height is 175 cm we have

$$NO = \frac{175}{109.2} \times 340.$$

That is

$$NO = 544.9 \text{ (to 4 sig. fig.)}.$$

This makes his distance from the eye 545 cm (to the nearest centimetre) and his distance behind the picture plane becomes 205 cm.

Similar calculations can be carried out for all other figures whose heads and feet we can see, and whose posture we are prepared to regard as adequately upright. We can find Federigo's standing height by adding on the length of his lower leg to that of his kneeling figure. The guesswork component will, of course, increase if we wish to include the angels in this process, though it is indeed possible that Piero always made angels the same size in relation to human beings, which would enable one to use the angels in *The Baptism of Christ* (National Gallery, London) as a standard. The neat sets of footprints for the figures shown in many reconstructions of the plan of the scene in the *Montefeltro Altarpiece* must have some basis such as the above. That is, their positions contain a notable element of conjecture.

Including the Virgin and Child in this computation leads to difficulties. The Virgin's eye height is more or less the same as that of the saints standing close to her, and since the low dais cannot compensate completely for the fact that she is sitting while they are standing, it is clear that she must be on a larger scale than they are. If we assume that the saints are 'average' (that is, 165 cm) and the Virgin is 'perfect' (175 cm) we shall be able to make up some of the discrepancy, but further calculation suggests that is not a complete solution.

Let us assume that we can find the Virgin's standing height by adding on the length of the lower leg to the height that we see in the seated figure. This gives us a standing height of 136.9 cm, that is M′N′ = 136.9. If the height of the Virgin is 175 cm, then we have

$$NO = \frac{175}{136.9} \times 340.$$

That is,

$$NO = 436.4 \text{ (to 4 sig. fig.)}.$$

This makes her distance from the eye 435 cm (to the nearest centimetre) and her distance behind the picture plane is 95 cm. This puts her well in front of St John the Baptist, which is clearly not the case. So the true height of the Virgin must be more than 175 cm.

The difficulty in finding the size of the Virgin is not very important because we have the alternative of finding her position by considering her relation to the saints. However, the matter does assume some importance if we wish to proceed to an investigation of the architecture. The only clue we have as to the scale of the architecture is that the width of the vaulted area appears to be about 16.6 times the length of the egg (ignoring the fact that the egg is slightly closer than the back of the vault). So it would be useful to know the size of the ostrich egg. No doubt ostrich eggs come in a range of sizes, but Piero can hardly have expected his viewers to be experts on the matter. He may, of course, have expected that there would be a nearby example of an ostrich egg to provide a size. If we are to continue to prefer guesswork to despair, we need to make another guess. This one is an aesthetic judgement. Several scholars have noted the visual analogy between the egg and the head of the Virgin.[7] Let us suppose that the visual parallel extended to their really being the same size. This is lent a certain plausibility by the fact that the image of the egg appears to be about a third of the size of that of the head of the Virgin. In any case, the image of the head of the Virgin is of height 14.4 cm, so that if we estimate the distance of the figure, say putting it 180 cm behind the picture plane – a little further back than St Jerome – we can find the actual size of the head by using the equation

$$\frac{MN}{M'N'} = \frac{NO}{N'O}$$

(employing the lettering of Fig. A9.8). In this case we have $M'N' = 14.4$, $NO = 520$ (= 180 + 340), and $N'O = 340$. So we have

$$MN = \frac{520}{340} \times 14.4.$$

that is,

$$MN = 22.0 \text{ (to 3 sig. fig.)}.$$

So the size of the Virgin's head, that is its vertical dimension, is 22 cm. The result is, however, less reliable than earlier ones, since it involved measurements of smaller lengths.

7 See for example Carlo Bertelli, *Piero della Francesca*, trans. E. Farelly, New Haven and London: Yale University Press, 1992, p.138.

Assuming the egg is of length 22 cm, the diameter of the vaulted area is 16.6 × 22 cm, that is, 365 cm (to the nearest centimetre). However, we also know that the image of the egg is one-third of the size of that of the head, so the distance of the egg from the eye must be three times that of the head. The distance of the head was 520 cm, so the distance of the egg is 1560 cm. Since our ground plan already contains two squares, the vaulted area and the area under what seems to be the crossing, it is of interest how many such squares lie between the egg and the eye. The answer is 1560/365, that is 4.27 (to 3 sig. fig.). Between the egg and the picture plane we have (1560 – 340)/365, that is 3.34 squares.

So if the egg really is meant to be the same size as the Virgin's head we are seeing rather large architecture at a rather large distance. Making the egg smaller will scale things down, because it will allow the egg to be nearer. However, my purpose here is not to produce a reasonable ground plan for the scene shown in the altarpiece, but rather to show that we cannot produce a ground plan at all unless we make so many guesses that it seems an economical and hardly less valid alternative to simply guess much of the ground plan itself. In particular, there seems to be no way to calculate how far the group of figures is in front of the 'crossing', and various scholars do, in fact, show slightly different distances.[8] What no one seems to do is to explain where the plan came from. Piero himself most probably did know the ground plan, but in the present state of the picture we have not got enough information to find out what it was. It is not clear whether Piero wished to put us in this fix, but we may note that there are other pictures by him that also defy exact reconstruction, for instance the *Madonna di Senigallia* (Galleria Nazionale delle Marche, Urbino). This aspect of the matter is discussed in more detail in Chapter 6.

8 Among more recent scholars, the claim is usually that such a plan is based on Shearman, 'The Logic and Realism of Piero' (full ref. note 5), or Meiss, '*Ovum struthionis*' (full ref. note 2).

Bibliography

Alberti, Leon Battista, *La Pittura . . . tradotta . . . per M. Lodovico Domenichi*, Venice, 1547.

Alberti, Leon Battista, *Della pittura*, trans. Cosimo Bartoli, in *Opuscoli morali*, Venice, 1568.

Alberti, Leon Battista, *'On Painting' and 'On Sculpture'*, edited and translated by Cecil Grayson, London: Phaidon, 1972. English text of *On Painting* reprinted with an introduction by Martin Kemp, London: Penguin, 1990.

Alberti, H. J. von, *Mass und Gewicht*, Berlin: Akademie-Verlag, 1957.

Alchuno caso sottile: La quinta distinzione della 'Pratica di Geometria' dal Codice Ottoboniano Latino 3307 della Bibloteca Apostolica Vaticana, ed. Annalisa Simi, Quaderni del Centro Studi della Matematica Medioevale, series eds. L. Toti Rigatelli and R. Franci, no.23, Siena: Servizio Editoriale dell'Università di Siena, 1998.

Bambach Cappel, Carmen, *Drawing and Painting in the Italian Renaissance Workshop: Theory and Practice, 1300–1600*, Cambridge: Cambridge University Press, 1999.

Banker, James R., 'Piero della Francesca's S. Agostino Altarpiece: Some New Documents', *Burlington Magazine* 129, 1987, pp.642–51.

Banker, James R., 'Un documento inedito del 1432 sull'attività di Piero della Francesca per la chiesa di San Francesco in Borgo S. Sepolcro', *Rivista d'Arte*, 4th series, vol.6, 1990, pp.245–47.

Banker, James R., 'Piero della Francesca's Friend and Translator: Maestro Matteo di Ser Paolo d'Anghiari', *Rivista d'Arte*, 4th series, vol.8, 1992, pp.331–40.

Banker, James R., 'Piero della Francesca, il fratello Don Francesco di Benedetto e Francesco dal Borgo', *Prospettiva* 68, October 1992, pp.54–6.

Banker, James R., 'Piero della Francesca as Assistant to Antonio d'Anghiari in the 1430s: Some Unpublished Documents', *Burlington Magazine* 135, 1993, pp.16–21.

Banker, James R., 'A Legal and Humanistic Library in Borgo San Sepolcro in the Middle of the Fifteenth Century', *Rinascimento*, 2nd series, vol.33, 1993, pp.163–91.

Banker, James R., 'The Altarpiece of the Confraternity of the Misericordia in Borgo Sansepolcro', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no. 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art, 1995, pp.21–35.

Banker, James R., 'Piero della Francesca: gli anni giovanili e l'inizio della sua carriera', in *Città e Corte nell' Italia di Piero della Francesca. Atti del Convegno Internazionale di Studi Urbino, 4–7 ottobre 1992*, ed. Claudia Cieri Via, Venice: Marsilio, 1996, pp.85–95.

Banker, James R., *The Culture of San Sepolcro During the Youth of Piero della Francesca*, Ann Arbor: University of Michigan Press, 2003.

Battisti, Eugenio, *Piero della Francesca*, 2nd edn, 2 vols, ed. M. Dalai Emiliani, Milan, 1992.

Baxandall, Michael, *Patterns of Intention: On the Historical Explanation of Pictures*, New Haven and London: Yale University Press, 1985.

Baxandall, Michael, *Giotto and the Orators*, Oxford: Clarendon Press, 1986.

Baxandall, Michael, *Words for Pictures: Seven Papers on Renaissance Art and Criticism*, New Haven and London: Yale University Press, 2003.

Bellosi, Luciano, ed., *Una scuola per Piero: Luce, colore e prospettiva nella formazione fiorentina di Piero della Francesca*, Venice: Marsilio, 1992.

Bellucci, Roberto, and Cecilia Frosinini, 'Ipotesi sul metodo di restituzione dei disegni preparatori di Piero della Francesca: il caso dei ritratti di Federigo da Montfeltro', in *La pala di San Bernardino di Piero della Francesca. Nuovi Studi oltre il restauro*, ed. Emanuela Daffra and Filippo Trevisani (Quaderni di Brera 9), Florence: Centro Di, 1997, pp.167–87.

Bellucci, Roberto, and Cecilia Frosinini, 'Piero della Francesca's Process: Panel Painting Technique', in *Painting Techniques, History, Materials and Studio Practice* (Contributions to the Dublin Congress of the International Institute for Conservation, 7–11 September 1998), ed. A. Roy and P. Smith, London: International Institute for

Conservation of Historic and Artistic Works, 1998, pp.89–93.

Benedetti, Giovanni Battista, *De rationibus operationum perspectivae* in *Diversarum speculationum . . . liber*, Turin, 1585.

Bennett, J. A., 'The Mechanics' Philosophy and the Mechanical Philosophy', *History of Science* 24, 1986, pp.1–28.

Bertelli, Carlo, 'La pala di San Bernardino e il suo restauro', *Notizie da Palazzo Albani* 11, 1–2, 1982, pp.13–20.

Bertelli, Carlo, *Piero della Francesca*, trans. E. Farelly, New Haven and London: Yale University Press, 1992.

Bianconi, Piero, *Tutta la pittura di Piero della Francesca*, Milan: Rizzoli, 1957.

Black, Robert, 'Humanism and Education in Renaissance Arezzo', *I Tatti Studies (Essays in the Renaissance)* 2, 1987, pp.171–237.

Bomford, D., J. Dunkerton, D. Gordon, A. Roy and J. Kirby, *Art in the Making: Italian Painting before 1400*, London: National Gallery Publications, 1989.

Bomford, D., ed., with contributions from R. Billings, L. Campbell, J. Dunkerton, S. Foister, J. Kirby, C. Plazzota, A. Roy and M. Spring, *Art in the Making: Underdrawings in Renaissance Paintings*, London: National Gallery Publications, 2002.

Brahe, Tycho, *Astronomiæ instauratæ mechanica*, Wandsbeck, 1598, reprint, Nuremberg, 1602.

Brown, Jonathan, *Velázquez: Painter and Courtier*, New Haven and London: Yale University Press, 1986, third reprint, 1990.

Burke, Peter, *The Italian Renaissance: Culture and Society in Italy*, Cambridge: Polity Press, 1986. (First edn 1972.)

Burns, Howard, 'San Bernardino a Urbino', in *Francesco di Giorgio architetto*, ed. Francesco Paolo Fiore and Manfredo Tafuri, Milan: Electa, 1993, pp.230–38.

Bussagli, Marco, 'Note sulla *Natività* di Londra: la gazza e l'asino, due motivi dissonanti', in *Città e Corte nell' Italia di Piero della Francesca. Atti del Convegno Internazionale di Studi Urbino, 4–7 ottobre 1992*, ed. Claudia Cieri Via, Venice: Marsilio, 1996, pp.233–41.

Buteo, Joannes [Jean Borrel], *De quadratura circuli Libri duo, ubi multorum quadraturae confutantur, & ab omnium impugnatione defenditur Archimedes*, Lyon, 1559.

Casazza, O., 'Il ciclo delle storie di San Pietro e la "Historia Salutis". Nuova lettura della Capella Brancacci', *Critica d'Arte* year 51, 4th series, no.9, April–June 1986, pp.69–84.

Casini, Paolo, *L'Antica Sapienza in Italia: Cronistoria di un Mito*, Rome: Il Mulino, 1999.

Castiglione, Baldassare, *Il Libro del Cortegiano*, with an introduction by Amedeo Quondam, notes by Nicola Longo, Milan: Garzanti, 2000. (First edn Venice, 1528)

Cennini, Cennino, *Il Libro dell'Arte*, ed. Fabio Frezzato, Vicenza: Neri Pozza, 2003.

Cennini, Cennino, *The Craftsman's Handbook'*, trans. Daniel V. Thompson, Jr, New Haven: Yale University Press 1933, reprint, New York: Dover Press, 1954.

Ceriana, Matteo, 'Sull'architettura dipinta della pala', in *La pala di San Bernardino di Piero della Francesca. Nuovi Studi oltre il restauro*, ed. Emanuela Daffra and Filippo Trevisani (Quaderni di Brera 9), Florence: Centro Di, 1997, pp.115–66.

Cheles, Luciano, *The Studiolo of Urbino: An Iconographical Study*, Wiesbaden: Dr Ludwig Reichert Verlag, 1986.

Cieri Via, Claudia, 'Il polittico della *Madonna della Misericordia* di Piero della Francesca: tradizione iconografica e tradizione cutuale', in *Città e Corte nell' Italia di Piero della Francesca. Atti del Convegno Internazionale di Studi Urbino, 4–7 ottobre 1992*, ed. Claudia Cieri Via, Venice: Marsilio, 1996, pp.167–82.

Clark, Nicholas, *Melozzo da Forlì*, ed. Amanda Lillie et al., London: Sothebys, 1990.

Copernicus, Nicolaus, *De revolutionibus orbium cœlestium*, Nuremberg, 1543.

Criminisi, Antonio, *Accurate Visual Metrology from Single and Multiple Uncalibrated Images*, Distinguished Dissertation Series, London: Springer-Verlag, 2001.

Cusanus, Nicolaus, *Reparatio Calendarii*, in *Opera*, ed. Jacques Lefèvre d'Étaples, Paris, 1514, vol.2, pp.xxii–xxix.

Cusanus, Nicolaus, *Complementum theologicum*, in *Opera*, ed. Jacques Lefèvre d'Étaples, Paris, 1514, vol.2, pp.xcii–c.

Dabell, Frank, 'Antonio d'Anghiari e gli inizi di Piero della Francesca', *Paragone* 35.2, 1984, pp.71–94.

Daffra, Emanuela, and Filippo Trevisani, eds., *La pala di San Bernardino di Pïero della Francesca. Nuovi Studi oltre il restauro* (Quaderni di Brera 9), Florence: Centro Di, 1997.

Davis, M. D., *Piero della Francesca's Mathematical Treatises: The 'Trattato d'abaco' and 'Libellus de quinque corporibus regularibus'*, Ravenna: Longo Editore, 1977.

Derenzini, Giovanna, 'Note autografe di Piero della Francesca nel codice 616 della Bibliothèque Municipale di Bordeaux. Per la storia testuale del *De prospectiva pingendi*', *Filologia Antica e Moderna* 9, 1995, pp.29–55.

Desargues, Girard, *see* Field and Gray.

Descartes, René, *Correspondence*, ed. C. Adam and G. Milhaud, Paris: Vrin, vol.3, 1940.

Dürer, Albrecht, *Underweysung der Messung mit dem Zirkel und Richtscheyt*, Nuremberg, 1525.

Edgerton, S. Y., *The Renaissance Rediscovery of Linear Perspective*, New York: Basic Books, 1975.

Edgerton, S. Y., *The Heritage of Giotto's Geometry: Art and Science on the Eve of the Scientific Revolution*, Ithaca: Cornell University Press, 1991.

Elkins, James, 'Piero della Francesca and the Renaissance Proof of Linear Perspective', *Art ulletin* 69.2, 1987, pp.220–30.

Euclid, *Opera omnia*, J. L. Heiberg and H. Menge (eds), 8 vols, Leipzig, 1883–1916.

Euclid, *The Thirteen Books of Euclid's Elements*, ed. and trans. T. L. Heath, Cambridge: Cambridge University Press, 1908. (2nd edn, New York: Dover Press, 1956, many reprints.)

Euclid, *L'Optique et la Catoptrique*, ed. and trans. Paul Ver Eecke, Paris: Librarie scientifique Albert Blanchard, 1959.

Fauvel, John, and Jeremy Gray, eds., *The History of Mathematics: A Reader*, London: Macmillan, 1987.

Federici-Vescovini, Graziella, 'Contributo per la storia della fortuna di Alhazen in Italia: il volgarizzamento del ms. Vat. 4595 e il commentario terzo del Ghiberti', *Rinascimento*, 2nd series, 5, 1965, pp.17–49.

Federici-Vescovini, Graziella, 'Il problema delle fonti ottichi medievali del Commentario terzo di Lorenzo Ghiberti', in *Lorenzo Ghiberti nel suo tempo. Atti del Convegno Internazionale di Studi (Firenze 18–21 ottobre 1978)*, [no name of editor], Florence, 1980, pp.347–87.

Federici-Vescovini, Graziella, 'Alhazen vulgarisé: *Le De li aspecti* d'un manuscrit du Vatican (moitié du XIVe siècle) et le troisième Commentaire sur l'optique de Lorenzo Ghiberti', *Arabic Sciences and Philosophy* 8(1), 1998, pp.67–96.

Fibonacci, *see* Leonardo of Pisa.

Field, Arthur, *The Origins of the Platonic Academy of Florence*, Princeton: Princeton University Press, 1988.

Field, J. V., 'Kepler's Rejection of Numerology', in *Occult and Scientific Mentalities in the Renaissance*, ed. B. W. Vickers, Cambridge: Cambridge University Press, 1984, pp.273–96.

Field, J. V., 'Giovanni Battista Benedetti on the Mathematics of Linear Perspective', *Journal of the Warburg and Courtauld Institutes* 48, 1985, pp.71–99.

Field, J. V., 'Two Mathematical Inventions in Kepler's *Ad Vitellionem paralipomena*', *Studies in History and Philosophy of Science* 17(4), 1986, pp.449–68.

Field, J. V., 'Piero della Francesca's Treatment of Edge Distortion', *Journal of the Warburg and Courtauld Institutes* 49, 1986, pp.66–99 and plate 21c.

Field, J. V., 'Linear Perspective and the Projective Geometry of Girard Desargues', *Nuncius* 2.2, 1987, pp.3–40.

Field, J. V., 'What is Scientific about a Scientific Instrument?', *Nuncius* 3.2, 1988, pp.3–26.

Field, J. V., 'The Relation Between Geometry and Algebra: Cardano and Kepler on the Regular Heptagon', in *Girolamo Cardano: Philosoph, Naturforscher, Arzt* (Proceedings of a conference held in the Herzog August Bibliothek, Wolfenbüttel, in October 1989), ed. E. Keßler, Wiesbaden, 1994, pp.219–42.

Field, J. V., 'Piero della Francesca and the "Distance Point Method" of Perspective Construction', *Nuncius* 10.2, 1995, pp.509–30.

Field, J. V., Piero della Francesca as a Practical Mathematician: The Painter as Teacher', in *Piero della Francesca tra arte e scienza. (Atti de convegno internazionale di studi, Arezzo, 8–11 ottobre 1992, Sansepolcro, 12 ottobre 1992)*, Marisa Dalai Emiliani and Valter Curzi (eds), Venice: Marsillo, 1996, pp.331–54.

Field, J. V., *The Invention of Infinity: Mathematics and Art in the Renaissance*, Oxford: Oxford University Press, 1997, reprint, 1999.

Field, J. V., 'Alberti, the Abacus, and Piero della Francesca's Proof of Perspective', *Renaissance Studies* 11/2, 1997, pp.61–88.

Field, J. V., 'Rediscovering the Archimedean Polyhedra: Piero della Francesca, Luca Pacioli, Leonardo da Vinci, Albrecht Dürer, Daniele Barbaro, and Johannes Kepler', *Archive for History of Exact Sciences* 50 (nos. 3–4), 1997, pp.241–89.

Field, J. V., 'When is a Proof not a Proof? Some Reflections on Piero della Francesca and Guidobaldo del Monte', in *La Prospettiva: Fondamenti teorici ed esperienze figurative dall'Antichità al mondo moderno. Atti del Convegno Internazionale di Studi, Istituto Svizzero di Roma (Roma, 11–14 settembre 1995)*, ed. R. Sinisgalli, Florence: Edizioni Cadmo, 1998, pp.120–32.

Field, J. V., 'Why Translate Serlio?', in *Thomas Gresham and Gresham College: Studies in the Intellectual History of London in the Sixteenth and Seventeenth Centuries*, ed. F. Ames-Lewis, Aldershot: Ashgate, 1999, pp.198–221.

Field, J. V., 'Renaissance Mathematics: Diagrams for Geometry, Astronomy and Music', *Interdisciplinary Science Reviews*, 29 (3), 2004, pp.259–77.

Field, J. V., 'Tycho Brahe, Johannes Kepler and the Concept of Error', in *Miscellanea Kepleriana. Festschrift für Volker Bialas*, ed. D. Di Liscia et al., Munich: Beck, forthcoming.

Field, J. V., 'Mathematical Books in the Library of Diego Velázquez (1599–1660)', forthcoming.

Field, J. V., and J. J. Gray, *The Geometrical Work of Girard Desargues*, London and New York: Springer-Verlag, 1987.

Field, J. V., and F. A. J. L. James, Introduction to *Renaissance and Revolution: Humanists, Craftsmen and Natural Philosophers in Early Modern Europe*, ed. J. V. Field and F. A. J. L. James, Cambridge: Cambridge University Press, 1993, reprint 1997.

Field, J. V., R. Lunardi and T. B. Settle, 'The Perspective Scheme of Masaccio's *Trinity* Fresco', *Nuncius* 4.2, 1988, pp.31–118.

Folkerts, Menso, 'New Results on the Mathematical Activity of Regiomontanus', in Ernst Zinner (trans. Ezra Brown), *Regiomontanus: His Life and Work* (Studies in the History and Philosophy of Mathematics, vol.1), Amsterdam and New York: North-Holland, 1990.pp.363–72.

Folkerts, Menso, and Richard Lorch, 'Some Geometrical Theorems Attributed to Archimedes and their Appearance in the West', in *Archimede: Mito Tradizione Scienza*, ed. Corrado Dollo (Proceedings of a conference held in Syracuse, October 1989), Biblioteca di Nuncius, Studi e Testi IV, Florence: Olschki, 1992, pp.61–79.

Fracastoro, Girolamo, *Syphilis sive morbus Gallicus*, Padua, 1530.

Fracastoro, Girolamo, *Homocentrica*, Padua, 1530.

Franci, R., and L. Toti Rigatelli, 'Towards a History of Algebra from Leonardo of Pisa to Luca Pacioli', *Janus* 72, 1985, pp.17–82.

Frangenberg, Thomas, *Der Betrachter: Studien zur florentinischen Kunstliteratur des 16. Jahrhunderts*, Berlin: Gebr. Mann Verlag, 1990.

Fusetti, Sergio, and Paola Virilli, 'Il restauro', in *Piero della Francesca. Il Polittico di Sant'Antonio*, ed. Vittoria Garibaldi, Perugia: Electa Editori, 1993, pp.137–51.

Galilei, Galileo, *Il saggiatore nel quale con bilancia esquisita e giusta si ponderano le cose contenute nella Libra astronomica e filosofica di Lotario Sarsi*, Rome, 1623.

Galilei, Galileo, *Dialogo sopra i due massimi sistemi del mondo*, Florence, 1632.

Galilei, Galileo, *Discorsi e dimostrazioni matematiche intorno a due nuove scienze*, Leiden, 1638.

Galilei, Vincenzo, *Dialogo . . . della musica antica e della moderna*, Florence, 1581, reprint, 1602.

Galluzzi, Paolo, *The Art of Invention: Leonardo and Renaissance Engineers*, trans. M. Mandelbaum, M. Gorman, L. Otten and K. Singleton, Florence: Giunti, 1999.

Garibaldi, Vittoria, ed., *Piero della Francesca. Il Polittico di Sant'Antonio*, Perugia: Electa Editori, 1993.

Garibaldi, Vittoria, Introduction to *Piero della Francesca. Il Polittico di Sant'Antonio*, ed. Vittoria Garibaldi, Perugia: Electa Editori, 1993, pp.19–44.

Ghiberti, Lorenzo, *Commentaria*, ed. Ottavio Morisani, Naples, 1947.

Lorenzo Ghiberti nel suo tempo. Atti del Convegno Internazionale di Studi (Firenze 18–21 ottobre 1978) [no name of editor], Florence: Olschki, 1980.

Gilbert, Creighton E., *Change in Piero della Francesca*, Locust Valley, New York: J. J. Augustin, 1968.

Gilbert, Creighton E., 'Melozzo: his Status, his Drawings', in *Arte d'Occidente: Studi in Onore di Gloria Maria Romanini*, Rome: Sintesi d' Informazione, 1999, pp.1043–50.

Gilbert, Creighton E., 'Piero at Work for the Confraternity of Mercy', in *Città e Corte nell' Italia di Piero della Francesca. Atti del Convegno Internazionale di Studi Urbino, 4–7 ottobre 1992*, ed. Claudia Cieri Via, Venice: Marsilio, 1996, pp.69–84.

Giusti, Enrico, 'L'algebra nel *Trattato d'abaco* di Piero della Francesca: osservazioni e congetture', *Bolletino di Storia delle Scienze Matematiche* 11.2, 1991, pp.55–83.

Giusti, Enrico, ed., *Luca Pacioli e la matematica del Rinascimento. Atti del convegno internazionale di studi. Sansepolcro 13–16 aprile 1994*, Città di Castello: Petruzzi Editore, 1998.

Goldthwaite, R., *The Building of Renaissance Florence*, Baltimore and London: The Johns Hopkins University Press, 1980.

Gombrich, E. H., 'From the Revival of Letters to the Reform of the Arts: Niccolò Niccoli and Filippo Brunelleschi', in *The Heritage of Apelles*, Oxford, 1976, pp.93–110.

Grendler, Paul, 'What Piero Learned in School: Fifteenth-Century Vernacular Education', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no.48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art, 1995, pp.161–74.

Hall, A. R., *Ballistics in the Seventeenth Century*, Cambridge: Cambridge University Press, 1952.

Hall, A. R., *The Scientific Revolution*, London: Longmans, Green and Co., 1954.

Hall, A. R., 'The Scholar and the Craftsman in the Scientific Revolution', in *Critical Problems in the History of Science*, ed. M. Clagett, Madison: University of Wisconsin Press, 1959 (a lecture given in 1957).

Hall, A. R., 'Gunnery, Science and the Royal Society', in *The Uses of Science in the Age of Newton*, ed. John G. Burke, Berkeley and Los Angles: University of California Press, 1983, pp.111–41.

Hall, A. R., 'Retrospection on the Scientific Revolution', in *Renaissance and Revolution: Humanists, Craftsmen and Natural Philosophers in Early Modern Europe*, ed. J. V. Field and F. A. J. L. James, Cambridge: Cambridge University Press, 1993, pp.239–49, reprint, 1997.

Hargittai, I., and M. Hargittai, *Symmetry: A Unifying Concept*, Bolinas, California: Shelter Publications Inc., 1994.

Hartt, Frederick, *Giulio Romano*, 2 vols, New Haven and London: Yale University Press, 1958.

Helden, A. van, *Measuring the Universe: Cosmic Dimensions from Aristarchus to Halley*, Chicago and London: The University of Chicago Press, 1985.

Hempel, Eberhard, 'Nikolaus von Cues in seinen Beziehung zur bildenden Kunst', *Berichte über die verhandlung der Sächischen Akademie der Wissenschaften zu Leipzig, Philologisch-historische Klasse*. Band 100, Heft 3, 1953, Berlin: Akademie Verlag, 1953, p.42. (Presented 20 March 1950.)

Hill, Donald R., trans., *Al-Jazzari. The Book of Knowledge of Ingenious Mechanical Devices*, Dordrecht: Reidel, 1974.

Hills, Paul, *The Light of Early Italian Painting*, London and New Haven: Yale University Press, 1987.

Hills, Paul, *Venetian Colour: Marble, Mosaic, Painting and Glass 1250–1550*, New Haven and London: Yale University Press, 1999.

Hobson, E. W., *Squaring the Circle: a History of the Problem*, Cambridge: Cambridge University Press, 1913.

Hope, Charles, 'Vasari's *Vita* of Piero della Francesca and the Date of the Arezzo Frescoes', in *Città e Corte nell' Italia di Piero della Francesca. Atti del Convegno Internazionale di Studi Urbino, 4–7 ottobre 1992*, ed. Claudia Cieri Via, Venice: Marsilio, 1996, pp.119–34.

Ivins, W. M., *On the Rationalisation of Sight, with an Examination of Three Renaissance Texts on Perspective*, New York: Da Capo Press, 1973, first published in *Papers of the Metropolitan Museum of Art*, no.8, 1938.

Janson, H. W., *The Sculpture of Donatello*, Princeton: Princeton University Press, 1963.

Jayawardene, S. A., 'The *Trattato d'abaco* of Piero della Francesca', in *Cultural Aspects of the Italian Renaissance: Essays in Honour of Paul Oskar Kristeller*, ed. C. H. Clough, Manchester: Manchester University Press, 1976, pp.229–43.

Joannides, Paul, *Masaccio and Masolino: A Complete Catalogue*, London: Phaidon, 1993.

Keller, A. G., *A Theatre of Machines*, London: Chapman and Hall, 1964.

Kemp, M. J., 'Science, Non-Science and Nonsense: the Interpretation of Brunelleschi's Perspective', *Art History* 1, 1978, pp.134–61.

Kemp, M. J., *Leonardo da Vinci: The Marvellous Works of Nature and Man*, London and Cambridge, Massachusetts: Harvard University Press, 1981.

Kemp, M. J., *The Science of Art: Optical Themes in Western Art from Brunelleschi to Seurat*, New Haven and London: Yale University Press, 1990.

Kemp, M. J., and A. Massing, with N. Christie and A. Groen, 'Paolo Uccello's "Hunt in the forest" ', *Burlington Magazine* 133, no.1056, March 1991, pp.164–78.

Kemp, M. J., 'Construction and Cunning: the Perspective of the Edinburgh Saenredam', in *Dutch Church Painters: Saenredam's 'Great Church at Haarlem' in Context*, ed. Hugh Macandrew, Edinburgh: National Gallery of Scotland, 1984.

Kepler, Johannes, *De fundamentis astrologiae certioribus*, Prague, 1602 (English translation in the second part of J. V. Field, 'A Lutheran Astrologer: Johannes Kepler', *Archive for History of Exact Sciences* 31, 1984, pp.189–272).

Kepler, Johannes, *Ad Vitellionem paralipomena quibus astronomiae pars optica traditur*, Frankfurt, 1604.

Kepler, Johannes, *Astronomia nova seu physica coelestis αιτιολογητος tradita commentariis de motibus planetæ Martis*, Heidelberg, 1609, reprinted in KGW, vol.3 (English translation *Johannes Kepler: New Astronomy*, trans. William H. Donahue, Cambridge: Cambridge University Press, 1992).

Kepler, Johannes, *Epitome astronomiae copernicanae*, Linz, 1618–21, reprinted in KGW, vol.7.

Kepler, Johannes, *Harmonices mundi libri V*, Linz, 1619, reprinted in KGW, vol.6 (English translation *The Harmony of the World*, trans. E. J. Aiton, A. M. Duncan and J. V. Field, Memoirs of the American Philosophical Society, vol.209, Philadelphia, 1997).

Kepler, Johannes, *Johannes Kepler gesammelte Werke*, ed. M. Caspar et al., Munich: Beck, 1938–.

Knorr, Wilbur R., 'On the [*sic*] Principle of Linear Perspective in Euclid's *Optics*', *Centaurus* 34, 1991, pp.193–210.

Koyré, Alexandre, *From the Closed World to the Infinite Universe*, Baltimore and London: The Johns Hopkins University Press, 1957.

Krautheimer, Richard, *Lorenzo Ghiberti*, Princeton: Princeton University Press, 1956.

Kubovy, Michael, *The Psychology of Perspective and Renaissance Art*, Cambridge: Cambridge University Press, 1986.

Lavin, M. A., *Piero della Francesca: The Flagellation of Christ*, New York: Viking Press, 1972, reprint, with additional bibliography, Chicago: University of Chicago Press, 1990.

Lavin, M. A., *Piero della Francesca's 'Baptism of Christ'*, New Haven and London: Yale University Press, 1981.

Lavin, M. A., *Piero della Francesca*, New York: Harry N. Abrams Inc., 1992.

Lavin, M. A., *Piero della Francesca: San Francesco, Arezzo*, New York: George Braziller, 1994.

Lavin, M. A., 'Piero's Meditation on the Nativity', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no.48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art, 1995, pp.127–41.

Leonardo of Pisa [Fibonacci], *Liber abaci*, in *Scritti di Leonardo Pisano*, ed. B. Boncompagni, vol.1, Rome, 1857–62.

Lightbown, R., *Piero della Francesca*, London: Abbeville Press, 1992.

Lindberg, D. C., *John Pecham and the Science of Optics*, Madison: University of Wisconsin Press, 1970.

Lindberg, D. C., *Theories of Vision from al-Kindi to Kepler*, Chicago: University of Chicago Press, 1976, reprint, 1981.

Lindberg, D. C., Introduction to *Reappraisals of the Scientific Revolution*, ed. D. C. Lindberg and R. S. Westman, Cambridge: Cambridge University Press, 1990.

Lindberg, D. C., *Roger Bacon and the Origins of 'Perspectiva' in the Middle Ages: A Critical Edition and English Translation of Bacon's 'Perspectiva' with Introduction and Notes*, Oxford: Oxford University Press, 1996.

Loach, Judi, 'Le Corbusier and the Creative Use of Mathematics', in *Science and the Visual*, ed. J. V. Field and F. A. J. L. James, *British Journal for the History of Science* 31(2), 1998, pp.185–215.

Long, Pamela O., 'Power, Patronage, and the Authorship of *Ars*: From Mechanical Know-how to Mechanical Knowledge in the Last Scribal Age', *Isis* 88/1, 1997, pp.1–41.

Longhi, Roberto, *Piero della Francesca*, Rome, 1927.

Mahnke, Dietrich, *Unendliche Sphäre und Allmittelpunkt*, Halle, 1937, reprint, Stuttgart, 1966.

Mancini, Francesco Federico, '"Depingi ac fabricari fecerunt quamdam tabulam". Un punto fermo per la chronologia del polittico di Perugia', in *Piero della Francesca. Il Polittico di Sant'Antonio*, ed. Vittoria Garibaldi, Perugia: Electa Editori, 1993, pp.65–72.

Manetti, Antonio di Tuccio, *Life of Brunelleschi*, ed. H. Saalman, trans. C. Engass, University Park and London: University of Pennsylvania Press, 1970.

Martone, Thomas, 'Piero della Francesca e la prospettiva dell'intelletto', in *Piero teorico d'Arte*, ed. Omar Calabrese, Rome: Gangemi, 1985, pp.173–86.

Meijer, Bert W., 'Piero and the North', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no.48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art, 1995, pp.143–59.

Meiss, Millard, '*Ovum struthionis:* Symbol and Allusion in Piero della Francesca's *Montefeltro Altarpiece*', in *Studies in Art and Literature for Belle da Costa Green*, ed. D. E. Miner, Princeton: Princeton University Press, 1954, pp.92–101.

Meiss, Millard, 'Addendum ovologicum', *Art Bulletin* 36, 1954, pp.221–4.

Munman, R., 'Optical Corrections in the Sculpture of Donatello', *Transactions of the American Philosophical Society* 75(2), 1985.

Newton, Isaac, *Philosophiae naturalis principia mathematica*, London, 1687.

Nutton, Vivian, 'Greek Science in the Sixteenth-Century Renaissance', in *Renaissance and Revolution: Humanists, Craftsmen and Natural Philosophers in Early Modern Europe*, ed. J. V. Field and F. A. J. L. James, Cambridge: Cambridge University Press, 1993, pp.15–28, reprint, 1997.

Pacioli, Luca, *Summa de arithmetica, geometria, proportioni e proportionalità*, Venice, 1494.

Panofsky, Erwin, *Perspective as Symbolic Form [Die Perspektive als 'symbolische Form'*, 1924], trans. Christopher S. Wood, New York: Zone Books, 1991.

Paolucci, Antonio, *Piero della Francesca. Catalogo completo dei dipinti*, Florence: Cantini, 1990.

Piero della Francesca, *Trattato d'abaco: Dal Codice Ashburnhamiano 280 (359*.291*) della Biblioteca Medicea Laurenziana di Firenze*, ed. G. Arrighi, Pisa: Domus Galilæana, 1970.

Piero della Francesca, 'Trattato d'abaco', Biblioteca Medicea Laurenziana, Florence, Codex Ashburnham 280 (359*.291*).

Piero della Francesca, *De prospectiva pingendi*, Parma, Biblioteca Palatina, MS no. 1576 (in Tuscan).

Piero della Francesca, *De prospectiva pingendi*, London, British Library, Add. MS 10366 (in Latin).

Piero della Francesca, *De prospectiva pingendi*, ed. G. Nicco Fasola, Florence: Sansoni, 1942, reprint, Florence: Casa Editrice Le Lettere, 1984.

Piero della Francesca, 'L'Opera "De corporibus regularibus" di Pietro dei Franceschi detto della Francesca, usurpata da Fra' Luca Pacioli', ed. G. Mancini, Memorie della R. Accademia dei Lincei, series 5, 14.8B, 1916, pp.441–580.

Piero della Francesca, *Libellus de quinque corporibus regularibus*, Vatican Library, Codex Urbinas 632.

Pirenne, Henri, *Optics, Painting and Photography*, Cambridge: Cambridge University Press, 1970.

Pope-Hennessy, J., 'Whose *Flagellation*?', *Apollo* 124, 1986, pp.162–5.

Pope-Hennessy, J., *The Piero della Francesca Trail*, London: Thames and Hudson, 1991.

Pozzo, Andrea, *Perspectiva pictorum et architectorum*, two parts, Rome, 1693, 1700 (in Latin and in Italian).

Ptolemy, Claudius, *Ptolemy's Almagest*, trans. G. J. Toomer, London: Duckworth, 1984.

Rasmo, Nicolò, *Michael Pacher*, trans. Philip Waley, London: Phaidon, 1971. (Italian original: *Michele Pacher*, Venice: Electa, 1969.)

Regiomontanus, *Epytoma Ioannis De monte regio In almagestum ptolomei*, Venice, 1496.

Regiomontanus, *De triangulis*, Nuremberg, 1533.

Regiomontanus, *De quadratura circuli*, Nuremberg, 1533.

Regiomontanus, *Regiomontanus on Triangles: De triangulis omnimodis by Johann Müller, Otherwise Known as Regiomontanus*, trans. Barnabas Hughes O. F. M., Madison and London: University of Wisconsin Press, 1967.

Reinhold, Erasmus, *Prutenicae tabulae coelestium motuum*, Tübingen, 1551.

Robertson, Giles, *Giovanni Bellini*, Oxford: Oxford University Press, 1968.

Roccasecca, Pietro, 'Il "modo optimo" di Leon Battista Alberti', *Studi di Storia dell'Arte* 4, 1993 [1995], pp.245–62.

Ronen, Avraham, 'L'*Annunciazione* nel ciclo della Croce di Piero della Francesca e la tradizione aretina', in *Città e Corte nell' Italia di Piero della Francesca. Atti del Convegno Internazionale di Studi Urbino, 4–7 ottobre 1992*, ed. Claudia Cieri Via, Venice: Marsilio, 1996, pp.205–18.

Rose, P. L., *The Italian Renaissance of Mathematics: Studies of Humanists and Mathematicians from Petrarch to Galileo*, Geneva: Droz, 1975.

Rosenauer, Artur, *Donatello*, Milan: Electa, 1993.

Rosenauer, Artur, ed., *Michael Pacher und sein Kreis* (Proceedings of Symposium, Brunico, Italy, 24–6 September 1998), Bolzano: Athesia, 1999.

Rossi, Paolo, *I filosofi e le macchine (1400–1700)*, Milan: Feltrinelli, 1962.

Rubin, P. L., *Giorgio Vasari: Art and History*, New Haven and London: Yale University Press, 1995.

Saalman, H., *Filippo Brunelleschi and the Cupola of Santa Maria del Fiore*, London: Zwemmer, 1980.

Sanchez Cantón, F. J., 'La librería de Velázquez', in *Homenaje ofrecido a Menéndez Pidal* vol.3, Madrid, 1925, pp.379–406.

Sandström, S., *Levels of Unreality: Studies in Structure and Construction in Italian Mural Paintings during the Renaissance*, Uppsala: Almqvist and Wiksell, 1963.

Schmarsow, A. von, *Melozzo da Forlì*, Berlin, 1886.

Serlio, Sebastiano, *Tutte l'Opere d'Architettura di S. Serlio . . . dove . . . hora di nuovo aggiunto (oltre il libro delle porte) gran numero di case private . . .*, ed. G. D. Scamozzi, Venice, 1584.

Serlio, Sebastiano, *Sebastiano Serlio On Architecture*, vol.1, *Books 1–5 of 'Tutte le Opere d'Architettura e Prospetiva' by Sebastiano Serlio*, trans. Vaughan Hart and Peter Hicks, New Haven and London: Yale University Press, 1996.

Settle, T. B., 'Brunelleschi's Horizontal Arches and Related Devices', *Annali dell'Istituto e Museo di Storia della Scienza* 3.1, 1978, pp.65–80.

Settle, T. B., 'The Tartaglia-Ricci Problem: Towards a Study of the Technical Professions in the Sixteenth Century, in *Cultura, scienze e techniche nella Venezia del cinquecento: Atti del convegno internazionale 'Giovan Battista Benedetti e il suo tempo'*, ed. A. Manno, Venice: Istituto Veneto di Scienze, Lettere ed Arti, 1987, pp.217–26.

Settle, T. B., 'Egnazio Danti and Mathematical Education in Late Sixteenth-Century Florence', in *New Perspectives on Renaissance Thought: Essays in the History of Science, Education and Philosophy in Memory of Charles B. Schmitt*, ed. J. Henry and S. Hutton, London: Duckworth, 1990, pp.24–37.

Settle, T. B., 'The Pendulum and Galileo, Conjectures and Constructions', in *Galileo's Experimental Research*, Preprint 52, Max-Planck-Institut für Wissenschaftsgeschichte, Berlin, 1996, pp.39–49.

Shapin, Steven, and Simon Schaffer, *Leviathan and the Air-pump: Hobbes, Boyle, and the Experimental Life*, including a translation of Thomas Hobbes, 'Dialogus physicus de natura aeris' by Simon Schaffer, Princeton and Oxford: Princeton University Press, 1985, reprint, 1989.

Shearman, John, 'The Logic and Realism of Piero della Francesca', in *Festschrift für Ulrich Middeldorf*, ed. Antije Kosegarten and Peter Tigler, Berlin: De Gruyter, 1968, pp.180–6.

Simms, D. L., 'Archimedes the Engineer', *History of Technology* 17, 1996, pp.45–111.

Simon, Gérard, *Le regard, l'être et l'apparence dans l'optique de l'antiquité*, Paris: Éditions du Seuil, 1988.

Simon, Gérard, 'Optique et perspective: Ptolomée, Alhazen, Alberti', *Revue d'Histoire des Sciences* 53–4, 2001, pp.325–50.

Simon, Gérard, *Archéologie de la vision*, Paris: Éditions du Seuil, 2003.

Smith, Christine, 'Piero's Painted Architecture: Analysis of His Vocabulary', in *Piero della Francesca and His Legacy*, ed. M. A. Lavin (Studies in the History of Art, no.48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), Washington, D.C.: National Gallery of Art, 1995, pp.223–53.

Steadman, J. P., *Vermeer's Camera*, Oxford: Oxford University Press, 2000.

Stewart, Ian, 'One Hundred Per Cent Proof', *Nature* 324, 4 December 1986, pp.406–7.

Tartaglia, Niccolò, *Quesiti e Inventioni diverse . . . di novo restampati con una gionta al sesto libro, nella quale si mostra duoi modi di redur una città, etc.*, Venetia, 1554.

Tartaglia, Niccolò, *General trattato di numeri et misure*, 6 parts, Venice, 1556–60.

Teuffel, Christa Gardner von, 'La collocazione originale e la struttura del polittico', in *Piero della Francesca. Il Polittico di Sant'Antonio*, ed. Vittoria Garibaldi, Perugia: Electa Editori, 1993, pp.89–92.

Thoren, V. E., *The Lord of Uraniborg: A Biography of Tycho Brahe*, Cambridge: Cambridge University Press, 1990.

Trevisani, Filippo, 'Struttura e pittura: i maestri legnaiuoli grossi e Piero della Francesca per la carpenteria della pala di San Bernardino', in *La pala di San Bernardino di Piero della Francesca. Nuovi Studi oltre il restauro*, ed. Emanuela Daffra and Filippo Trevisani (Quaderni di Brera 9), Florence: Centro Di, 1997, pp.31–83.

Ugolini, Guido, 'L'architettura della Pala di San Bernardino', *Notizie da Palazzo Albani* 11, 1–2, 1982, pp.36–41.

Ugolini, Guido, *La pala dei Montefeltro: una porta per il mausoleo dinastico di Federico*, Pesaro: Nobili, 1985.

Valerio, Vladimiro, 'Sul disegno e sulla forma degli anfiteatri', *Disegnare Idee Immagini* (Rivista semestrale del Dipartimento di Rappresentazione e Rilievo, Università degli Studi di Roma 'La Sapienza'), Anno 4, no.6, 1993, pp.25–34.

Vasari, G., *Le Vite dei piu eccellenti architetti, pittori e scultori italiani da Cimabue a tempi nostri*, ed. P. Barocchi and R. Bettarini, Florence: Sansoni, 1967–97.

Vasari, G., *The Lives of the Artists*, trans. George Bull, 2 vols, London: Penguin Books, 1985.

Veltman, Kim, 'Military Surveying and Topography: the Practical Dimension of Renaissance Linear Perspective', *Revista da Universidade de Coimbra* 27, 1979, pp.329–68.

Walker, D. P., *Studies in Musical Theory in the Late Renaissance*, London: The Warburg Institute, 1978.

White, J., *The Birth and Rebirth of Pictorial Space*, 3rd edn, London: Faber and Faber, 1987.

Wilde, J., 'Die "Pala di San Cassiano" von Antonello da Messina', *Jahrbuch der kunsthistorischen Sammlungen in Wien* N. F., 3, 1929, pp.57–72.

Winkler, Mary G., and Albert van Helden, 'Representing the Heavens: Galileo and Visual Astronomy', *Isis* 83, 1992, pp.195–217.

Winterberg, C., ed., *Petrus pictor burgensis de prospectiva pingendi*, Strasbourg, 1899.

Wittkower, R., 'Brunelleschi and "Proportion in Perspective"', *Journal of the Courtauld and Warburg Institutes* 16, 1951, pp.275–91.

Wittkower, R., and B. A. R. Carter, 'The Perspective of Piero della Francesca's *Flagellation*', *Journal of the Warburg and Courtauld Institutes* 16, 1953, pp.292–302.

Wohl, H., *The Paintings of Domenico Veneziano (c.1410–1461). A Study in Florentine Art of the Early Renaissance*, Oxford: Phaidon, 1980.

Zarlino, Gioseffo, *Istitutioni harmoniche*, Venice, 1558.

Zeuthen, H. G., *Geschichte der Mathematik in XVI. und XVII. Jahrhundert*, ed. B. Meyer, Leipzig, 1903.

Zilsel, Edgar, 'The Sociological Roots of Science', *American Journal of Sociology* 47, 1942, pp.544–62, reprinted in Edgar Zilsel, *The Social Origins of Modern Science*, Boston Studies in the Philosophy of Science, vol.20, ed. Diederick Raven, Wolfgang Krohn and Robert S. Cohen, Dordrecht: Kluwer Academic Publisher, 2000, pp.4–21.

Zinner, Ernst, *Regiomontanus: His Life and Work*, trans. Ezra Brown (Studies in the History and Philosophy of Mathematics, vol.1), Amsterdam and New York: North-Holland, 1990 (a translation of Zinner's second edition of 1968, with additional essays by various scholars to bring the work up to date; first edition was 1938).

Zwijnenberg, Robert, *The Writings and Drawings of Leonardo da Vinci: Order and Chaos in Early Modern Thought*, trans. Caroline van Eck, Cambridge: Cambridge University Press, 1999.

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Writings

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Jacket illustrations

Front: Piero della Francesca, *Sigismondo Malatesta before St Sigismund* (detail, see Fig. 6.1), 1451, Cappella delle Reliquie, Tempio Malatestiano (San Francesco), Rimini. Photograph © Alinari, Florence.

Back: Piero della Francesca, *The Nativity of Christ* (detail of Borgo San Sepolcro, see Fig. 6.38), National Gallery, London.

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