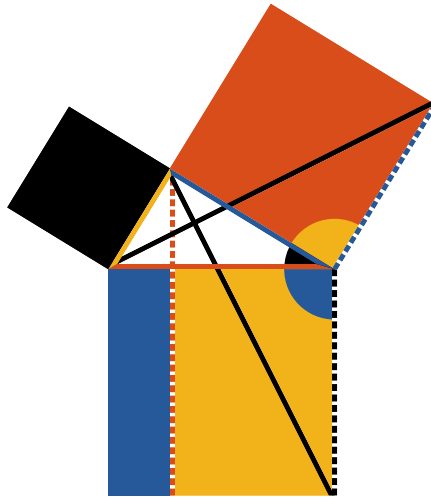


THE FIRST SIX BOOKS OF THE ELEMENTS OF EUCLID

IN WHICH COLOURED DIAGRAMS AND SYMBOLS ARE USED
INSTEAD OF LETTERS FOR THE GREATER EASE OF LEARNERS

BY OLIVER BYRNE

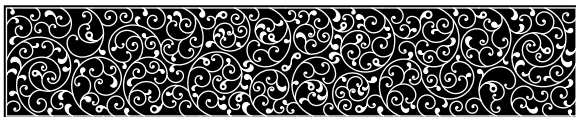


github.com/jemmybutton

2020 ed. 0.8a



This rendition of Oliver Byrne's "The first six books of the Elements of Euclid" is
made by Slyusarev Sergey and is distributed under CC-BY-SA 4.0 license



Introduction



THE arts and sciences have become so extensive, that to facilitate their acquirement is of as much importance as to extend their boundaries. Illustration, if it does not shorten the time of study, will at least make it more agreeable. This Work has a greater aim than mere illustration; we do not introduce colors for the purpose of entertainment, or to amuse *by certain combinations of tint and form*, but to assist the mind in its researches after truth, to increase the facilities of introduction, and to diffuse permanent knowledge. If we wanted authorities to prove the importance and usefulness of geometry, we might quote every philosopher since the day of Plato. Among the Greeks, in ancient, as in the school of Pestalozzi and others in recent times, geometry was adopted as the best gymnastic of the mind. In fact, Euclid's Elements have become, by common consent, the basis of mathematical science all over the civilized globe. But this will not appear extraordinary, if we consider that this sublime science is not only better calculated than any other to call forth the spirit of inquiry, to elevate the mind, and to strengthen the reasoning faculties, but also it forms the best introduction to most of the useful and important vocations of human life. Arithmetic, land-surveying, hydrostatics, pneumatics, optics, physical astronomy, &c. are all dependent on the propositions of geometry.

Much however depends on the first communication of any science to a learner, though the best and most easy methods are seldom adopted. Propositions are placed before a student, who though having a sufficient understanding, is told just as much about them on entering at the very threshold of the science, as given him a prepossession most unfavorable to his future study of this delightful subject; or “the formalities and paraphernalia of rigour are so ostentatiously put forward, as almost to hide the reality. Endless and perplexing repetitions, which do not confer greater exactitude on the reasoning, render the demonstrations involved and obscure, and conceal from the view of the student the consecution of evidence.” Thus an aversion is created in the mind of the pupil, and a subject so calculated to improve the reasoning powers, and give the habit of close thinking, is degraded by a dry and rigid course of instruction into an uninteresting exercise of the memory. To rise the curiosity, and to awaken the listless and dormant powers of younger minds should be the aim of every teacher; but where examples of excellence are wanting, the attempts to attain it are but few, while eminence excites attention and produces imitation. The object of this Work is to introduce a method of teaching geometry, which has been much approved of by many scientific men in this country, as well as in France and America. The plan here adopted forcibly appeals to the eye, the most sensitive and the most comprehensive of our external organs, and its pre-eminence to imprint its subject on the mind is supported by the incontrovertible maxim expressed in the well known words of Horace:—

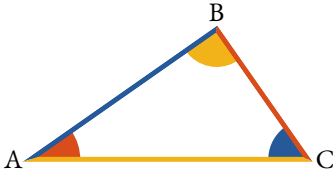
*Segnius irritant animos demissa per aurem
Quam quae sunt oculis subjecta fidelibus*

A feebler impress through the ear is made,
Than what is by the faithful eye conveyed.

All language consists of representative signs, and those signs are the best which effect their purposes with the greatest precision and dispatch. Such for all common purposes are the audible signs called words, which are still considered as audible, whether addressed immediately to the ear, or through the medium of letters to the eye. Geometrical diagrams are not signs, but the materials of geometrical science, the object of which is to show the relative quantities of their parts by a process of reasoning called Demonstration. This reasoning has been generally carried on by words, letters, and black or uncoloured diagrams; but as the use of coloured symbols, signs, and diagrams in the linear arts and sciences, renders the process of reasoning more precise, and the attainment more expeditious, they have been in this instance accordingly adopted.

Such is the expedition of this enticing mode of communicating knowledge, that the Elements of Euclid can be acquired in less than one third of the time usually employed, and the retention by the memory is much more permanent; these facts have been ascertained by numerous experiments made by the inventor, and several others who have adopted his plans. The particulars of which are few and obvious; the letters annexed to points, lines, or other parts of a diagram are in fact but arbitrary names, and represent them in the demonstration; instead of these, the parts being differently coloured, are made to name themselves, for their forms in corresponding colours represent them in the demonstration.

In order to give a better idea of this system, and of advantages gained by its adoption, let us take a right angled triangle, and express some of its properties both by colours and the method generally employed.



Some of the properties of the right angled triangle ABC, expressed by the method generally employed:

1. The angle BAC, together with the angles BCA and ABC are equal to two right angles, or twice the angle ABC.
2. The angle CAB added to the angle ACB will be equal to the angle ABC.
3. The angle ABC is greater than either of the angles BAC or BCA.
4. The angle BCA or the angle CAB is less than the angle ABC.
5. If from the angle ABC, there be taken the angle BAC, the remainder will be equal to the angle ACB.
6. The square of AC is equal to the sum of the squares of AB and BC.

The same properties expressed by colouring the different parts:

1.  +  +  = 2  = .

That is, the red angle added to the yellow angle added to the blue angle, equal twice the yellow angle, equal two right angles.

2.  +  = .

Or in words, the red angle added to the blue angle, equal the yellow angle.

3.  >  or > .

The yellow angle is greater than either the red or blue angle.

4.  or  < .

Either the red or blue angle is less than the yellow angle.

5.  -  = .

In other terms, the yellow angle made less be the blue angle equal red angle.

6.  ² =  ² +  ².

That is, the square of the yellow line is equal to the sum of the squares of the blue and red lines.

In oral demonstrations we gain with colours this important advantage, the eye and the ear can be addressed at the same moment, so that for teaching geometry, and other linear arts and sciences, in classes, the system is best ever proposed, this is apparent from the examples given.

Whence it is evident that a reference from the text to the diagram is more rapid and sure, by giving the forms and colours of the parts, or by naming the parts and their colours, than naming the parts and letters on the diagram. Besides the superior simplicity, this system is likewise conspicuous for concentration, and wholly excludes the injurious though prevalent practice of allowing the student to commit the demonstration to memory; until reason, and fact, and proof only make impressions of the understanding.

Again, when lecturing on the principles or properties of figures, if we mention the colour of the part or parts referred to, as in saying, the red angle, the blue line, or lines, &c, the part or parts thus named will be immediately seen by all the class at the same instant; not so if we say the angle ABC, the triangle PFQ, the figure EGKt, and so on; for the letters must be traced one by one before students arrange in their minds the particular magnitude referred to, which often occasions confusion and error, as well as loss of time. Also if the parts which are given as equal, have the same colours in any diagram, the mind will not wander from the object before it; that is, such an arrangement presents an ocular demonstration of the parts to be proved equal, and the learner retains the data throughout the whole of reasoning. But whatever may be the advantages of the present plan, if it be not substituted for, it can always be made a

powerful auxiliary to the other methods, for the purpose of introduction, or of a more speedy reminiscence, or of more permanent retention by the memory.

The experience of all who have formed systems to impress facts on the understanding, agree in proving that coloured representations, as pictures, cuts, diagrams, &c. are more easily fixed in the mind than mere sentences unmarked by any peculiarity. Curious as it may appear, poets seem to be aware of this fact more than mathematicians; many modern poets allude to this visible system of communicating knowledge, one of them has thus expressed himself:

Sounds which address the ear are lost and die
In one short hour, but these which strike the eye,
Live long upon the mind, the faithful sight
Engraves the knowledge with a beam of light.

This perhaps may be reckoned the only improvement which plain geometry has received since the days of Euclid, and if there were any geometers of note before that time, Euclid's success has quite eclipsed their memory, and even occasioned all good things of that kind to be assigned to him; like Æsop among the writers of Fables. It may also be worthy of remark, as tangible diagrams afford the only medium through which geometry and other linear arts can be taught to the blind, the visible system is no less adapted to the exigencies of the deaf and dumb.

Care must be taken to show that colour has nothing to do with the lines, angles, or magnitudes, except merely to name them. A mathematical line, which is length without breadth, cannot possess colour, yet the junction of two colours on the same plane gives a good idea of what is meant by a mathematical line; recollect we are speaking familiarly, such a junction is to be understood and not the colour, when we say the black line, the red line or lines, &c.

Colours and coloured diagrams may at first appear a clumsy method to convey proper notions of the properties and parts of mathematical figures and magnitudes, however they will be found to afford a means more refined and extensive than any that has hitherto proposed.

We shall here define a point a line, and a surface, and demonstrate a proposition in order to show the truth of this assertion.

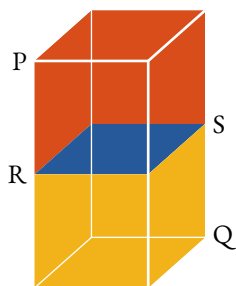
A point is that which has position, but not magnitude; or a point is position only, abstracted from the consideration of length, breadth, and thickness. Perhaps the following description is better calculated to explain the nature of mathematical point to those who have not acquired the idea, than the above specious definition.

Let three colours meet and cover a portion of the paper, where they meet is not blue, nor is it yellow, nor is it red, as it occupies no portion of the plane, for if it did, it would belong to the blue, the red, or the yellow part; yet it exists, and has position without magnitude, so that with a little reflection, this junction of three colours on a plane, gives a good idea of a mathematical point.

A line is length without breadth. With the assistance of colours, nearly in the same manner as before, an idea of a line may be thus given:—

Let two colours meet and cover a portion of paper; where they meet is not red, nor is it blue; therefore the junction occupies no portion of the plane, and therefore it cannot have breadth, but only length: from which we can readily form an idea of what is meant by a mathematical line. For the purpose of illustration, one colour differing from the colour of the paper, or plane upon which it is drawn, would have been sufficient; hence in future, if we say the red line, the blue line or lines, &c. it is the junctions with the plane upon which they are drawn are to be understood.



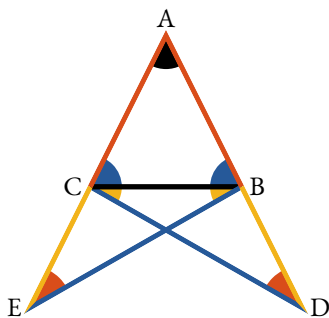








Surface is that which has length and breadth without thickness.

When we consider a solid body (PQ), we perceive at once that it has three dimensions, namely:— length, breadth, and thickness; suppose one part of this solid (PS) to be red, and the other part (QR) yellow, and that the colours be distinct without commingling, the blue surface (RS) which separates these parts, or which is the same thing, that which divides the solid without loss of material, must be without thickness, and only possesses length and breadth; this plainly appears from reasoning, similar to that just employed in defining, or rather describing a point and a line.

The proposition which we have selected to elucidate the manner in which the principles are applied, is the fifth of the first Book.

In an isosceles triangle ABC, the internal angles at the base ABC are equal, and when the sides AB, AC are produced, the external angles at the base BCE, CBD are also equal.



Produce  and ,
make  = , draw  and .



we have  = ,

 common and  = :

\therefore  = ,  = 

and  =  (pr. 1.4).

Again in  and ,

 = ,  = 

and  = .

$$\therefore \triangle ABC = \triangle ACB \text{ and } \angle ABC = \angle ACB \text{ (pr. 1.4)}$$

But $\triangle ABC = \triangle ACB$, $\therefore \angle ABC = \angle ACB$.

Q. E. D.

By annexing Letters to the Diagram.

Let the equal sides AB and AC be produced through the extremities BC, of the third side, and in the produced part BD of either, let any point D be assumed, and from the other let AE be cut off equal to AD (pr. 1.3). Let points E and D, so taken in the produced sides, be connected by straight lines DC and BE with the alternate extremities of the third side of the triangle.

In the triangles DAC and EAB the sides DA and AC are respectively equal to EA and AB, and the included angle A is common to both triangles. Hence (pr. 1.4) the line DC is equal to BE, the angle ADC to the angle AEB, and the angle ACD to the angle ABE; if from the equal lines AD and AE the equal sides AB and AC be taken, the remainders BD and CE will be equal. Hence in the triangles BDC and CEB, the sides BD and DC are respectively equal to CE and EB, and the angles D and E included by those sides are also equal. Hence (pr. 1.4) the angles DBC and ECB, which are those included by the third side BC and the productions of the equal sides AB and AC are equal. Also the angles DCB and ECB are equal if those equals be taken from the angles DCA and EBA before proved equal, the remainders, which are the angles ABC and ACB opposite to the equal sides, will be equal.

Therefore in an isosceles triangle, &c.

Q. E. D.

Our object in this place being to introduce system rather than to teach any particular set of propositions, we have therefore selected the foregoing out of the regular

course. For schools and other public places of instruction, dyed chalks will answer to describe the diagrams, &c. for private use coloured pencils will be found very convenient.

We are happy to find that the Elements of Mathematics now forms a considerable part of every sound female education, therefore we call the attention to those interested or engaged in the education of ladies to this very attractive mode of communicating knowledge, and to the succeeding work for its future developement.

We shall for the present conclude by observing, as the senses of sight and hearing can be so forcibly and instantaneously addressed alike with one thousand as with one, *the million* might be taught geometry and other branches of mathematics with great ease, this would advance the purpose of education more than any thing *might* be named, for it would teach the people how to think, and not what to think; it is in this particular the great error of education originates.

THE ELEMENTS OF EUCLID

BOOK I

Definitions

I

A *point* is that which has no parts.

2

A *line* is length without breadth.

3

The extremities of a line are points.

4

The straight or right line is that which lies evenly between its extremities.

5

A surface is that which has length and breadth only.

6

The extremities of a surface are lines.

7

A plane surface is that which lies evenly between its extremities.

8

A plane angle is the inclination of two lines to one another, in a plane, which meet together, but are not in the same direction.

9



A plane rectilinear angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

10



When one straight line standing on another straight line makes the adjacent angles equal, each of these angles is called a *right angle*, and each of these lines is said to be *perpendicular* to one another.

11



An obtuse angle is an angle greater than a right angle

12



An acute angle is an angle less than a right angle.

13

A term or boundary is the extremity of any thing.

14

A figure is a surface enclosed on all sides by a line or lines.

15



A circle is a plane figure, bounded by one continued line, called its circumference or periphery; and having a certain point within it, from which all straight lines drawn to its circumference are equal.

16

This point (from which the equal lines are drawn) is called the centre of the circle.

17



A diameter of a circle is a straight line drawn through the centre, terminated both ways in the circumference.

18

A semicircle is the figure contained by the diameter, and the part of the circle cut off by the diameter.



19

A segment of a circle is a figure contained by straight line and the part of the circumference which it cuts off.





20

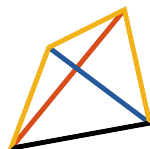
A figure contained by straight lines only, is called a rectilinear figure.

21

A triangle is a rectilinear figure included by three sides.

22

A quadrilateral figure is one which is bounded by four sides. The straight lines  and  connecting the vertices of the opposite angles of a quadrilateral figure, are called its diagonals.



23

A polygon is a rectilinear figure bounded by more than four sides.

24

A triangle whose sides are equal, is said to be equilateral.



25

A triangle which has only two sides equal is called an isosceles triangles.



26

A scalene triangle is one which has no two sides equal.

27

A right angled triangle is that which has a right angle.



28



An obtuse angled triangle is that which has an obtuse angle.

29



An acute angled triangle is that which has three acute angles.

30



Of four-sided figures, a square is that which has all its sides equal, and all its angles right angles.

31



A rhombus is that which has all its sides equal, but its angles are not right angles.

32



An oblong is that which has all its angles right angles, but has not all its sides equal.

33



A rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal nor its angles right angles.

34

All other quadrilateral figures are called trapeziums.

35



Parallel straight lines are such as are in the same plane, and which being produced continually in both directions would never meet.

Postulates

I

Let it be granted that a straight line may be drawn from any one point to any other point.

II

Let it be granted that a finite straight line may be produced to any length in a straight line.

III

Let it be granted that a circle may be described with any centre at any distance from that centre.

Axioms

I

Magnitudes which are equal to the same are equal to each other.

II

If equals be added to equals the sums will be equal.

III

If equals be taken away from equals the remainders will be equal.

IV

If equals be added to unequals the sums will be unequal.

V

If equals be taken away from unequals the remainders will be unequal.

VI

The doubles of the same or equal magnitudes are equal.

VII

The halves of the same or equal magnitudes are equal.

VIII

The magnitudes which coincide with one another, or exactly fill the same space, are equal.

IX

The whole is greater than its part.

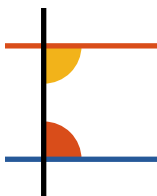
X





Two straight lines cannot include a space.

XI

All right angles are equal.

XII



If two straight lines () meet a third straight line () so as to make the two interior angles ( and ) on the same side less than two straight angles, these two straight lines will meet if they be produced on that side on which the angles are less than two right angles.

The twelfth axiom may be expressed in any of the following ways:

1. Two diverging straight lines cannot be both parallel to the same straight line.
2. If a straight line intersect one of the two parallel straight lines it must also intersect the other.
3. Only one straight line can be drawn through a given point, parallel to a given straight line.

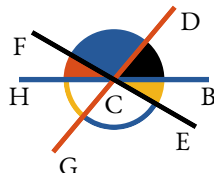
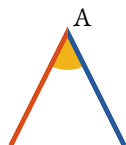
Elucidations

Geometry has for its principal objects the exposition and explanation of the properties of *figure*, and figure is defined to be the relation which subsists between the boundaries of space. Space or magnitude is of three kinds, *linear*, *superficial*, and *solid*.

Angles might properly be considered as a fourth species of magnitude. Angular magnitude evidently consists of parts, and must therefore be admitted to be a species of quantity. The student must not suppose that the magnitude of an angle is affected by the length of the straight lines which include it and of whose mutual divergence it is the measure. The *vertex* of an angle is the point the *sides* of the *legs* of the angle meet, as A.

An angle is often designated by a single letter when its legs are the only lines which meet together at its vertex. Thus the red and blue lines form the yellow angle, which in other systems would be called angle A. But when more than two lines meet in the same point, it was necessary by former methods, in order to avoid confusion, to employ three letters to designate an angle about that point, the letter which marked the vertex of the angle being always placed in the middle. Thus the black and red lines meeting together at C, form the blue angle, and has been usually denominated the angle FCD or DCF. The lines FC and CD are the legs of the angle; the point C is its vertex. In like manner the black angle would be designated the angle DCB or BCD. The red and blue angles added together, or the angle HCF added to FCD, make the angle HCD; and so of other angles.

When the legs of an angle are produced or prolonged beyond its vertex, the angles made by them on both sides of the vertex are said to be *vertically opposite* to each other:



thus the red and yellow angles are said to be vertically opposite angles.

Superposition is the process by which one magnitude may be conceived to be placed upon another, so as exactly to cover it, or so that every part of each shall exactly coincide.

A line is said to be *produced*, when it is extended, prolonged, or it has length increased, and the increase of length which it receives is called *produced part*, or its *production*.

The entire length of the line or lines which enclose a figure, is called its *perimeter*. The first six books of Euclid treat of plain figures only. A line drawn from the centre of a circle to its circumference, is called a *radius*. That side of a right angled triangle, which is opposite to the right angle, is called the *hypotenuse*. An oblong is defined in the second book, and called a *rectangle*. All lines which are considered in the first six books of the Elements are supposed to be in the same plane.

The *straight-edge* and *compasses* are the only instruments, the use of which is permitted in Euclid, or plain Geometry. To declare this restriction is the object of the *postulates*.

The *Axioms* of geometry are certain general propositions, the truth of which is taken to be self-evident and incapable of being established by demonstration.

Propositions are those results which are obtained in geometry by a process of reasoning. There are two species of propositions in geometry, *problems* and *theorems*.

A *Problem* is a proposition in which something is proposed to be done; as a line to be drawn under some given conditions, a circle to be described, some figure to be constructed, &c.

The *solution* of the problem consists in showing how the thing required may be done by the aid of the rule or straight-edge and compasses.

The *demonstration* consists in proving that the process indicated in the solution attains the required end.

A *Theorem* is a proposition in which the truth of some principle is asserted. This principle must be deduced from the axioms and definitions, or other truths previously and independently established. To show this is the object of demonstration.

A *Problem* is analogous to a postulate.

A *Theorem* resembles an axiom.

A *Postulate* is a problem, the solution to which is assumed.

An *Axiom* is a theorem, the truth of which is granted without demonstration.

A *Corollary* is an inference deduced immediately from a proposition.

A *Scholium* is a note or observation on a proposition not containing an inference of sufficient importance to entitle it to the name of *corollary*.

A *Lemma* is a proposition merely introduced for the purpose of establishing some more important proposition.

Symbols and abbreviations

- ∴ expresses the word *therefore*.
- ∵ expresses the word *because*.
- = expresses the word *equal*. This sign of equality may be read *equal to*, or *is equal to*, or *are equal to*; but the discrepancy in regard to the introduction of the auxiliary verbs *is*, *are*, &c. cannot affect the geometrical rigour.
- ≠ means the same as if the words '*not equal*' were written.
- > signifies *greater than*.
- < signifies *less than*.
- ⋈ signifies *not greater than*.
- ⋊ signifies *not less than*.
- + is read *plus* (*more*), the sign of addition; when interposed between two or more magnitudes, signifies their sum.
- − is read *minus* (*less*), signifies subtraction; and when placed between two quantities denotes that the latter is taken from the former.
- × this sign expresses the product of two or more numbers when placed between them in arithmetic and algebra; but in geometry it is generally used to express a *rectangle*, when placed between "two straight lines which contain one of its right angles." A *rectangle* may also be represented by placing a point between two of its conterminous sides.
- : :: expresses an *analogy* or *proportion*; thus if A, B, C and D represent four magnitudes, and A has to B the same ratio that C has to D, the proportion is thus briefly written

$$A : B :: C : D, A : B = C : D, \text{ or } \frac{A}{B} = \frac{C}{D}.$$

This equality or sameness of ratio is read,
as A is to B, so is C to D;
or A is to B, as C is to D.
- ∥ signifies *parallel to*.

\perp signifies *perpendicular to*.

\triangle signifies *angle*.

\sphericalangle signifies *right angle*.

\bigcirc signifies *two right angles*.

\uparrow or \downarrow briefly designates a *point*.

The square described on a line is concisely written thus,
 \square^2 .

In the same manner twice the square of, is expressed
 by $2 \cdot \square^2$.

def. signifies *definition*.

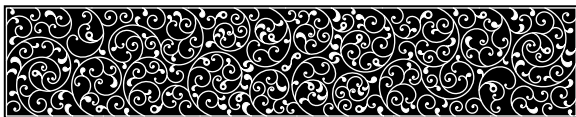
post. signifies *postulate*.

ax. signifies *axiom*.

hyp. signifies *hypothesis*. It may be necessary here to remark, that *hypothesis* is the condition assumed or taken for granted. Thus, the hypothesis of the proposition given in the Introduction, is that the triangle is isosceles, or that its legs are equal.

const. signifies *construction*. The *construction* is the change made in the original figure, by drawing lines, making angles, describing circles, &c. in order to adapt it to the argument of the demonstration or the solution of the problem. The conditions under which these changes are made, are as indisputable as those contained in the hypothesis. For instance, if we make an angle equal to a given angle, these two angles are equal by construction.

Q. E. D. signifies *Quod erat demonstrandum*. Which was to be demonstrated.

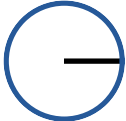
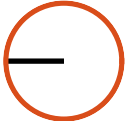


Book I

Prop. I. Prob.



ON a given finite straight line (—) to describe an equilateral triangle.

Describe  and  (post. III);


draw  and  (post. I).

Then will  be equilateral.

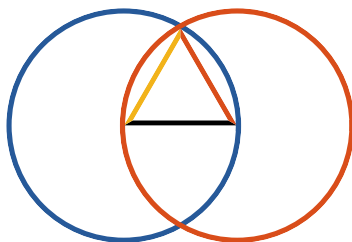
For  =  (def. 15);

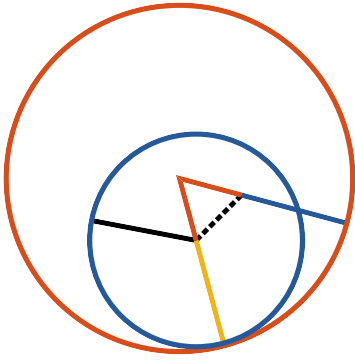
and  =  (def. 15);

\therefore  =  (ax. I);


and therefore  is the equilateral triangle required.



Q. E. D.





FROM a given point (————), to draw a straight line equal to a given straight line (————).

Draw ———— (post. I), describe  (pr. 1.1),
produce ———— (post. II),




describe  (post. III), and  (post. III);

produce ———— (post. II),
then ———— is the line required.

For  =  (def. 15),

and  =  (const.),

\therefore  =  (ax. III),

but (def. 15)  =  = ;

\therefore ———— drawn from the given point
(————), is equal to the given line ———— .

Q. E. D.



FROM the greater (—) of two given straight lines, to cut off a part equal to the less (—).

Draw — = — (pr. I.2);

describe (post. III),

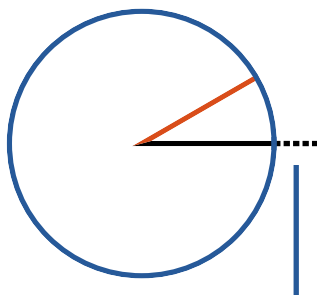
then — = —

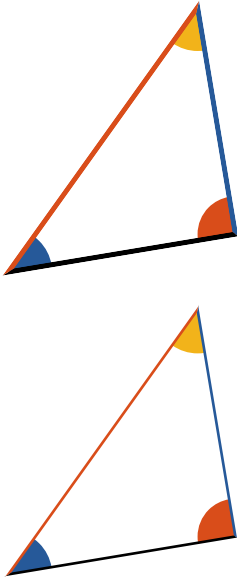
For — = — (def. I5),







and — = — (const.);









\therefore — = — (ax. I).

Q. E. D.












IF two triangles have two sides of the one respectively equal to two sides of the other, (— to — and — to —) and the angles ( and ) contained by those equal sides also equal; then their bases or their sides (— and —) are also equal: and the remaining angles opposite to equal sides are respectively equal ( =  and  = ): and the triangles are equal in every respect.

Let two triangles be conceived, to be so placed, that the vertex of the one of the equal angles,  or ; shall fall upon that of the other, and — to coincide with —, then will — coincide with — if applied: consequently — will coincide with —, or two straight lines will enclose a space, which is impossible (ax. X), therefore — = —,  =  and  = , and as the triangles  and  coincide, when applied, they are equal in every respect.



Q. E. D.






In any isosceles triangle  if the equal sides be produced, the external angles at the base are equal, and the internal angles at the base are also equal.




Produce  and  (post. II),
take  =  (pr. I.3);
draw  and .

Then in  and 

we have  =  (const.),

 common to both,

and  =  (hyp.)

\therefore  = ,  = 

and  =  (pr. I.4).

Again in  and 

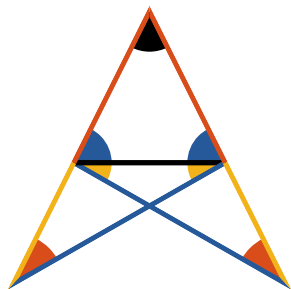
we have  = ,

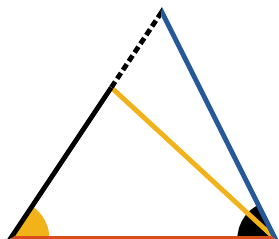
 =  and  = ,






\therefore  =  and  =  (pr. I.4)






but  = , \therefore  =  (ax. III).








Q. E. D.







IN any triangle () if two angles ( and ) are equal, the sides ( and ) opposite to them are also equal.

For if the sides be not equal, let one of them  be greater than the other , and from it to cut off  =  (pr. 1.3), draw .

Then in  and ,
 =  (const.),
 =  (hyp.)
 and  common,

\therefore the triangles are equal (pr. 1.4)
 a part equal to the whole, which is absurd;
 \therefore neither of the sides  or  is greater than the other,
 hence they are equal.

Q. E. D.



QUON the same base (—), and on the same side of it there cannot be two triangles having their conterminous sides (— and —, — and —) at both extremities of the base, equal to each other.

When two triangles stand on the same base, and on the same side of it, the vertex of the one shall either fall outside of the other triangle, or within it; or, lastly, on one of its sides.

If it be possible let the two triangles be constructed so that $\left\{ \begin{array}{l} \text{—} = \text{—} \\ \text{—} = \text{—} \end{array} \right\}$, then draw and,

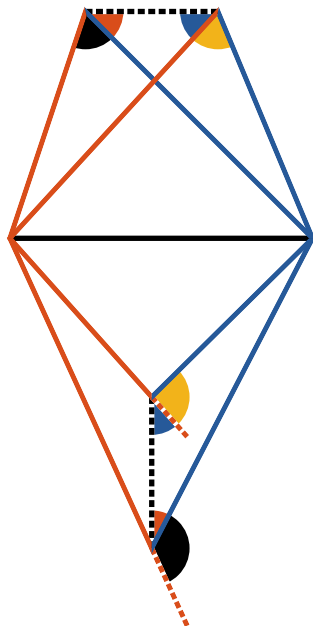
$$\triangle ABC = \triangle DEF \quad (\text{pr. I.5})$$

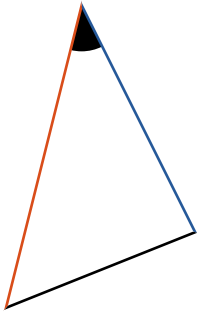
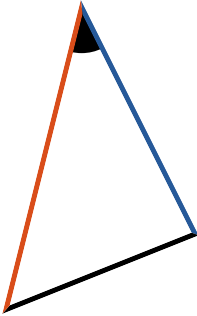
$$\therefore \angle ABC < \angle DEF \quad \text{and}$$









$$\left. \begin{array}{l} \therefore \angle ABC < \angle DEF \\ \text{but (pr. I.5) } \angle ABC = \angle DEF \end{array} \right\} \text{which is absurd,}$$







therefore the two triangles cannot have their conterminous sides equal at both extremities of the base.







Q. E. D.





IF two triangles have two sides of the one respectively equal to two sides of the other ( =  and  = ) and also their bases ( = ), equal; then the angles ( and ) contained by their equal sides are also equal.



If the equal bases  and  be conceived to be placed one upon the other, so that the triangles shall lie at the same side of them, and that the equal sides  and  ,  and  be conterminous, the vertex of the one must fall on the vertex of the other; for to suppose them not coincident would contradict the last proposition.



Therefore sides  and  , being coincident with  and  , \therefore  =  .

Q. E. D.




o bisect a given rectilinear angle ().



Take  =  (pr. 1.3)

draw , upon which describe  (pr. 1.1),

draw .

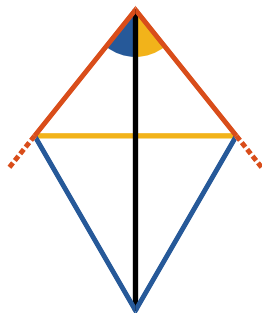
\therefore  =  (const.)

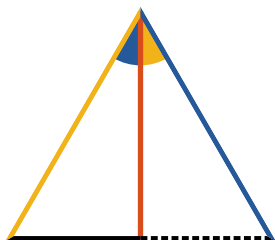
and  common to the two triangles

and  =  (const.),





\therefore  =  (pr. 1.8).

Q. E. D.










P bisect a given finite straight line (—....).

Construct  (pr. 1.1),
draw , making  =  (pr. 1.9).

Then  =  by (pr. 1.4),

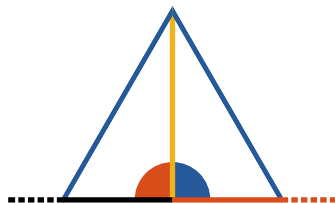
for  =  (const.)  = 
and  common to the two triangles.

Therefore the given line is bisected.


Q. E. D.



FROM a given point (———), in a given straight line (———), to draw a perpendicular.



Take any point (———) in the given line,
cut off ——— = ——— (pr. 1.3),

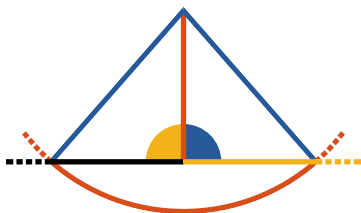
construct  (pr. 1.1),
draw ——— and it shall be
perpendicular to the given line.


For ——— = ——— (const.)
——— = ——— (const.)
and ——— common to the two triangles.



Therefore  =  (pr. 1.8)





\therefore ——— \perp ——— (def. 10).




Q. E. D.





o draw a straight line perpendicular to a given indefinite straight line (———) from a given point () without.

With the given point  as centre, at one side of the line, and any distance ——— capable of extending to the other side, describe .

Make ——— = ——— (pr. I.10),
draw ,  and ,
then  \perp ———.

For (pr. I.8) since ——— = ——— (const.),
 common to both,
and  =  (def. 15),




\therefore  = , and

\therefore  \perp ——— (def. 10).




Q. E. D.







WHEN a straight line (—) standing upon another straight line (—) makes angles with it; they are either two right angles or together equal to two right angles.

If — be \perp to — then,
 and  =  (def. 10),

but if — be not \perp to — ,
 draw — \perp — (pr. 1.11);

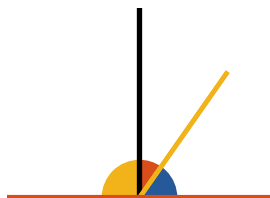
 +  =  (const.),

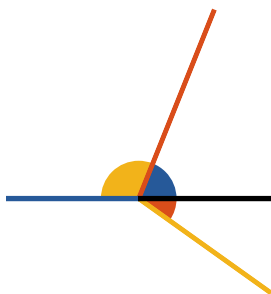
 =  =  + 



\therefore  +  =  +  +  (ax. II)

=  +  = .

Q. E. D.












IF two straight lines (— and —), meeting a third straight line (—), at the same point, and at opposite sides of it, make with it adjacent angles ( and ) equal to two right angles; these straight lines lie in one continuous straight line.

For, if possible let , and not , be the continuation of ,

then  +  = 




but by the hypothesis  +  = 

\therefore  =  (ax. III); which is absurd (ax. IX).

\therefore  is not the continuation of , and the like may be demonstrated of any other straight line except , \therefore  is the continuation of .

Q. E. D.



F two right lines ( and ) intersect one another, the vertical angles  and

,  and  are equal.

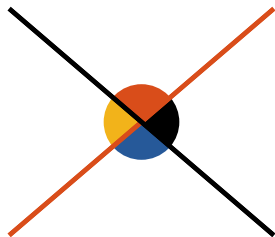


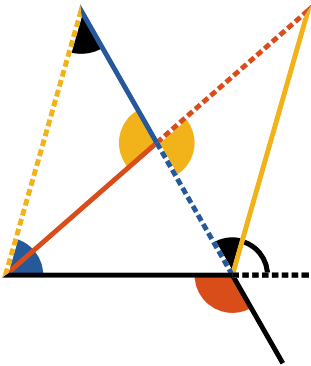
$$\therefore \text{yellow angle} = \text{black angle}.$$





In the same manner it may be shown that







$$\text{orange angle} = \text{blue angle}.$$

Q. E. D.





If a side of a triangle () is produced, the external angle () is greater than either of the internal remote angles ( or ).

Make  =  (pr. I.10);
 Draw  and produce it until  =  ;
 draw .

In  and  ;




 = ,  =  and




 =  (const., pr. I.15),

\therefore  =  (pr. I.4),

\therefore  > .


In like manner it can be shown, that if

 be produced,  > 

and therefore  which is =  is > .

Q. E. D.



ANY two angles of a triangle  are together less than two right angles.

Produce , then will

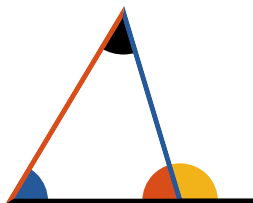


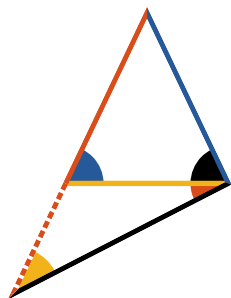
But  >  (pr. 1.16)


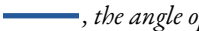


\therefore  +  < ,

and in the same manner it may be shown that any other two angles of the triangle taken together are less than two right angles.



Q. E. D.





IN any triangle if one side  be greater than another , the angle opposite to the greater side is greater than the angle opposite to the less. I. e.  > .

Make  =  (pr. 1.3), draw .

Then will  =  (pr. 1.5);






but  >  (pr. 1.16)

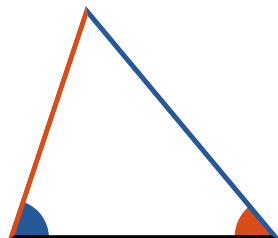
\therefore  >  and much more





is  > .





Q. E. D.









F in any triangle  one angle  be greater than another  the side  which is opposite to the greater angle, is greater than the side  opposite the less.



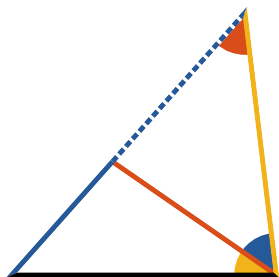
If  be not greater than  then must  = or < .





If  =  then  =  (pr. I.5); which is contrary to the hypothesis.





 is not less than ; for if it were,  <  (pr. I.18) which is contrary to the hypothesis:



\therefore  > .

Q. E. D.



ANY two sides  and  of a triangle  taken together are greater than the third side (.

Produce , and
make  =  (pr. I.3);
draw .

Then \therefore  =  (const.),

 =  (pr. I.5)

\therefore  >  (ax. IX)

\therefore  +  >  (pr. I.19)


and \therefore  +  > .


Q. E. D.



From any point () within a triangle



straight lines be drawn to the extremities of one side (), these lines must be together less than the other two sides, but must contain a greater angle.

Produce ,

$$\text{blue} + \text{red} > \text{black} \cdots \text{ (pr. I.20),}$$

add  to each,

$$\text{blue} + \text{red} \cdots > \text{black} \cdots + \text{red} \cdots \text{ (ax. IV)}$$



in the same manner it may be shown that

$$\text{black} \cdots + \text{red} \cdots > \text{black} + \text{yellow},$$

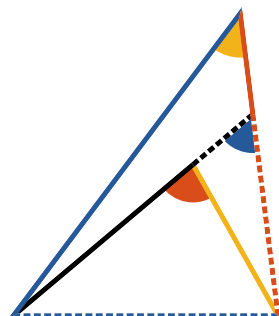
$$\therefore \text{blue} + \text{red} \cdots > \text{black} + \text{yellow},$$

which was to be proved.

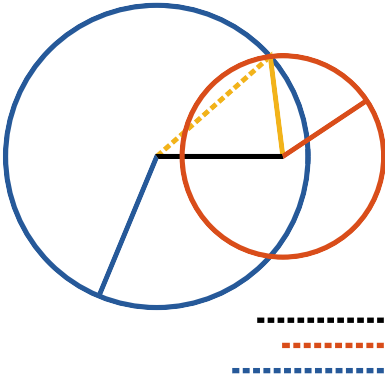
Again  $>$  (pr. I.16),

and also  $>$  (pr. I.16),

$$\therefore \text{red angle} > \text{yellow angle}.$$



Q. E. D.

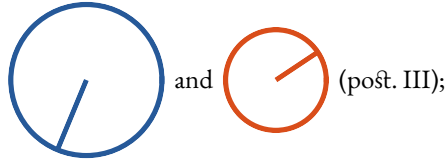


IVEN three right lines $\left\{ \begin{array}{l} \text{black dotted} \\ \text{orange dotted} \\ \text{blue dotted} \end{array} \right.$ the sum of any two greater than the third, to construct a triangle whose sides shall be respectively equal to the given lines.


Assume $\text{black solid} = \text{black dotted}$ (pr. 1.3).

Draw $\text{orange solid} = \text{orange dotted}$
and $\text{blue solid} = \text{blue dotted}$ } (pr. 1.2).

With blue solid and orange solid as radii, describe





draw yellow dotted and yellow solid ,


then will  be the triangle required.






For $\text{black solid} = \text{black dotted}$,
 $\text{yellow solid} = \text{orange solid} = \text{orange dotted}$,
and $\text{yellow dotted} = \text{blue solid} = \text{blue dotted}$. } (const.)



Q. E. D.



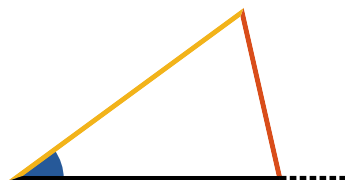
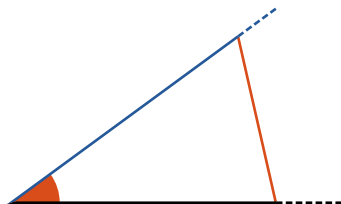
At a given point () in a given straight line (), to make an angle equal to a given rectilinear angle ().

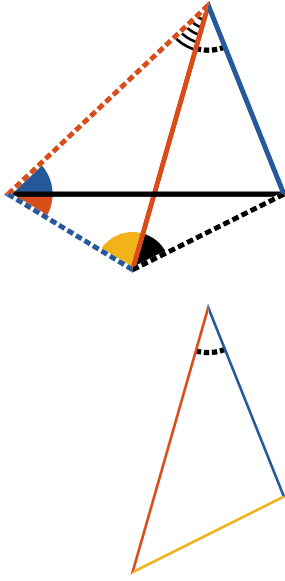
Draw  between any two points in the legs of the given angle.

Construct  (pr. 1.22)
so that  =  ,  = 
and  = .

Then  =  (pr. 1.8).

Q. E. D.





IF two triangles have two sides of the one respectively equal to two sides of the other (——— to ——— and to ———), and if one of the angles () contained by the equal sides be greater than the other (), the side (———) which is opposite to the greater angle is greater than the side (———) which is opposite to the less angle.

Make = (pr. I.23),
and = (pr. I.3),
draw and .

\therefore = (ax. I, hyp., const.)

\therefore = (pr. I.5)

but < ,

and \therefore < ,

\therefore > (pr. I.19)

but = (pr. I.4)

\therefore > .

Q. E. D.



F two triangles have two sides (— blue and — red) of the one respectively equal to two sides (— blue and — red) of the other, but their bases unequal, the angle subtended by the greater base (— black) of the one, must be greater than the angle subtended by the less base (— yellow) of the other.

$$\triangle = , > \text{ or } < \triangle$$

$$\triangle \text{ is not equal to } \triangle$$

for if $\triangle = \triangle$ then — black = — yellow (pr. 1.4)
which is contrary to the hypothesis;

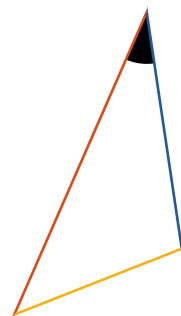
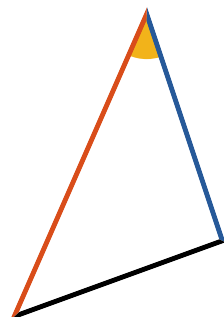
$$\triangle \text{ is not less than } \triangle$$

$$\text{for if } \triangle < \triangle$$





then — black < — yellow (pr. 1.24),
which is also contrary to the hypothesis:

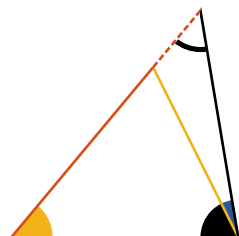
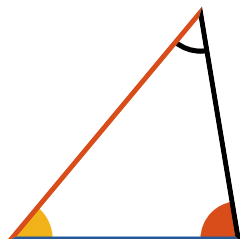
$$\therefore \triangle > \triangle.$$

Q. E. D.

























If two triangles have two angles of the one respectively equal to two angles of the other ( =  and  = ), and a side of the one equal to a side of the other similarly placed with respect to the equal angles, the remaining sides and angles are respectively equal to one another.











Case I.

Let  and  which lie between the equal angles be equal, then  = .










For if it be possible, let one of them  be greater than the other; make  = , draw .









In  and  we have
 = ,  = ,  = ;
 \therefore  =  (pr. 4.)
 but  =  (hyp.)

and therefore  = , which is absurd;
 hence neither of the sides  and  is greater than the other;
 and \therefore they are equal;



\therefore  = , and  = , (pr. 1.4).


Case II.



Again, let  = , which lie opposite the equal angles  and .
 If it be possible, let  > , then
 take  = , draw .

Then in  and 
 we have  = ,
 =  and  = ,

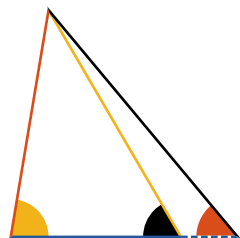
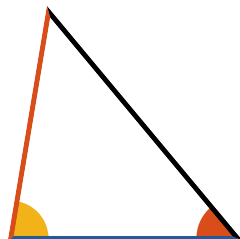
\therefore  =  (pr. I.4)

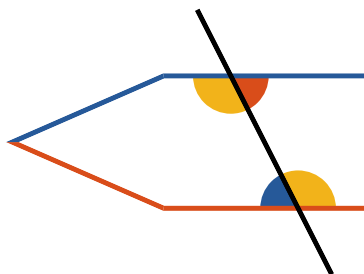
but  =  (hyp.)

\therefore  =  which is absurd (pr. I.16).

Consequently, neither of the sides  or  is greater than the other, hence they must be equal. It follows (by pr. I.4) that the triangles are equal in all respects.

Q. E. D.













IF a straight line (—) meeting two other straight lines (— and —) makes with them the alternate angles (▲ and ▼ ; ◐ and ◑) equal, these two straight lines are parallel.

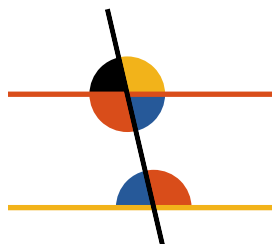
If — be not parallel to — they shall meet when produced.



If it be possible, let those lines be not parallel, but meet when produced; then the external angle ▼ is greater than ▲ (pr. I.16), but they are also equal (hyp.), which is absurd: in the same manner it may be shown that they cannot meet on the other side; ∴ they are parallel.











Q. E. D.





F a straight line (—), cutting two other straight lines (— and —), makes the external equal to the internal and opposite angle, at the same side of the cutting line (namely  =  or  = ) , or if it makes the two internal angles at the same side ( and  , or  and ) together equal to two right angles, those two straight lines are parallel.



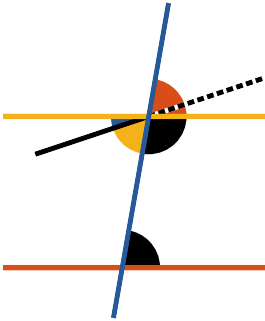
First, if  =  , then  =  (pr. I.15),
 \therefore  =  \therefore — \parallel — (pr. I.27).

Secondly, if  +  =  ,
 then  +  =  (pr. I.13),
 \therefore  +  =  +  (ax. III)




$$\therefore \text{  =  }$$

$$\therefore \text{ — } \parallel \text{ — (pr. I.27).}$$







Q. E. D.








STRAIGHT line (———) falling on two parallel straight lines (——— and ———), makes the alternate angles equal to one another; and also the external equal to the internal and opposite angle on the same side; and the two internal angles on the same side together equal to two right angles.

For if the alternate angles  and  be not equal, draw ———, making  =  (pr. 1.23).

Therefore ——— || ——— (pr. 1.27) and therefore two straight lines which intersect are parallel to the same straight line, which is impossible (ax. XII).

Hence  and  are not unequal, that is, they are equal:  =  (pr. 1.15); \therefore  = , the external angle equal to the internal and opposite on the same side:

if  be added to both, then  +  =  =

 (pr. 1.13). That is to say, the two internal angles at the same side of the cutting line are equal to two right angles.

Q. E. D.



STRAIGHT lines (— and —) which
are parallel to the same straight line (—),
are parallel to one another.

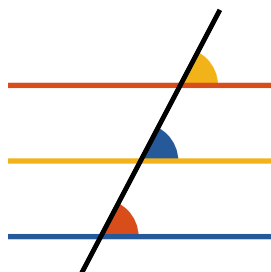
Let — intersect { — };

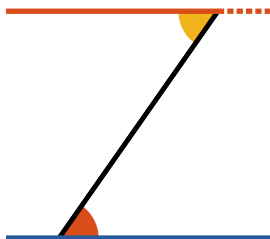
Then,  =  =  (pr. I.29),

\therefore  = 









\therefore — \parallel — (pr. I.28).

Q. E. D.








FROM a given point  to draw a straight line parallel to a given straight line ().

Draw  from the point 
to any point  in ,
make  =  (pr. I.23),
then  \parallel  (pr. I.27).

Q. E. D.



*F any side (—) of a triangle be produced,
the external angle () is equal to the sum of
the two internal and opposite angles ( and
), and the three internal angles of any triangle taken
together are equal to two right angles.*

Through the point  draw
 \parallel  (pr. I.31).

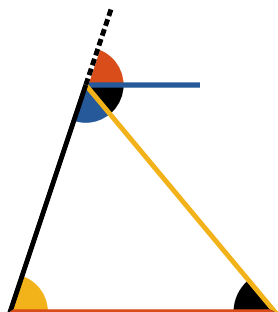
Then $\left\{ \begin{array}{l} \text{red angle} = \text{yellow angle} \\ \text{black angle} = \text{black angle} \end{array} \right\}$ (pr. I.29),

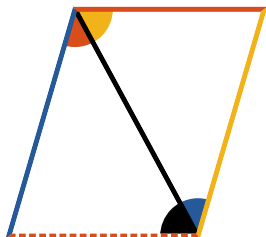
$$\therefore \text{yellow angle} + \text{black angle} = \text{red angle} \text{ (ax. II),}$$

and therefore

$$\text{yellow angle} + \text{blue angle} + \text{black angle} = \text{red angle} + \text{blue angle} = \text{straight angle} \text{ (pr. I.13).}$$

Q. E. D.





TRAIGHT lines (——— and ———) which
join the adjacent extremities of two equal and
parallel straight lines (——— and ———),
are themselves equal and parallel.

Draw ——— the diagonal.

——— = ——— (hyp.)

▲ = ▲ (pr. 1.29)

and ——— common to the two triangles;

\therefore ——— = ———, and ▲ = ▲ (pr. 1.4);

and \therefore ——— \parallel ——— (pr. 1.27).

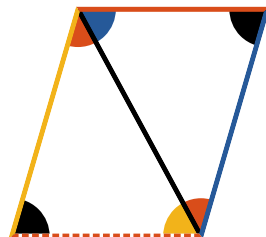
Q. E. D.





HE *opposite sides and angles of any parallelogram are equal, and the diagonal (—) divides it into two equal parts.*

Since $\left\{ \begin{array}{l} \text{blue triangle} = \text{yellow triangle} \\ \text{red triangle} = \text{red triangle} \end{array} \right\}$ (pr. I.29)
and — common to the two triangles.

$\therefore \left\{ \begin{array}{l} \text{red line} = \text{dotted red line} \\ \text{yellow line} = \text{blue line} \\ \text{black angle} = \text{black angle} \end{array} \right\}$ (pr. I.26)
and $\text{blue triangle} = \text{yellow triangle}$ (ax. II).



Therefore the opposite sides and angles of the parallelogram are equal: and as the triangles  and  are equal in every respect (pr. I.4), the diagonal divides the parallelogram into two equal parts.

Q. E. D.



PARALLELOGRAMS *on the same base, and between the same parallels, are (in area) equal.*

On account of the parallels,

$$\left. \begin{array}{l} \text{red triangle} = \text{blue triangle}; \\ \text{black triangle} = \text{white triangle}; \end{array} \right\} \begin{array}{l} \text{(pr. I.29)} \\ \text{(pr. I.29)} \end{array}$$

and $\text{blue line} = \text{red line}$ (pr. I.34).

But $\text{yellow triangle} = \text{yellow triangle}$ (pr. I.8)



$$\therefore \text{yellow triangle} - \text{blue triangle} = \text{yellow triangle} - \text{white triangle},$$

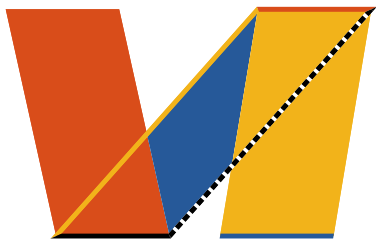
and $\text{yellow triangle} - \text{blue triangle} = \text{yellow triangle} - \text{white triangle};$




$$\therefore \text{yellow triangle} = \text{yellow triangle}.$$



Q. E. D.





PARALLELOGRAMS ( and ) on equal bases, and between the same parallels, are equal.



Draw  and 
 =  =  by (pr. I.34, and hyp.);

\therefore  = and \parallel ;

\therefore  = and \parallel  (pr. I.33)



And therefore  is a parallelogram:

but  =  =  (pr. I.35)

\therefore  =  (ax. I).



Q. E. D.



TRIANGLES  and  on the same base
 (—) and between the same parallels are
 equal.

Draw  ||  } (pr. I.31)
 || 

Produce



 and  are parallelograms
 on the same base and between the same
 parallels, and therefore equal. (pr. I.35)

$\therefore \left\{ \begin{array}{l} \text{yellow triangle} = \text{twice } \text{yellow triangle} \\ \text{black triangle} = \text{twice } \text{yellow triangle} \end{array} \right\} \text{ (pr. I.34)}$

$\therefore \text{yellow triangle} = \text{black triangle} .$

Q. E. D.



TRIANGLES ( and ) on equal bases and between the same parallels are equal.

Draw  \parallel  } (pr. I.31)
and  \parallel  }

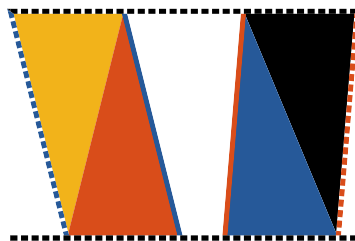
$$\begin{array}{c} \text{yellow triangle} \\ \text{red triangle} \end{array} = \begin{array}{c} \text{blue triangle} \\ \text{black triangle} \end{array} \quad (\text{pr. I.36});$$

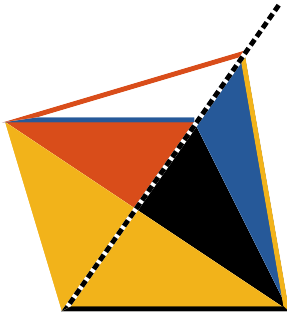
$$\text{but } \begin{array}{c} \text{yellow triangle} \\ \text{red triangle} \end{array} = \text{twice } \begin{array}{c} \text{red triangle} \end{array} \quad (\text{pr. I.34}),$$



$$\text{and } \begin{array}{c} \text{blue triangle} \\ \text{black triangle} \end{array} = \text{twice } \begin{array}{c} \text{blue triangle} \end{array} \quad (\text{pr. I.34}),$$






$$\therefore \begin{array}{c} \text{red triangle} \end{array} = \begin{array}{c} \text{blue triangle} \end{array} \quad (\text{ax. VII}).$$

Q. E. D.





QUAL triangles  and  on the same base (—) and on the same side of it, are between the same parallels.



If , which joins the vertices of the triangles, be not \parallel , draw  \parallel  (pr. I.31), meeting .







Draw .

\therefore  \parallel  (const.)

 =  (pr. I.37);



but  =  (hyp.);







\therefore  = , a part equal to the whole, which is absurd.

\therefore  \nparallel ; and in the same manner it can be demonstrated, that no other line except  is \parallel ; \therefore  \parallel .

Q. E. D.











QUAL triangles ( and ) on equal bases, and on the same side, are between the same parallels.

If  which joins the vertices of triangles be not \parallel ,
draw  \parallel  (pr. I.31),
meeting .
Draw .

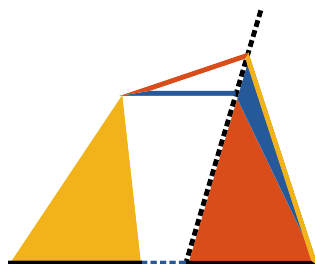
\therefore  \parallel  (const.)

 =  but  = 






\therefore  = , a part equal to the whole, which is absurd.

\therefore  \parallel : and in the same manner it can be demonstrated, that no other line except  is \parallel : \therefore  \parallel .



Q. E. D.







L^F a parallelogram  and a triangle  are upon the same base  and between the same parallels  and , the parallelogram is double the triangle.

Draw  the diagonal.


Then  =  (pr. 1.37)



 = twice  (pr. 1.34)

\therefore  = twice  .


Q. E. D.



o construct a parallelogram equal to a given triangle  and having an angle equal to a given rectilinear angle .



Make  =  (pr. I.10)

Draw .

Make  =  (pr. I.23)

Draw $\left\{ \begin{array}{l} \text{---} \parallel \text{---} \\ \text{---} \parallel \text{---} \end{array} \right\}$ (pr. I.31)

 = twice  (pr. I.41)



but  =  (pr. I.38)

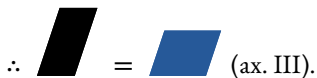
\therefore  = .

Q. E. D.











HE complements  and  of the parallelograms which are about the diagonal of a parallelogram are equal.




Q. E. D.




On a given straight line (—) to apply a parallelogram equal to a given triangle () and having an angle equal to a given rectilinear angle () .


Make  =  with  =  (pr. I.42)

and having one of its sides conterminous with and in continuation of — .

Produce — till it meets  ||

draw  produce it till it meets continued;





draw || meeting

 produced and produce

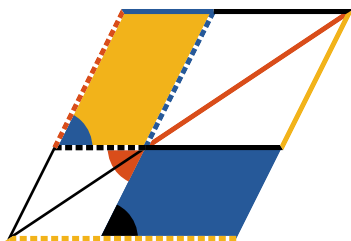
 =  (pr. I.43)

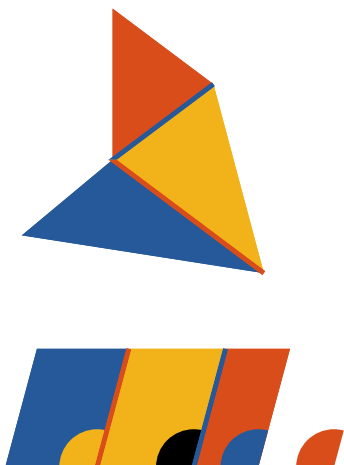
but  =  (const.)



\therefore  =  ;



and  =  =  =  (pr. I.29 and const.).

Q. E. D.





○ *construct a parallelogram equal to a given rectilinear figure () and having an angle equal to a given rectilinear angle ().*




Draw  and  dividing the rectilinear figure into triangles.



Construct  = 

having  =  (pr. I.42)

to  apply  = 

having  =  (pr. I.44)

to  apply  = 

having  =  (pr. I.44)

∴  = 

and  is a parallelogram. (pr. I.29, I.14, I.30)

having  = .



Q. E. D.





PON a given straight line (—) to construct a square.

Draw — \perp and = — (pr. I.II, I.3)

Draw — \parallel — ,
and meeting — drawn \parallel — .

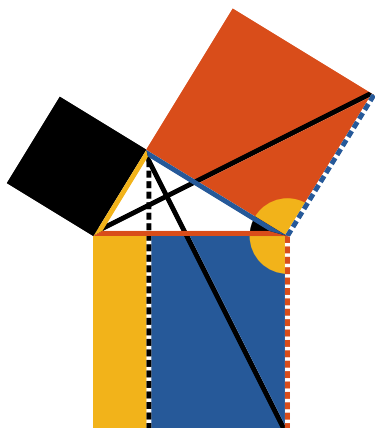
In  — = — (const.)
 = a right angle (const.)

\therefore  =  = a right angle (pr. I.29),
and the remaining sides and angles
must be equal (pr. I.34).

And \therefore  is a square (def. 30).

Q. E. D.

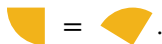




IN a right angled triangle the square on the hypotenuse is equal to the sum of the squares of the sides (and).

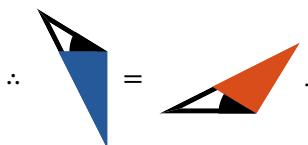
On , ,
describe squares, (pr. I.46)

Draw \parallel (pr. I.31)
also draw and .

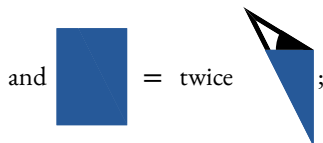
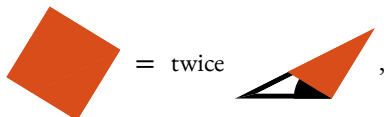




To each add \therefore = ,

= and = ;



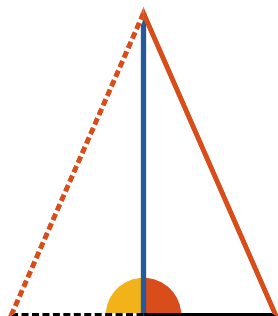
Again, \therefore \parallel



In the same manner it may be shown that  =  ;

hence  =  .

Q. E. D.



If the square of one side (———) of a triangle is equal to the squares of the other two sides (——— and ———), the angle (◡) subtended by that side is a right angle.

Draw ——— \perp ——— and = ——— (pr. I.11, I.3)
and draw ——— also.

Since ——— = ——— (const.)
—————² = ———²;

$$\therefore \text{—————}^2 + \text{—————}^2 = \text{—————}^2 + \text{—————}^2$$

but ———² + ———² = ———² (pr. I.47),
and ———² + ———² = ———² (hyp.)

$$\therefore \text{—————}^2 = \text{—————}^2,$$

$$\therefore \text{—————} = \text{—————};$$

and \therefore ◡ = ◡ (pr. I.8),

consequently ◡ is a right angle.

Q. E. D.








Book II







Definition I



RECTANGLE or a right angled parallelogram is said to be contained by any two of its adjacent or conterminous sides.



Thus: the right angled parallelogram  is said to be contained by the sides  and ; or it may be briefly designated by  · .

If the adjacent sides are equal; i. e.  = , then  ·  which is the expression for the rectangle under  and  is a square, and

is equal to $\left\{ \begin{array}{l} \text{---} \cdot \text{---} \text{ or } \text{---}^2 \\ \text{---} \cdot \text{---} \text{ or } \text{---}^2 \end{array} \right.$

Definition II



N a parallelogram, the figure composed of one of the parallelograms about the diagonal, together with the two complements, is called a Gnomon.

Thus



and



are called Gnomons.



THE rectangle contained by two straight lines, one of which is divided into any number of parts,

$$\text{---} \cdot \text{---} = \left\{ \begin{array}{l} \text{---} \cdot \text{---} \\ + \text{---} \cdot \text{---} \\ + \text{---} \cdot \text{---} \end{array} \right.$$

is equal to the sum of the rectangles contained by the undivided line, and the several parts of the divided line.

Draw $\text{---} \perp \text{---}$ and $= \text{---}$ (pr. I.II, pr. I.3); complete the parallelograms, that is to say,

$$\text{draw } \left\{ \begin{array}{l} \text{---} \parallel \text{---} \\ \text{---} \parallel \text{---} \end{array} \right\} \text{ (pr. I.31)}$$

$$\text{---} = \text{---} + \text{---} + \text{---}$$

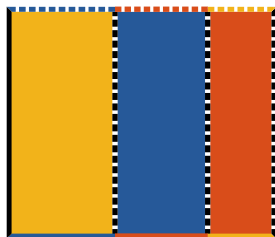
$$\text{---} = \text{---} \cdot \text{---}$$

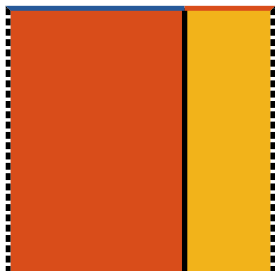
$$\text{---} = \text{---} \cdot \text{---}, \text{---} = \text{---} \cdot \text{---},$$


$$\text{---} = \text{---} \cdot \text{---}$$

$$\therefore \text{---} \cdot \text{---} = \text{---} \cdot \text{---} + \text{---} \cdot \text{---} + \text{---} \cdot \text{---}.$$

Q. E. D.







If a straight line be divided into any two parts
, the square of the whole line is equal
 to the sum of the rectangles contained by the
 whole line and each of its parts.

$$\text{---}^2 = \left\{ \begin{array}{l} \text{---} \cdot \text{---} \\ + \text{---} \cdot \text{---} \end{array} \right.$$

Describe  (pr. I.46)

Draw  parallel to  (pr. I.31)

$$\text{---} = \text{---}^2$$

$$\text{---} = \text{---} \cdot \text{---} = \text{---} \cdot \text{---}$$


$$\text{---} = \text{---} \cdot \text{---} = \text{---} \cdot \text{---}$$

$$\text{---} = \text{---} + \text{---}$$

$$\therefore \text{---}^2 = \text{---} \cdot \text{---} + \text{---} \cdot \text{---}$$

Q. E. D.



F a straight line be divided into any two parts
, the rectangle contained by the
 whole line and either of its parts, is equal to
 the square of that part, together with the rec-
 tangle under the parts.

$$\text{---} \cdot \text{---} = \text{---}^2 + \text{---} \cdot \text{---}, \text{ or}$$

$$\text{---} \cdot \text{---} = \text{---}^2 + \text{---} \cdot \text{---}.$$








Describe  (pr. I.46)

Describe  (pr. I.31)

Then  =  + , but

 =  ·  and

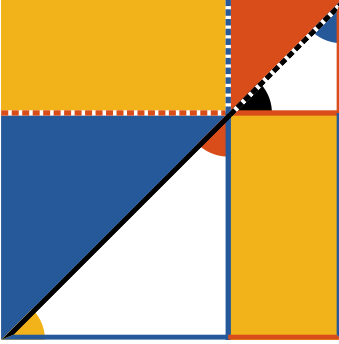
 = ²,  =  · ,

$$\therefore \text{---} \cdot \text{---} = \text{---}^2 + \text{---} \cdot \text{---}.$$

In a similar manner it may be readily shown that

$$\text{---} \cdot \text{---} = \text{---}^2 + \text{---} \cdot \text{---}.$$

Q. E. D.



*If a straight line be divided into any two parts
 —, the square of the whole line is equal
 to the squares of the parts, together with twice
 the rectangle contained by the parts.*

$$\text{—}^2 = \text{—}^2 + \text{—}^2 + \text{twice } \text{—} \cdot \text{—}$$

Describe  (pr. I.46)

draw (post. I)


and $\left\{ \begin{array}{l} \text{—} \parallel \text{—} \\ \text{—} \parallel \text{—} \end{array} \right\}$ (pr. I.31)

 =  (pr. I.5),

 =  (pr. I.29),






\therefore  = 

\therefore by (pr. I.6, pr. I.29, pr. I.34)

 is a square = —^2 .

For the same reasons  is a square = —^2 ,

$$\text{—} = \text{—} = \text{—} \cdot \text{—} \quad (\text{pr. I.43})$$

But  =  +  +  + ,

$$\therefore \text{—}^2 = \text{—}^2 + \text{—}^2 + \text{twice } \text{—} \cdot \text{—}.$$

Q. E. D.



F a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half that line

$$\text{red} \cdot \text{blue} + \text{blue}^2 = \text{yellow}^2 = \text{red}^2.$$

Describe  (pr. I.46),

draw 

$$\text{and } \left\{ \begin{array}{l} \text{---} \parallel \text{---} \\ \text{---} \parallel \text{---} \\ \text{---} \parallel \text{---} \end{array} \right\} \text{ (pr. I.31)}$$

$$\text{black} = \text{blue} \cdot \text{yellow} \text{ (pr. I.36)}$$

$$\text{yellow} = \text{red} \cdot \text{blue} \text{ (pr. I.43)}$$

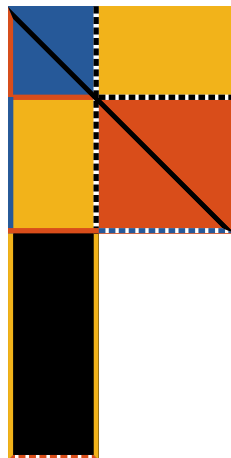
$$\therefore \text{(ax. II)} \quad \text{blue} \cdot \text{yellow} = \text{black} = \text{red} \cdot \text{blue}$$

$$\text{but } \text{red} = \text{blue}^2 \text{ (pr. 2.4)}$$

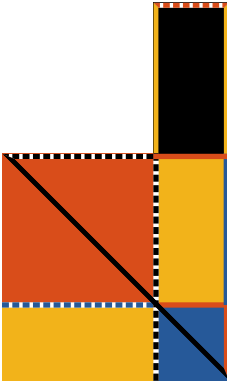
$$\text{and } \text{blue} \cdot \text{yellow} = \text{red}^2 \text{ (const.)}$$

$$\therefore \text{(ax. II)} \quad \text{blue} \cdot \text{yellow} = \text{black}$$

$$\therefore \text{red} \cdot \text{blue} + \text{blue}^2 = \text{yellow}^2 = \text{red}^2.$$





Q. E. D.



F a straight line be bisected and produced to any point, the rectangle contained by the whole line so increased, and the part produced, together with the square of half the line, is equal to the square of the line made up of the half, and the produced part.

$$\text{---} \cdot \text{---} + \text{---}^2 = \text{---}^2$$

Describe  (pr. I.46), draw 

and $\left\{ \begin{array}{l} \text{---} \parallel \text{---} \\ \text{---} \parallel \text{---} \\ \text{---} \parallel \text{---} \end{array} \right\}$ (pr. I.31)

$$\text{---} = \text{---} = \text{---} \quad (\text{pr. I.36, pr. I.43})$$

$$\therefore \text{---} = \text{---} = \text{---} \cdot \text{---};$$


$$\text{but } \text{---} = \text{---}^2 \quad (\text{pr. 2.4})$$

$$\therefore \text{---} = \text{---}^2 = \text{---} \quad (\text{const., ax. II})$$

$$\therefore \text{---} \cdot \text{---} + \text{---}^2 = \text{---}^2.$$



Q. E. D.



F a straight line be divided into any two parts
, the squares of the whole line and
 one of the parts are equal to twice the rectan-
 gle contained by the whole line and that part,
 together with the square of the other parts.


$$\text{blue}^2 + \text{red}^2 = 2 \cdot \text{blue} \cdot \text{red} + \text{blue}^2$$



Describe  (pr. I.46),
 draw  (post. I),

and $\left\{ \begin{array}{l} \text{blue} \cdots \parallel \text{yellow} \\ \cdots \text{red} \parallel \text{blue} \end{array} \right\}$

 =  (pr. I.43),

add  = red^2 to both (pr. 2.4)

$$\text{yellow} + \text{blue} = \text{black} = \text{blue} \cdot \text{red}$$

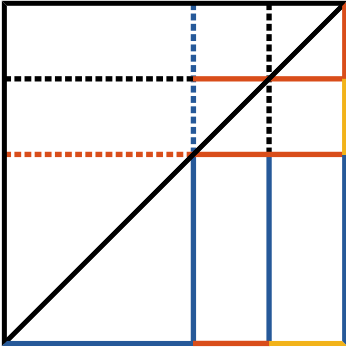
$$\text{red} = \text{blue}^2 \text{ (pr. 2.4)}$$

$$\text{yellow} + \text{blue} + \text{black} + \text{red} =$$

$$2 \cdot \text{blue} \cdot \text{red} + \text{blue}^2 = \text{yellow} + \text{black} + \text{blue};$$

$$\text{blue}^2 + \text{red}^2 = 2 \cdot \text{blue} \cdot \text{red} + \text{blue}^2.$$

Q. E. D.



F a straight line be divided into any two parts
 —, the square of the sum of the whole
 line and any one of its parts is equal to four
 times the rectangle contained by the whole line,
 and that part together with the square of the other part.

$$\text{—}^2 = 4 \cdot \text{—} \cdot \text{—} + \text{—}^2$$

Produce — and make — = —



Construct — (pr. I.46);
 draw —,

$$\left. \begin{array}{l} \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} \parallel \left. \begin{array}{l} \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} \text{ (pr. I.31)}$$

$$\text{---}^2 = \text{---}^2 + \text{---}^2 + 2 \cdot \text{---} \cdot \text{---} \quad \text{(pr. 2.4)}$$

$$\text{but } \text{---}^2 + \text{---}^2 = 2 \cdot \text{---} \cdot \text{---} + \text{---}^2 \quad \text{(pr. 2.7)}$$

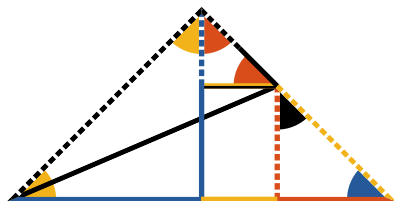
$$\therefore \text{---}^2 = 4 \cdot \text{---} \cdot \text{---} + \text{---}^2.$$

Q. E. D.



F a straight line be divided into two equal parts
 and also into two unequal parts
 the squares of the unequal
 parts are together double the squares of half
 the line, and of the part between the points of section.

$$\text{blue}^2 + \text{red}^2 = 2 \cdot \text{blue}^2 + 2 \cdot \text{yellow}^2$$



Make $\text{blue} \perp$ and $= \text{blue}$ or yellow ,

Draw dotted and black ,

$\text{dotted} \parallel \text{blue}$, $\text{yellow} \parallel \text{red}$ and draw black .

$\text{yellow} = \text{yellow}$ (pr. I.5) = half a right angle (pr. I.32)

$\text{blue} = \text{black}$ (pr. I.5) = half a right angle (pr. I.32)

$\therefore \text{yellow} = \text{black}$ = a right angle.

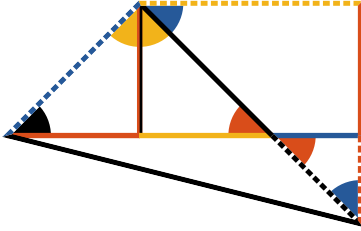
$\text{blue} = \text{red} = \text{red} = \text{black}$ (pr. I.5, pr. I.29).

hence $\text{dotted} = \text{red}$, $\text{blue} = \text{yellow} = \text{yellow}$ (pr. I.6, pr. I.34)

$$\left\{ \begin{array}{l} \text{black}^2 = \\ \text{blue}^2 + \text{dotted}^2, \text{ or } + \text{red}^2 \\ = \left\{ \begin{array}{l} \text{dotted}^2 + \text{black}^2 \\ = 2 \cdot \text{blue}^2 + 2 \cdot \text{yellow}^2 \end{array} \right. \quad (\text{pr. I.47}) \end{array} \right.$$

$$\therefore \text{blue}^2 + \text{red}^2 = 2 \cdot \text{blue}^2 + 2 \cdot \text{yellow}^2.$$

Q. E. D.



IF a straight line be bisected and produced to any point , the squares of the whole produced line, and of the produced part, are together double of the squares of the half line, and of the line made up of the half and produced part.

$$\text{red}^2 + \text{yellow}^2 + \text{red}^2 = 2 \cdot \text{red}^2 + 2 \cdot \text{red} \cdot \text{yellow}^2$$

Make \perp and $=$ to or ,
draw and ,

and $\left\{ \begin{array}{l} \text{red} \dots \parallel \text{red} \\ \text{yellow} \dots \parallel \text{yellow} \end{array} \right\}$ (pr. 1.31)
draw also.

$=$ (pr. 1.5) = half a right angle (pr. 1.32)

$=$ (pr. 1.5) = half a right angle (pr. 1.32)

\therefore = a right angle.

$=$ $=$ $=$ $=$ $=$

half a right angle (pr. 1.5, pr. 1.32, pr. 1.29, pr. 1.34),

and $=$, $=$ $=$,
(pr. 1.6, pr. 1.34).


Hence by (pr. 1.47)

$$\text{solid black}^2 = \left\{ \begin{array}{l} \text{red} \text{ yellow}^2 + \text{red}^2 \text{ or } \text{blue}^2 \\ \left\{ \begin{array}{l} + \text{dotted blue}^2 = 2 \cdot \text{red}^2 \\ + \text{solid black}^2 = 2 \cdot \text{dotted yellow}^2 \end{array} \right. \end{array} \right.$$

$$\therefore \text{red} \text{ yellow}^2 + \text{blue}^2 = 2 \cdot \text{yellow}^2 + 2 \cdot \text{red} \text{ yellow}^2.$$

Q. E. D.





o divide a given straight line  in such a manner, that the rectangle contained by the whole line and one of its parts may be equal to the square of the other.



$$\text{---} \cdot \text{---} = \text{---}^2$$





Describe  (pr. I.46),

make  =  (pr. I.10),

draw ,

take  =  (pr. I.3),

on  describe  (pr. I.46).

Produce  (post. II).

Then, (pr. 2.6)   +  $^2 =$

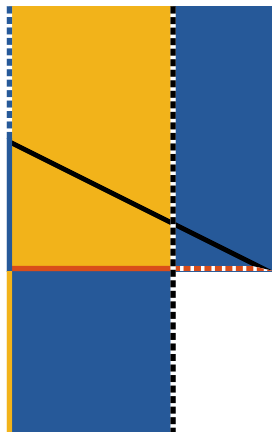
$$\text{---}^2 = \text{---}^2 = \text{---}^2 + \text{---}^2 \therefore$$

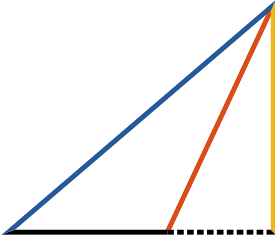
$$\text{---} \cdot \text{---} = \text{---}^2, \text{ or,}$$



$$\therefore \text{---} \cdot \text{---} = \text{---}^2.$$

Q. E. D.





IN any obtuse angled triangle, the square of the side subtending the obtuse angle exceeds the sum of the squares of the sides containing the obtuse angle, by twice the rectangle contained by either of these sides and the produced parts of the same from the obtuse angle to the perpendicular let fall on it from the opposite acute angle.

$$\text{blue}^2 > \text{black}^2 + \text{red}^2 \text{ by } 2 \cdot \text{black} \cdot \text{white}$$

By pr. 2.4

$$\text{white}^2 = \text{black}^2 + \text{white}^2 + 2 \cdot \text{black} \cdot \text{white};$$

add yellow^2 to both

$$\text{white}^2 + \text{yellow}^2 = \text{blue}^2 \text{ (pr. I.47)}$$

$$= 2 \cdot \text{black} \cdot \text{white} + \text{black}^2 + \left\{ \frac{\text{white}^2}{\text{yellow}^2} \right\} \text{ or} \\ + \text{red}^2 \text{ (pr. I.47)}.$$

$$\therefore \text{blue}^2 = 2 \cdot \text{black} \cdot \text{white} + \text{black}^2 + \text{red}^2;$$

$$\text{hence } \text{blue}^2 > \text{black}^2 + \text{red}^2 \\ \text{by } 2 \cdot \text{black} \cdot \text{white}.$$

Q. E. D.



IN any triangle, the square of the side subtending an acute angle, is less than the sum of the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the part of it intercepted between the foot of the perpendicular let fall on it from the opposite angle, and the angular point of the acute angle.

First.

$$\text{Blue}^2 < \text{Black}^2 + \text{Red}^2 \text{ by } 2 \cdot \text{Black} \cdot \text{Black}.$$

Second.

$$\text{Blue}^2 < \text{Red}^2 + \text{Black}^2 \text{ by } 2 \cdot \text{Black} \cdot \text{Black}.$$

First, suppose the perpendicular to fall within the triangle, then (pr. 2.7)

$$\text{Black}^2 + \text{Black}^2 = 2 \cdot \text{Black} \cdot \text{Black} + \text{Black}^2,$$

add to each Yellow^2 then,

$$\text{Black}^2 + \text{Black}^2 + \text{Yellow}^2 = 2 \cdot \text{Black} \cdot \text{Black} + \text{Black}^2 + \text{Yellow}^2$$

\therefore (pr. 1.47)

$$\text{Black}^2 + \text{Red}^2 = 2 \cdot \text{Black} \cdot \text{Black} + \text{Blue}^2,$$

$$\text{and } \therefore \text{Blue}^2 < \text{Black}^2 + \text{Red}^2 \text{ by } 2 \cdot \text{Black} \cdot \text{Black}$$

Next suppose the perpendicular to fall without the triangle, then (pr. 2.7)

$$\text{Black}^2 + \text{Black}^2 = 2 \cdot \text{Black} \cdot \text{Black} + \text{Black}^2,$$

add to each Yellow^2 then

$$\text{Black}^2 + \text{Black}^2 + \text{Yellow}^2 = 2 \cdot \text{Black} \cdot \text{Black} + \text{Black}^2 + \text{Yellow}^2$$

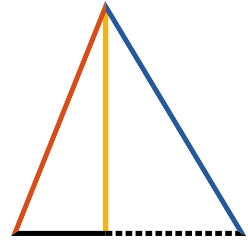
\therefore (pr. 1.47)

$$\text{Red}^2 + \text{Black}^2 = 2 \cdot \text{Black} \cdot \text{Black} + \text{Blue}^2,$$

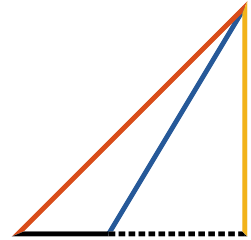
$$\therefore \text{Blue}^2 < \text{Red}^2 + \text{Black}^2 \text{ by } 2 \cdot \text{Black} \cdot \text{Black}.$$

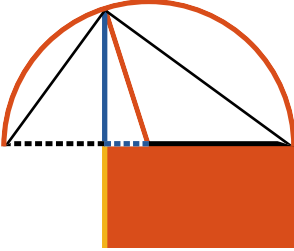
Q. E. D.

First






Second










o draw a right line of which the square shall be equal to a given rectilinear figure.

To draw  such that, ² = .

Make  =  (pr. I.45),

produce  until  = ;

take  =  (pr. I.10).

Describe  (post. III),
and produce  to meet it: draw .

² or ² =  ·  + ² (pr. 2.5),

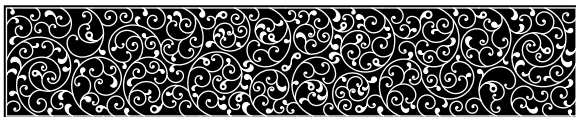
but ² = ² + ² (pr. I.47);

∴ ² + ² =  ·  + ²

∴ ² =  · , and

∴ ² =  = .

Q. E. D.



Book III

Definitions

I

Equal circles are those whose diameters are equal.

2

A right line is said to touch a circle when it meets the circle, and being produced does not cut it.

3

Circles are said to touch one another which meet, but do not cut one another.

4

Right lines are said to be equally distant from the centre of a circle when the perpendiculars drawn to them from the centre are equal.

5

And the straight line on which the greater perpendicular falls is said to be farther from the centre.



6



A segment of a circle is the figure contained by a straight line and the part of the circumference it cuts off.

7

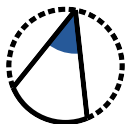
An angle of a segment is that contained by a straight line and a circumference of a circle.

8



An angle in a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment.

9



An angle is said to stand on the part of the circumference, or the arch, intercepted between the right lines that contain the angle.

10



A sector of a circle is the figure contained by two radii and the arch between them.

11



Similar segments of circles are those which contain equal angles.



12












Circles which have the same centre are called *concentric circles*.



















o find the centre of a given circle

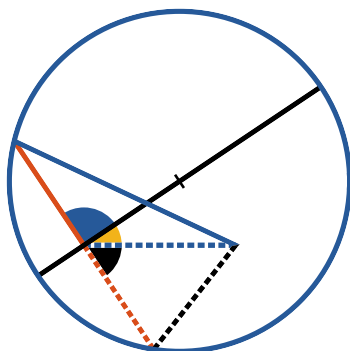


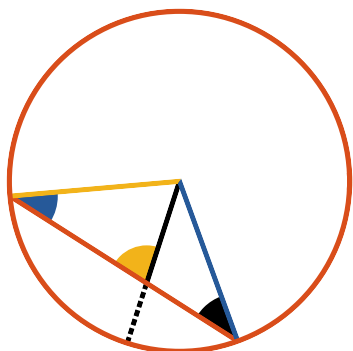
Draw within the circle any straight line , make  = , draw  \perp ; bisect , and the point of bisection is the centre.

For, if it be possible, let any other point as the point of concurrence of ,  and  be the centre.

Because in  and   =  (hyp. and def. 15),  =  (const.) and  common,  =  (pr. 1.8), and are therefore right angles; but  =  (const.)  =  (ax. XI) which is absurd; therefore the assumed point is not the centre of the circle; and in the same manner it can be proved that no other point which is not on  is the centre, therefore the centre is in , and therefore the point where  is bisected is the centre.

Q. E. D.







STRAIGHT line (—) joining two points
in the circumference of a circle (○), lies
wholly within the circle.

Find the centre of (○) (pr. 3.1);

from the centre draw — to any point in — ,
meeting the circumference from the centre;
draw — and — .

Then  =  (pr. 1.5)

but  >  or >  (pr. 1.16)

∴ — > — (pr. 1.19)

but — = — ,

∴ — > — ;

∴ — < — ;

∴ every point in — lies within the circle.

Q. E. D.



F a straight line (—) drawn through the centre of a circle (○) bisect a chord (—) which does not pass through the centre, it is perpendicular to it; or, if perpendicular to it, it bisects it.

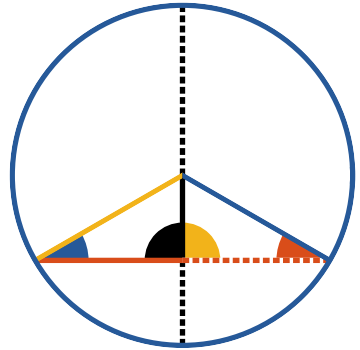
Draw — and — to the centre of the circle.

In and common,
 and = \therefore = (pr. 1.8)
 and \therefore — \perp — (def. 10)

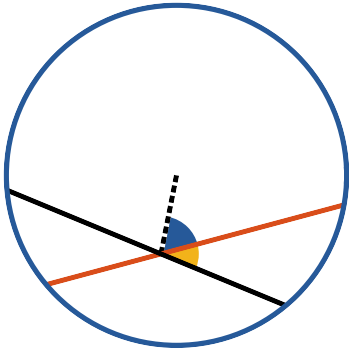
Again let — \perp —
 Then in and
 = (pr. 1.5)
 = (hyp.)
 and — = —

\therefore — = — (pr. 1.26)

and \therefore — bisects —.








Q. E. D.






If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect one another.

If one of the lines pass through the centre, it is evident that it cannot be bisected by the other, which does not pass through the centre.

But if neither of the lines  or  pass through the centre, draw  from the centre to their intersection.

If  be bisected,  \perp to it (pr. 3.3)



$$\therefore \text{blue angle} = \text{right angle}$$

and if  be bisected,  \perp  (pr. 3.3)

$$\therefore \text{blue angle} = \text{right angle};$$

$$\text{and } \therefore \text{blue angle} = \text{yellow angle};$$



a part equal to the whole, which is absurd:

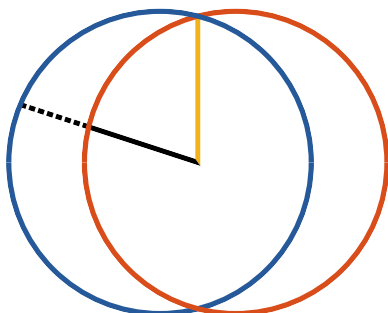
\therefore  and  do not bisect one another.



Q. E. D.





F two circles intersect , they have
not the same centre.

Suppose it possible that two intersecting circles have a common centre; from such supposed centre draw  to the intersecting point, and  meeting the circumferences of the circles.



Then  =  (def. 15)

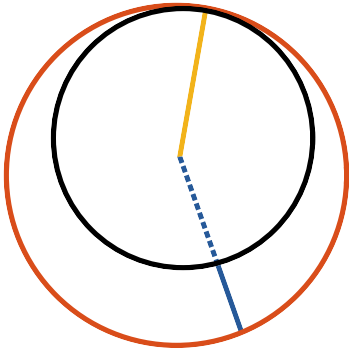
and  =  (def. 15)

\therefore  = 



a part equal to the whole, which is absurd:



\therefore circles supposed to intersect in any point cannot have the same centre.



Q. E. D.





If two circles touch one another internally, they have not the same centre.

For, if it be possible, let both circles have the same centre; from such a supposed centre draw , and  to the point of contact.

Then  =  (def. 15)

and  =  (def. 15)

\therefore  = ;






a part equal to the whole, which is absurd; therefore the assumed point is not the centre of both circles, and in the same manner it can be demonstrated that no other point is.

Q. E. D.



F from any point within a circle



which is not the centre, lines {    } are drawn to the circumference; the greatest of those lines is that () which passes through the centre, and the least is the remaining part () of the diameter.



Of the others, that () which is nearer to the line passing through the centre, is greater than that () which is more remote.





























Fig. 2. The two lines ( and ) which make equal angles with that passing through the centre, on opposite sides of it, are equal to each other; and there cannot be drawn a third line equal to them, from the same point to the circumference.

Figure I.

To the centre of the circle draw  and ; then  =  (def. 15)  =  +  >  (pr. 1.20) in like manner  may be shown to be greater than , or any other line drawn from the same point to the circumference. Again, by (pr. 1.20)  +  >  =  + , take  from both; \therefore  >  (ax. III), and in like manner it may be shown that  is less than any other line drawn from the same point to the

circumference. Again, in  and ,  —

common,  > , and  >  \therefore

 >  (pr. 1.24) and  may in like manner to be proved greater than any other line drawn

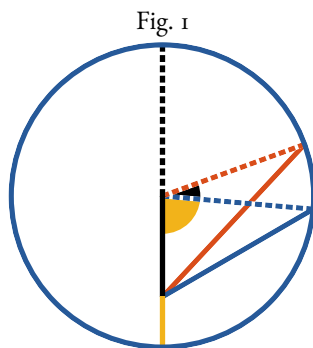
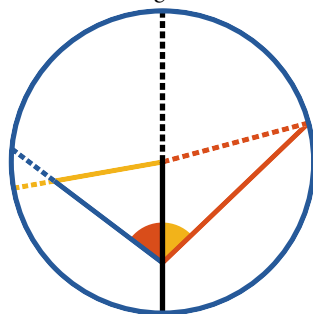
























Fig. 2



from the same point to the circumference more remote
from —......

Figure II.

If  =  then  = ,
if not take  =  draw 
then in  and ,  common,
 =  and  = 
 \therefore  =  (pr. I.4)
 \therefore  =  = 
a part equal to the whole, which is absurd:





\therefore  = ; and no other line is equal to
 drawn from the same point to the circumference;
for if it were nearer to the one passing through the centre
it would be greater, and if it were more remote it would be
less.

Q. E. D.









THE original text of this proposition is here divided into three parts.


I.

If from a point without a circle, straight lines {  } are drawn to the circumference; of those falling upon the concave circumference the greatest is that () which passes through the centre, and the line () which is nearer the greatest is greater than that () which is more remote.




Draw  and  to the centre.

Then,  which passes through the centre, is greatest;

for since  = , if  be added to both  =  + ;
but >  (pr. I.20)

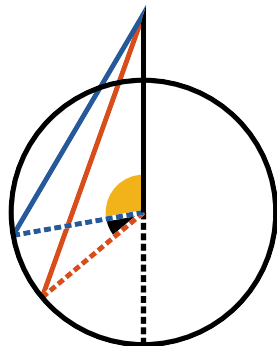
∴  is greater than any other line drawn from the same point to the concave circumference.

Again in  and ,

 = , and  common,

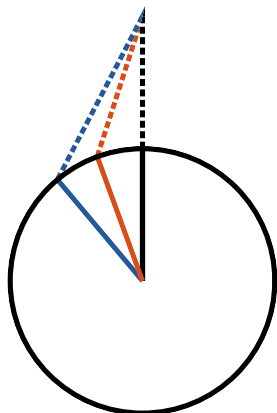
but  > , ∴  >  (pr. I.24);

and in like manner  may be shown > than any other line more remote from .



II.

Of those lines falling on the convex circumference the least is that (.....) which being produced would pass through the centre, and the line which is nearer to least is less than that which is more remote.

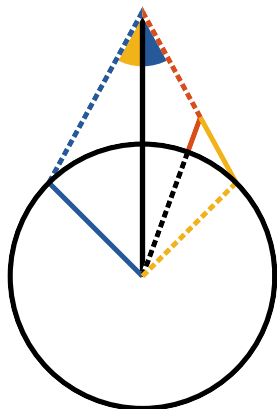


For, since ——— + > (pr. I.20)
 and ——— = ——— ,
 \therefore > (ax. V)

And again, since
 ——— + > ——— + (pr. I.21),
 and ——— = ——— ,
 \therefore < And so of others.

III.

Also the lines making equal angles with that which passes through the centre are equal, whether falling on the concave or convex circumference; and no third line can be drawn equal to them from the same point to the circumference.





For if > , but making $\angle = \angle$;
 make = , and draw

Then in and we have =
 and common, and also $\angle = \angle$,

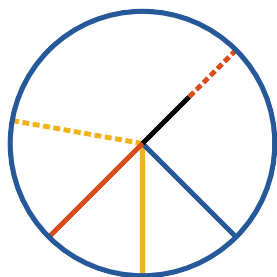
\therefore = (pr. I.4);
 but = ;
 \therefore = , which is absurd.





\therefore  is \neq , nor to any part
of , \therefore  is \nless .




Neither is  $>$ ,
they are $\therefore =$ to each other.









And any other line drawn from the same point to the circumference must lie at the same side with one of these lines, and be more or less remote than it from the line passing through the centre, and cannot therefore be equal to it.


Q. E. D.



IF a point be taken within a circle , from which more than two equal straight lines (, , ) can be drawn to the circumference, that point must be the centre of the circle.



For if it be supposed that the point  in which more than two equal straight lines meet is not the centre, some other point  must be; join these two points by  and produce it both ways to the circumference.





Then since more than two equal straight lines are drawn from a point which is not the centre, to the circumference, two of them at least must lie at the same side of the diameter ; and since from a point , which is not the centre, straight lines are drawn to the circumference; the greatest is , which passes through the centre: and  which is nearer to , >  which is more remote (pr. 3.8); but  =  (hyp.) which is absurd.




The same may be demonstrated of any other point, different from , which must be the centre of the circle.





Q. E. D.



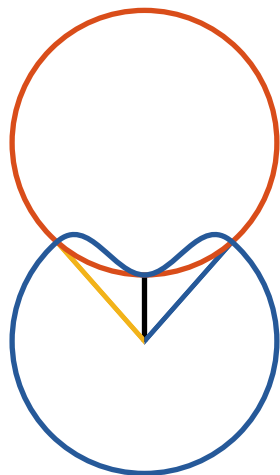
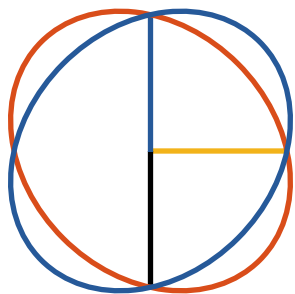
NE circle  cannot intersect another 
in more points than two.

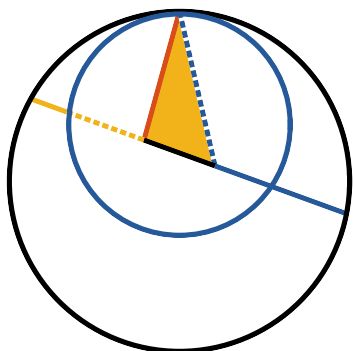
For, if it be possible, let it intersect in three points; from the centre of  draw ,  and  to the points of intersection;



\therefore  =  =  (def. 15), but as the circles intersect, they have not the same centre (pr. 3.5):






\therefore the assumed point is not the centre of , and \therefore as ,  and  are drawn from a point not the centre, they are not equal (pr. 3.7, pr. 3.8); but it was shown before that they were equal, which is absurd; the circles therefore do not intersect in three points.

Q. E. D.









IF two circles  and  touch one another internally, the right line joining their centres, being produced, shall pass through a point of contact.


For, if it be possible, let  join their centres, and produce it both ways; from a point of contact draw  to the centre of , and from the same point of contact draw  to the centre of .



Because in  ;  +  >  (pr. 1.20),

and  =  


as they are radii of ,



but  +  >   ;

take away  which is common,

and  >  ;

but  = ,




because they are radii of ,

and \therefore  >  a part greater than the whole, which is absurd.

The centres are not therefore so placed, that a line joining them can pass through any point but a point of contact.

Q. E. D.



If two circle  and  touch one another externally, the straight line  joining their centres, passes through the point of contact.

If it be possible, let  joining the centres, and not pass through a point of contact; then from a point of contact draw  and  to the centres.

$$\therefore \text{dotted yellow} + \text{solid yellow} > \text{red-blue} \quad (\text{pr. I.20})$$

$$\text{and } \text{red} = \text{dotted yellow} \quad (\text{def. 15}),$$

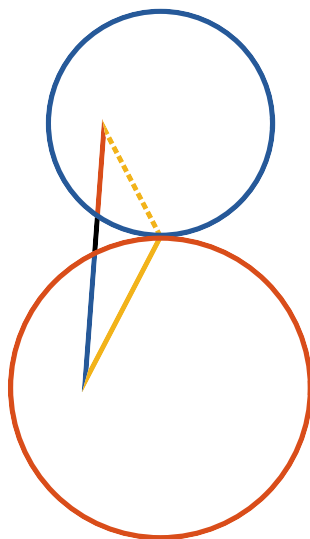
$$\text{and } \text{blue} = \text{solid yellow} \quad (\text{def. 15}),$$

$$\therefore \text{red} + \text{blue} > \text{red-blue},$$

a part greater than the whole, which is absurd.

The centres are not therefore so placed, that the line joining them can pass through any point but the point of contact.

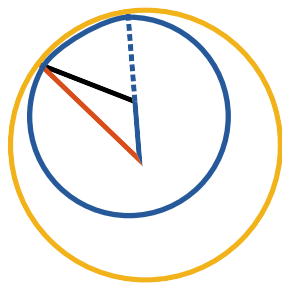
Q. E. D.










NE circle cannot touch another, either externally or internally, in more points than one


Figure I.



For, if it possible, let  and  touch one another internally in two points; draw  joining their centres, and produce it until it pass through one of the points of contact (pr. 3.11);

draw  and .

But  =  (def. 15),

\therefore if  be added to both,

 =  + .

but  =  (def. 15),

and \therefore  +  = .




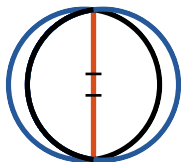





but \therefore  +  >  (pr. 1.20),
which is absurd.











Figure II.



But if the points of contact be the extremities of the right line joining the centres, this straight line must be bisected in two different points for the two centres; because it is the diameter of both circles, which is absurd.

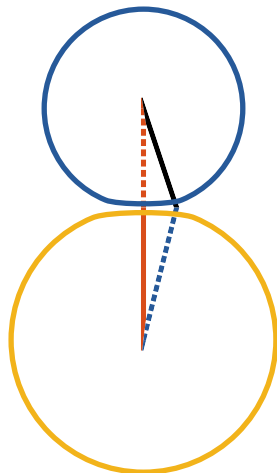
Figure III.

Next, if it be possible, let  and  touch externally in two points; draw  joining the centres of the circles, and passing through one of the points of contact, and draw  and .

 =  (def. 15);
 and  =  (def. 15);
 \therefore  +  = ;
 but  +  > 
 (pr. 1.20), which is absurd.

There is therefore no case in which two circles can touch one another in two points.



Q. E. D.





THE diameter is the greatest straight line in a circle: and, of all others, that which is nearest to the centre is greater than the more remote.

Figure I.

The diameter  is $>$ any line .



For draw  and .

Then  = 



and  = ,

\therefore  +  = 

but  +  $>$  (pr. 1.20)

\therefore  $>$ .

Again, the line which is nearer the centre is greater than the one more remote.

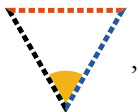
First, let the given lines be  and , which are at the same side of the centre and do not intersect;





draw $\left\{ \begin{array}{l} \text{yellow solid line} \\ \text{yellow dotted line} \\ \text{black solid line} \\ \text{blue dotted line} \end{array} \right.$

In



and



 and  =  and .

but  $>$ ,

\therefore  $>$  (pr. 1.24)

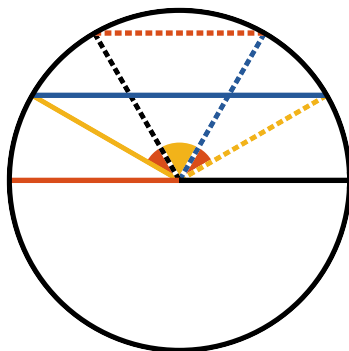
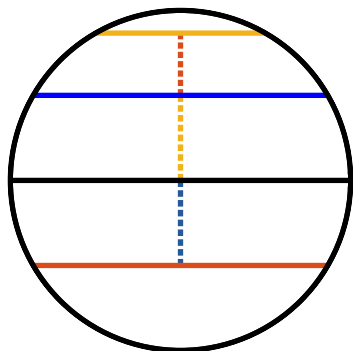


Figure II.



Let the given lines be — and — which either are at different sides of the centre, or intersect; from the centre draw ⋯ and ⋯ \perp — and —,





make ⋯ = ⋯,
and draw — \perp ⋯.

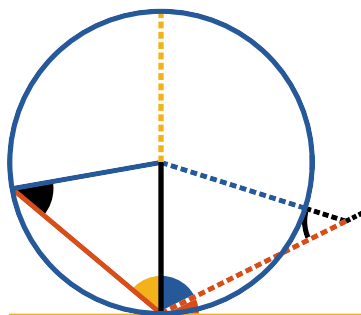
Since — and — are equally distant from the centre, — = — (pr. 3.14);
but — > — (pr. 3.15),

\therefore — > —.

Q. E. D.



 HE *Straight line*  drawn from the extremity of the diameter  of a circle perpendicular to it falls without the circle And if any straight line  by drawn from a point within that perpendicular to the point of contact, it cuts the circle.



Part I.



If it be possible, let , which meets the circle again, be \perp , and draw .

Then, \because $=$,






 =  (pr. 1.5),

and \therefore each of these angles is acute (pr. I.17)

but $\triangleleft = \cap$ (hyp.), which is absurd,

therefore  drawn \perp 
does not meet the circle again.


Part II.

Let  be \perp  and let  be drawn from a point  between  and the circle, which, if it be possible, does not cut the circle.

\vdots  $=$ ,

$\therefore \angle A$ is an acute angle;

suppose  \perp , drawn from the centre of the circle, it must fall at the side of  the acute angle.

$\therefore \triangle$ which is supposed to be a right angle, is $>$ ,


$$\therefore \text{————} > \text{.....};$$




$$\text{but } \text{.....} = \text{————},$$

and $\therefore \text{.....} > \text{.....}$, a part greater than the whole, which is absurd. Therefore the point does not fall outside the circle, and therefore the straight line cuts the circle.



Q. E. D.






o draw a tangent to a given circle  from a given point, either in or outside of its circumference.


If a given point be in the circumference, as at  it is plain that the straight line  \perp  the radius, will be the required tangent (pr. 3.16).

But if the given point  be outside of the circumference,




draw  from it to the centre, cutting ;


and draw  \perp ,













describe  concentric with  radius = ,

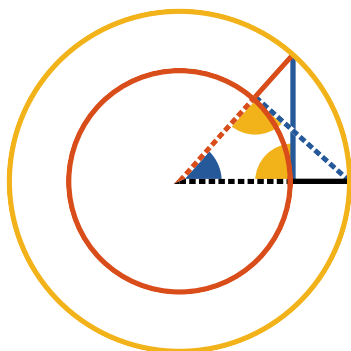
draw  to the centre from the point

where  falls on  circumference,

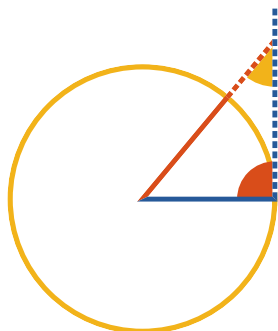
draw  \perp  from the point where it cuts ,

Then  will be the tangent required.

For in  and 
 = ,  common,
 and  = ,
 \therefore (pr. 1.4)  =  = ,
 \therefore  is a tangent to .



Q. E. D.



If a straight line be a tangent to a circle, the straight line — drawn from the centre to the point of contact, is perpendicular to it.

For, if it be possible, let — be \perp ,

then \therefore  = ,  is acute (pr. 1.17)

\therefore — > — (pr. 1.19);

but — = —,



and \therefore — > —, a part greater than the whole, which is absurd.



\therefore — is not \perp

and in the same manner it can be demonstrated, that no other line except — is perpendicular to

Q. E. D.






F a straight line  be a tangent to a circle, the straight line , drawn perpendicular to it from point of the contact, passes through the centre of the circle.


For, if it be possible, let the centre be without , and draw  from the supposed centre to the point of contact.

\therefore  \perp  (pr. 3.18)

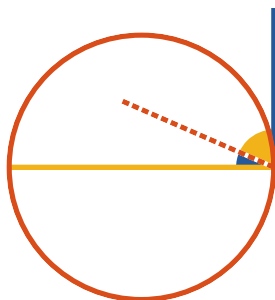
\therefore  = , a right angle;

but  =  (hyp.),

and \therefore  = , a part equal to the whole, which is absurd.

Therefore the assumed point is not the centre; and in the same manner it can be demonstrated, that no other point without  is the centre.

Q. E. D.





HE angle at the centre of a circle is double the angle at the circumference, when they have the same part of the circumference for their base.

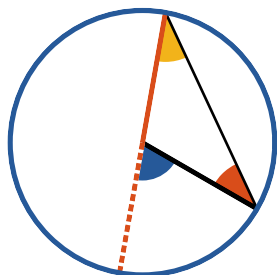




Figure I.

Let the centre of the circle be on  a side of .

$$\therefore \text{—} = \text{—},$$

$$\text{yellow triangle} = \text{red triangle} \text{ (pr. 1.5).}$$

$$\text{But } \text{blue triangle} = \text{yellow triangle} + \text{red triangle},$$

$$\text{or } \text{blue triangle} = \text{twice yellow triangle} \text{ (pr. 1.32).}$$

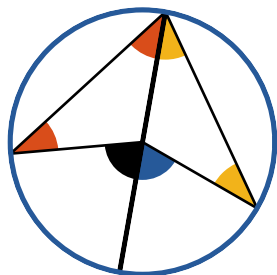








Figure II.

Let the centre be within ,
the angle at the circumference;

draw  from the angular point
through the centre of the circle;

then  = , and  = , because
of the equality of the sides (pr. 1.5).



$$\text{Hence } \text{red triangle} + \text{red triangle} + \text{yellow triangle} + \text{yellow triangle} = \text{twice } \text{yellow triangle}.$$

$$\text{But } \text{black triangle} = \text{red triangle} + \text{red triangle},$$

$$\text{and } \text{blue triangle} = \text{yellow triangle} + \text{yellow triangle},$$

$$\therefore \text{black triangle} + \text{blue triangle} = \text{twice } \text{yellow triangle}.$$

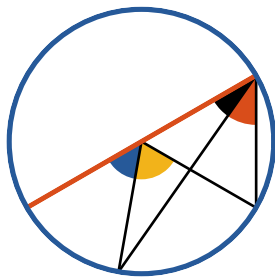
Figure III.

Let the centre be without 
and draw , the diameter.

$$\therefore \text{blue triangle} + \text{yellow triangle} = \text{twice black triangle};$$

$$\text{and blue triangle} = \text{twice black triangle (case I.)};$$

$$\therefore \text{yellow triangle} = \text{twice red triangle}.$$



Q. E. D.



THE angles in the same segment of a circle are equal.

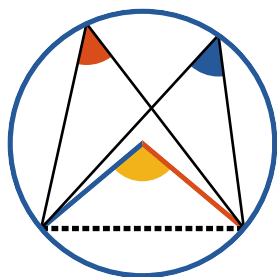


Figure I.

Let the segment be greater than a semicircle, and draw
— and — to the centre.

$$\text{yellow sector} = \text{twice } \triangle \text{ (red)} \text{ or twice } = \triangle \text{ (blue)} \text{ (pr. 3.20);}$$

$$\therefore \triangle \text{ (red)} = \triangle \text{ (blue)}$$

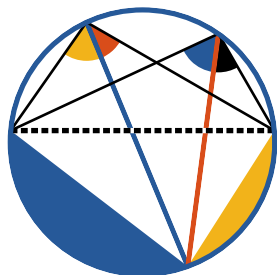


Figure II.





Let the segment be a semicircle, or less than a semicircle,
 draw — the diameter, also draw —.



$$\text{yellow sector} = \triangle \text{ (blue)} \text{ and } \triangle \text{ (red)} = \triangle \text{ (black)} \text{ (case I.)}$$

$$\therefore \text{yellow sector} = \triangle \text{ (blue)} + \triangle \text{ (black)}.$$

Q. E. D.



THE opposite angles  and ,  and  of any quadrilateral figure inscribed in a circle, are together equal to two right angles.

Draw  and  the diagonals;
and because angles in the same segment are equal

$$\begin{array}{c} \text{blue sector} = \text{blue sector}, \\ \text{and red sector} = \text{red sector}; \end{array}$$

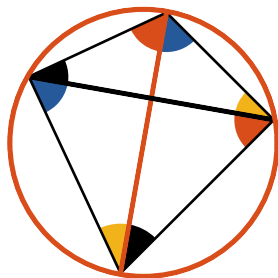
add  to both.

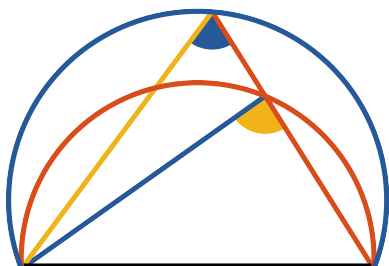
$$\begin{array}{c} \text{red and blue sectors} + \text{yellow and black sectors} = \text{yellow and black sectors} + \text{blue sector} + \text{red sector} = \text{two right angles} \\ \text{(pr. I.32).} \end{array}$$

In like manner it may be shown that,

$$\text{blue and black sectors} + \text{yellow and red sectors} = \text{semicircle}.$$

Q. E. D.





PON the same straight line, and upon the same side of it, two similar segments of circles cannot be constructed, which do not coincide.

For if it be possible, let two similar segments





draw any right line  cutting both the segments,

draw  and .

Because the segments are similar,



but  $>$  (pr. 1.16)
which is absurd;













therefore no point in either of the segments falls without the other, and therefore the segments coincide.

Q. E. D.



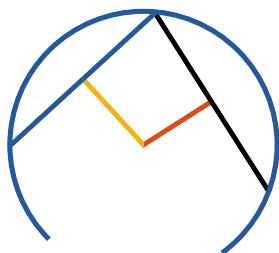
SIMILAR segments  and ,
of circles upon equal straight lines (
and ) are each equal to the other.



For, if  be so applied to ,
that  may fall on ,
the extremities of  may
be on the extremities 
and  at the same side as ;
 \therefore  = ,
 must wholly coincide with ;

and the similar segments being then upon the same straight line and at the same side of it, must also coincide (pr. 3.23), and are therefore equal.

Q. E. D.







SEGMENT of a circle being given, to describe the circle of which it is the segment.





From any point in the segment
draw ——— and ——— ,
biseĉt them, and from the points of biseĉtion
draw ——— \perp ———
and ——— \perp ———
where they meet is the centre of the circle.

Because ——— terminated in the circle is biseĉted perpendicularly by ——— , it passes through the centre (pr. 3.1), likewise ——— passes through the centre, therefore the centre is in the intersection of these perpendiculars.

Q. E. D.







*I*N equal circles  and , the arcs ,  on which stand equal angles, whether at the centre of circumference, are equal.

For, let  =  at the centre,
draw  and .

Then since  = ,



 and  have

 =  =  = ,

and  = ,

\therefore  =  (pr. 1.4).

But  =  (pr. 3.20);

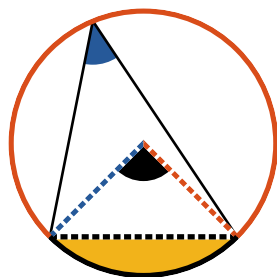
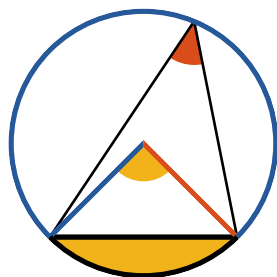
\therefore  and  are similar (def. 3.11);
they are also equal (pr. 3.24)

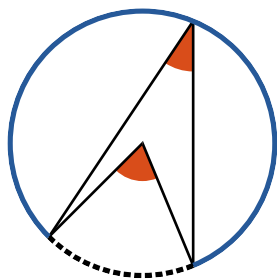
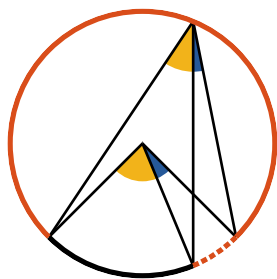
If therefore the equal segments be taken from the equal circles, the remaining segments will be equal;





hence  =  (ax. III);
and \therefore  = .

But if the given equal angles be at the circumference, it is evident that the angles at the centre, being double of those at the circumference, are also equal, and therefore the arcs on which they stand are equal.



Q. E. D.









IN equal circles  and  the angles  and  which stand upon equal arches are equal, whether they be at the centres or at the circumferences.



For if it be possible, let one of them

 be greater than the other 

and make  = 

\therefore  =  (pr. 3.26)

but  =  (hyp.)

\therefore  =  a part equal to the whole, which is absurd;





\therefore neither angle is greater than the other,



and \therefore they are equal.





Q. E. D.





Nequal circles  and , equal
chords ,  cut off equal arches.

From the centres of the equal circles,
draw ,  and , ;

and \therefore  = 

,  = , 

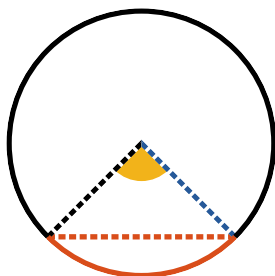
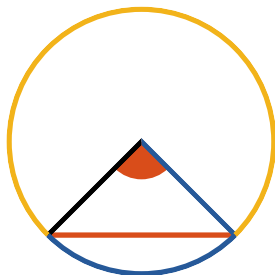
also  =  (hyp.)

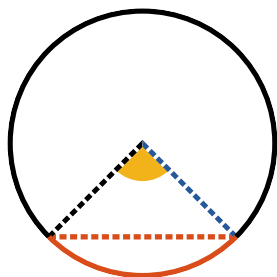
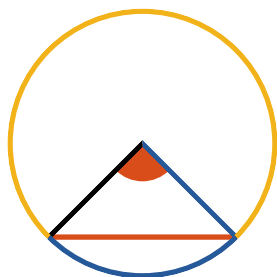
\therefore  = 





\therefore  =  (pr. 3.26)

and \therefore  =  (ax. III).





Q. E. D.





IN equal circles  and  the
chords  and  which subtend
equal arcs are equal.

If the equal arcs be semicircles the proposition is evident. But if not,

let ,  and ,
 be drawn to the centres;

\therefore  =  (hyp.)

and  =  (pr. 3.27);


but  and  =  and 

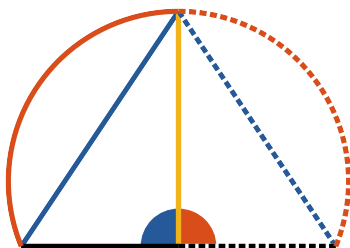
\therefore  =  (pr. I.4);


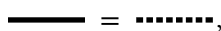



but these are the chords subtending the equal arcs.

Q. E. D.




o *bisect a given arc* 



Draw ;
 make  = ,
 draw  \perp , and it bisects the arc.

Draw  and .

 =  (const.)

 is common,

and  =  (const.)

\therefore  =  (pr. I.4)

 =  (pr. 3.28),

and therefore the given arc is bisected.

Q. E. D.



IN a circle the angle in a semicircle is a right angle, the angle in a segment greater than a semicircle is acute, and the angle in a segment less than a semicircle is obtuse.

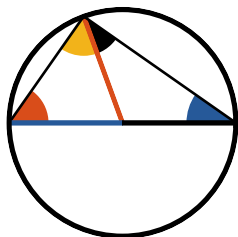



Figure I.

The angle  in a semicircle is a right angle.

Draw  and 

$$\begin{aligned}
 & \text{orange triangle} = \text{yellow triangle} \text{ and } \text{blue triangle} = \text{black triangle} \quad (\text{pr. I.5}) \\
 & \text{blue triangle} + \text{orange triangle} = \text{yellow triangle} = \text{the half of } \left(\text{semicircle} \right) = \text{right angle} \\
 & \quad \quad \quad (\text{pr. I.32})
 \end{aligned}$$

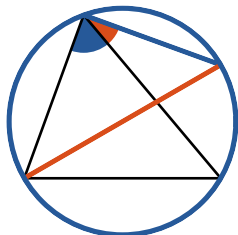




Figure II.

The angle  in a segment greater than a semicircle is acute

Draw  the diameter, and 

\therefore  = a right angle, \therefore  is acute.

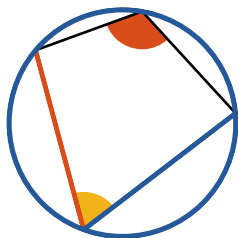





Figure III.

The angle  in a segment less than a semicircle is obtuse.

Take in the opposite circumference any point, to which draw  and .

$$\therefore \text{yellow triangle} + \text{red triangle} = \text{semicircle} \quad (\text{pr. 3.22})$$

but  <  (figure II.), \therefore  is obtuse.

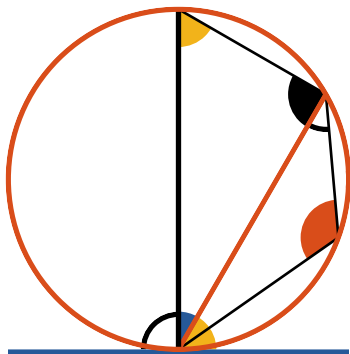
Q. E. D.



F a right line — be a tangent to a circle, and from the point of contact a right line — be drawn cutting the circle, the angle made by this line with the tangent is equal to the angle in the alternate segment of the circle.

If the chord should pass through the centre, it is evident the angles are equal, for each of them is a right angle. (pr. 3.16, pr. 3.31)

But if not, draw \perp from the point of contact, it must pass through the centre of the circle, (pr. 3.19)



$$\therefore \text{black sector} = \text{white sector} \quad (\text{pr. 3.31})$$

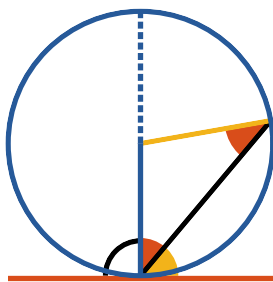
$$\text{yellow sector} + \text{blue sector} = \text{white sector} = \text{blue sector} + \text{yellow sector} \quad (\text{pr. 1.32})$$



$$\therefore \text{yellow sector} = \text{yellow sector} \quad (\text{ax. III}).$$

$$\text{Again } \text{white sector} = \text{white sector} = \text{yellow sector} + \text{red sector} \quad (\text{pr. 3.22})$$

$$\therefore \text{blue sector} = \text{red sector}, (\text{ax. III}), \text{ which is the angle in the alternate segment.}$$

Q. E. D.



IN a given straight line ——— to describe a segment of a circle that shall contain an angle equal to a given angle , , Δ .


If a given angle be a right angle, bisect the given line, and describe a semicircle on it, this will evidently contain a right angle. (pr. 3.31)


If the given angle be acute or obtuse, make with the given line, at its extremity,




$$\text{yellow triangle} = \Delta,$$

draw ——— \perp ———

and make  = ,



describe  with ——— or ——— as radius, for they are equal.


————— is tangent to  (pr. 3.16)

\therefore ——— divides the circle into two segments capable of containing angles equal to  and  which were made respectively equal to  and Δ (pr. 3.32).

Q. E. D.




Let \odot cut off from a given circle  a segment which shall contain an angle equal to a given angle .

Draw  (pr. 3.17), a tangent to the circle at any point;

at the point of contact make

 =  the given angle;

and  contains an angle = the given angle.



Because  is a tangent,
and  cuts it,

the angle in  =  (pr. 3.32),

but  =  (const.).

Q. E. D.



IF two chords {   } in a circle intersect each other, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

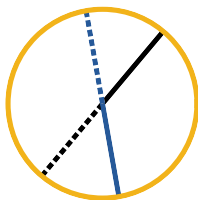


Figure I.

If the given right lines pass through the centre, they are bisected in the point of intersection, hence the rectangles under their segments are squares of their halves, and therefore equal.

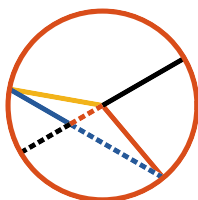















Figure II.

Let   pass through the centre,
and  not; draw  and .

Then  \times  = ² - ²
(pr. 2.6), or  \times  = ² - ².

\therefore  \times  =   (pr. 2.5).

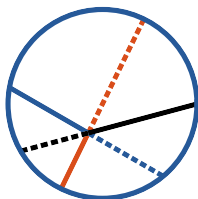















Figure III.

Let neither of the given lines pass through the centre, draw through their intersection a diameter ,

and  \times  =  \times  (part 2.),

also  \times  =  \times  (part 2.);

\therefore  \times  =  \times .

Q. E. D.










From a point without a circle two straight lines be drawn to it, one of which  is a tangent to the circle, and the other  cuts it; the rectangle under the whole cutting line  and the external segment  is equal to the square of the tangent .

Figure I.

Let  pass through the centre;
draw  from the centre to the point of contact;

$$\text{blue}^2 = \text{black dotted}^2 - \text{yellow}^2 \text{ (pr. 1.47),}$$

$$\text{or } \text{blue}^2 = \text{black dotted}^2 - \text{black dotted}^2,$$

$$\therefore \text{blue}^2 = \text{black dotted} \times \text{red} \text{ (pr. 2.6).}$$

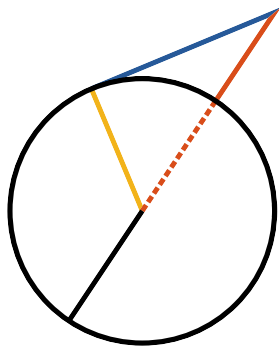



Figure II.

If  do not pass through the centre,

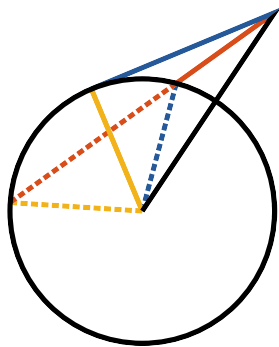
draw  and .

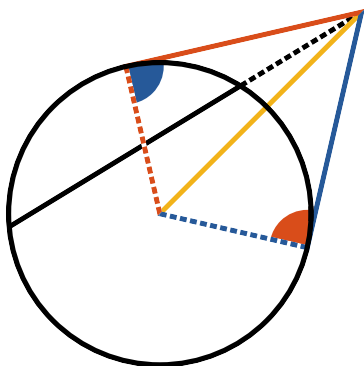
$$\text{Then } \text{black dotted} \times \text{red} = \text{black}^2 - \text{blue dotted}^2 \text{ (pr. 2.6),}$$

$$\text{that is, } \text{black dotted} \times \text{red} = \text{black}^2 - \text{yellow}^2,$$

$$\therefore \text{black dotted} \times \text{red} = \text{blue}^2 \text{ (pr. 3.18).}$$

Q. E. D.





From a point outside of a circle two straight lines be drawn, the one ———— cutting the circle, the other ———— meeting it, and if the rectangle contained by the whole cutting line ———— and its external segment ———— be equal to the square of the line meeting the circle, the latter ———— is a tangent to the circle.

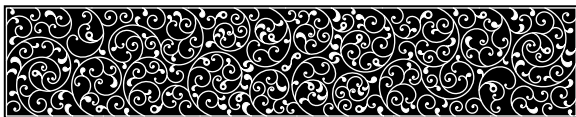
Draw from the given point
 ————, a tangent to the circle,
 and draw from the centre ————, ————, and ————,
 $————^2 = ———— \times ————$ (pr. 3.36)
 but $————^2 = ———— \times ————$ (hyp.),
 $\therefore ————^2 = ————^2$,
 and $\therefore ———— = ————$.



Then in ———— and ————,
 ———— and ———— = ———— and ————,
 and ———— is common,
 $\therefore \angle ———— = \angle ————$ (pr. 1.8);
 but $\angle ———— = \angle ————$ a right angle (pr. 3.18),

$\therefore \angle ———— = \angle ————$ a right angle,
 and $\therefore ————$ is a tangent to the circle (pr. 3.16).

Q. E. D.



Book IV

Definitions

I

A rectilinear figure is said to be *inscribed in* another, when all the angular points of the inscribed figure are on the sides of the figure in which it is said to be inscribed.



2

A figure is said to be *described about* another figure, when all the sides of the circumscribed figure pass through the angular points of the other figure.



3

A rectilinear figure is said to be *inscribed in* a circle, when the vertex of each angle of the figure is in the circumference of the circle.

4

A rectilinear figure is said to be *circumscribed about* a circle, when each of its sides is a tangent to the circle.

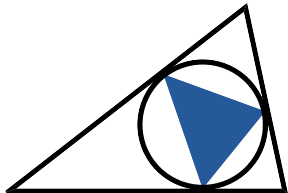


5



A circle is said to be *inscribed in* a rectilinear figure, when each side of the figure is a tangent to the circle.

6

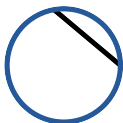


A circle is said to be *circumscribed about* a rectilinear figure, when the circumference passes through the vertex of each angle of the figure.



is circumscribed.

7





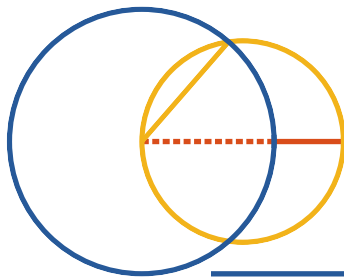
A straight line is said to be *inscribed in* a circle, when its extremities are in the circumference.



The Fourth Book of the Elements is devoted to the solution of problems, chiefly relating to the inscription and circumscription of regular polygons and circles.



A regular polygon is one whose angles and sides are equal.










N a given circle  to place a straight line,
equal to a given straight line () , not
greater than the diameter of the circle.






Draw , the diameter of  ;




and if  =  , then the problem is solved.

But if  be not equal to  ,
 >  (hyp.);

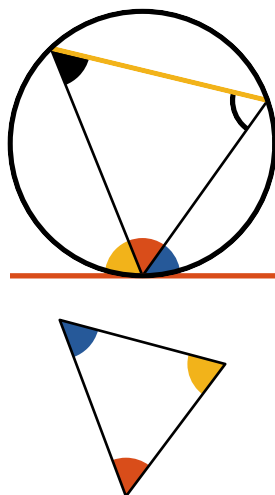
make  =  (pr. I.3)
with  as radius,


describe , cutting ,


and draw , which is the line required.



For  =  =  (def. I.15, const.).



Q. E. D.



N a given circle  to inscribe a triangle
equiangular to a given triangle.

To any point of the given circle
draw , a tangent (pr. 3.17);



and at the point of contact make  =  (pr. 1.23)

and in like manner  = ,

and draw .

\therefore  =  (const.)

and  =  (pr. 3.32)

\therefore  = ;

also  =  for the same reason.

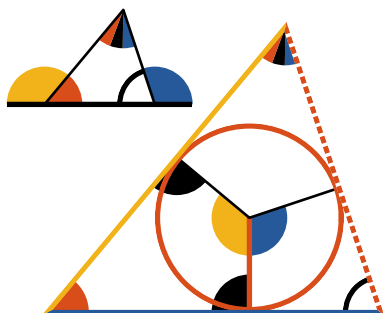
\therefore  =  (pr. 1.32),


and therefore the triangle inscribed in the circle is equian-
gular to the given one.

Q. E. D.



ABOUT a given circle  to circumscribe a triangle equiangular to a given triangle.






Produce any side , of the given triangle both ways;



from the centre of the given circle draw , any radius.

Make  =  (pr. 1.23)



and  = .





At the extremities of the radii, draw ,  and , tangents to the given circle. (pr. 3.17)

The four angles of , taken together are equal to four right angles. (pr. 1.32)

but  and  are right angles (const.)

\therefore  +  = , two right angles

but  =  (pr. 1.13)

and  =  (const.) and \therefore  = .

In the same manner it can be demonstrated that

$$\triangle = \triangle;$$











$$\therefore \triangle ABC = \triangle DEF \text{ (pr. I.32)}$$

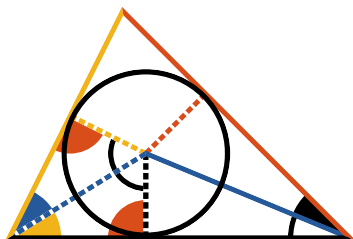
and therefore the triangle circumscribed about the given circle is equiangular to the given triangle.

Q. E. D.





In a given triangle  to inscribe a circle.

Bisect  and  (pr. 1.9) by  and ; from the point where these lines meet draw ,  and  respectively perpendicular to ,  and .



 = ,  =  and  common,
 \therefore  =  (pr. 1.4, pr. 1.26).

In like manner, it may be shown
 also that  = ,

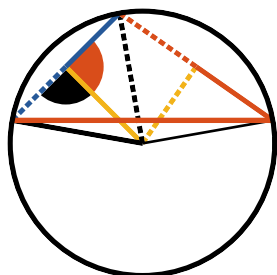
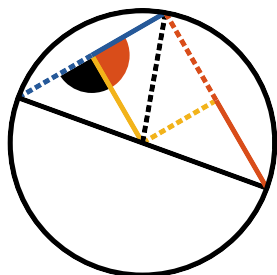
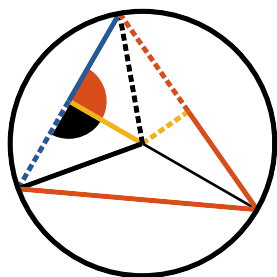
\therefore  =  = .

hence with any one of these lines as radius, describe



and it will pass through the extremities of the other two; and the sides of the given triangle, being perpendicular to the three radii at their extremities, touch the circle (pr. 3.16), which is therefore inscribed in the given triangle.

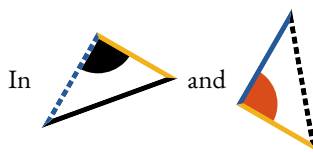
Q. E. D.



o describe a circle about a given triangle.

Make — = ---- and — = ----
(pr. 1.10)

From the points of bisection draw — and ---- \perp — and — respectively (pr. 1.11), and from their point of concurrence draw —, ---- and — and describe a circle with any one of them, and it will be the circle required.



---- = — (const.),

— common,

— = — (const.),

\therefore ---- = — (pr. 1.4).

In like manner it may be shown
that — = ----.

\therefore ---- = — = —; and therefore
a circle described from the concurrence of these three lines
with any one of them as a radius will circumscribe the given
triangle.




Q. E. D.



IN a given circle





to inscribe a square.

Draw the two diameters of the circle \perp to each other, and draw , ,  and .





is a square.

For, since  and  are, each of them, in a semicircle, they are right angles (pr. 3.31),

\therefore  \parallel  (pr. 1.28):


and in like manner  \parallel .


And \because  =  (const.),

and  =  =  (def. 15).

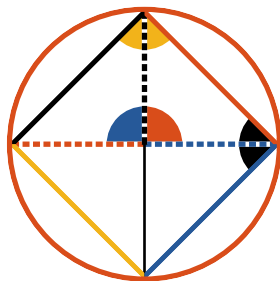
\therefore  =  (pr. 1.4);

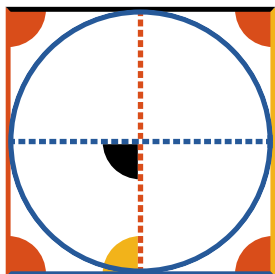
and since the adjacent sides and angles of the parallelo-

gram  are equal, they are all equal (pr. 1.34); and





\therefore , inscribed in the given circle, is a square.

Q. E. D.
















ABOUT a given circle  to circumscribe a square.


Draw two diameters of the given circle perpendicular to each other, and through their extremities draw , ,  and  tangents to the circle;

and  is a square.





 =  a right angle, (pr. 3.18)

also  =  (const.),

\therefore  \parallel ; in the same manner it can be demonstrated that  \parallel , and also that  and  \parallel .

\therefore  is a parallelogram,

and \because  =  =  =  = 
they are all right angles (pr. 1.34);

it is also evident that , ,  and  are equal.


\therefore  is a square.

Q. E. D.




o inscribe a circle in a given square.



Make $\text{---} = \text{---}$,
 and $\text{---} = \text{---}$,
 draw $\text{---} \parallel \text{---}$,
 and $\text{---} \parallel \text{---}$
 (pr. 1.31)

\therefore  is a parallelogram;

and since $\text{---} = \text{---}$ (hyp.)
 $\text{---} = \text{---}$

\therefore  is equilateral (pr. 1.34)

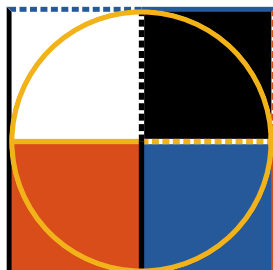
In like manner it can be shown that

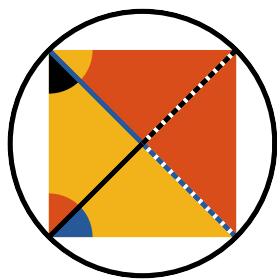
 =  are equilateral parallelograms;

$\therefore \text{---} = \text{---} = \text{---} = \text{---}$.

and therefore if a circle be described from the concourse of these lines with any one of them as radius, it will be inscribed in the given square (pr. 3.16).

Q. E. D.






Q describe a circle about a given square




Draw the diagonals  and  intersecting each other;

then, because  and  have their sides equal, and the base  common to both,

$$\text{yellow triangle} = \text{red triangle} \text{ (pr. 1.8),}$$

or  is bisected:

in like manner it can be shown that  is bisected;

$$\text{but } \text{yellow triangle} = \text{red triangle},$$

hence  =  their halves,

$$\therefore \text{black dashed line} = \text{blue dashed line} \text{ (pr. 1.6);}$$

and in like manner it can be proved that

$$\text{blue dashed line} = \text{black dashed line} = \text{dotted line} = \text{dashed line}.$$

If from the confluence of these lines with any one of them as radius, a circle be described, it will circumscribe the given square.

Q. E. D.



o construct an isosceles triangle, in which each of the angles at the base shall be double of the vertical angle.

Take any straight line and divide it so that
 \times = ² (pr. 2.11)

With as radius, describe
 and place in it from the extremity
 of the radius, = (pr. 4.1);
 draw .

Then is the required triangle.

For, draw and describe
 about (pr. 4.5)

Since \times = ² = ²,

\therefore is tangent to (pr. 3.37)

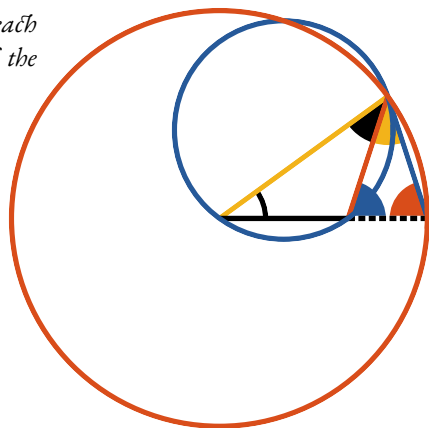
\therefore = (pr. 3.32),

add to each,

\therefore + = + ;

but + or + = (pr. 1.5):



since = (pr. 1.5)









consequently  =  +  =  (pr. I.32)

\therefore  =  (pr. I.6)

\therefore  =  =  (const.)

\therefore  =  (pr. I.5)

\therefore  =  =  =  +  = twice ;

and consequently each angle at the
base is double of the vertical angle.

Q. E. D.




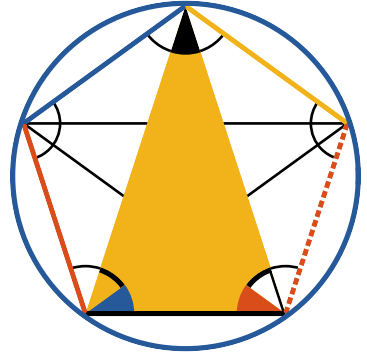
On a given circle  to inscribe an equilateral and equiangular pentagon.













Construct an isosceles triangle, in which each of the angles at the base shall be double of the angle at the vertex,



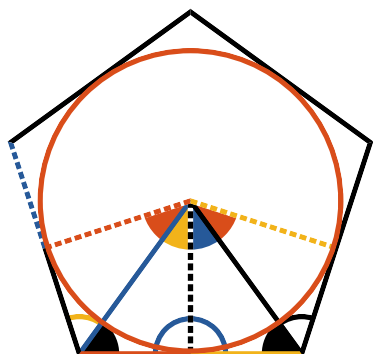
and inscribe in the given circle a triangle  equiangular to it (pr. 4.2);





Bisect  and  (pr. I.9),
draw , ,  and .

Because each of the angles , , ,  and  are equal, the arcs upon which they stand are equal (pr. 3.26); and \therefore , , ,  and  which subtend these arcs are equal (pr. 3.29) and \therefore the pentagon is equilateral, it is also equiangular, as each of its angles stand upon equal arcs (pr. 3.27).

Q. E. D.



o describe an equilateral and equiangular pentagon about a given circle .




Draw five tangents through the vertices of the angles of any regular pentagon inscribed in the given circle 
(pr. 3.17).





These five tangents will form the required pentagon.





Draw .

In  and 





 =  (pr. 1.47)

 = , and  common;






\therefore  =  and \therefore  =  (pr. 1.8)


\therefore  = twice , and  = twice .



In the same manner it can be demonstrated that

 = twice , and  = twice .



but  =  (pr. 3.27),

\therefore their halves  = , also  = ,
and  common;



\therefore  =  and  = ,

\therefore  = twice .





In the same manner it can be demonstrated

that  = twice ,



but  = 

\therefore  =  ;

In the same manner it can be demonstrated that the other sides are equal, and therefore the pentagon is equilateral, it is also equiangular, for

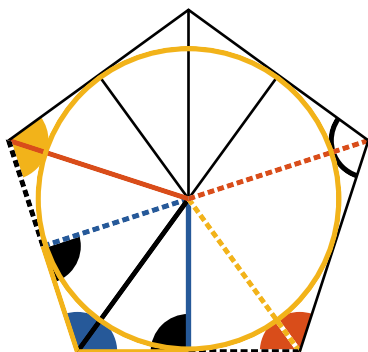
 = twice  and  = twice ,

and therefore  = ,

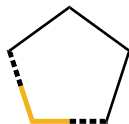
\therefore  =  ;


in the same manner it can be demonstrated that the other angles of the described pentagon are equal.





Q. E. D.












o inscribe a circle in a given equiangular and equilateral pentagon.

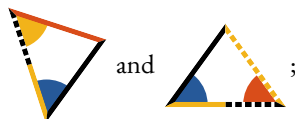




Let  be a given equiangular and equilateral pentagon; it is required to inscribe a circle in it.

Make  = , and  =  (pr. I.9)

Draw , , , , &c.



\therefore  = ,  = , and
 common to the two triangles



\therefore  =  and  =  (pr. I.4)


And \therefore  =  = twice 



\therefore twice , hence  is bisected by .

In like manner it may be demonstrated that  is bisected by , and that the remaining angle of the polygon is bisected in a similar manner.

Draw ———, , &c. perpendicular
to the sides of the pentagon.


Then in the two triangles  and 

we have  =  (const.), ——— common,

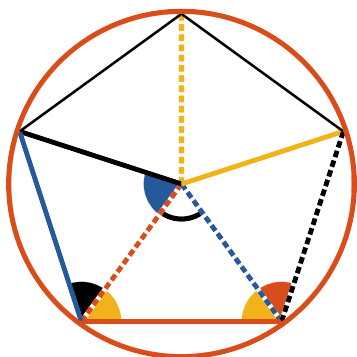
and  =  = a right angle ;

\therefore ——— = (pr. I.26)








In the same way it may be shown that the five perpendiculars on the sides of the pentagon are equal to one another.

Describe  with any one of the perpendiculars as radius, and it will be the inscribed circle required. For if it does not touch the sides of the pentagon, but cut them, then a line drawn from the extremity at right angles to the diameter of a circle will fall within the circle, which has been shown to be absurd (pr. 3.16).

Q. E. D.



o describe a circle about a given equilateral and equiangular pentagon.

Bisect  and  by  and , and from the point of section, draw ,  and .

$$\begin{aligned} \text{black sector} &= \text{yellow sector}, \\ \text{yellow sector} &= \text{yellow sector}, \end{aligned}$$

$$\therefore \text{dashed blue line} = \text{dashed red line} \text{ (pr. I.6);}$$

and since in  and ,

$$\text{solid blue line} = \text{solid red line}, \text{ and dashed red line common,}$$

$$\text{also black sector} = \text{yellow sector};$$

$$\therefore \text{solid black line} = \text{dashed blue line} \text{ (pr. I.4).}$$

In like manner it may be proved that

$$\text{dashed red line} = \text{solid yellow line} = \text{solid black line}.$$

and therefore

$$\text{dashed red line} = \text{solid black line} = \text{dashed red line} = \text{dashed blue line} = \text{solid yellow line}.$$

Therefore if a circle be described from the point where these five lines meet, with any one of them as a radius, it will circumscribe the given pentagon.

Q. E. D.



o *inscribe an equilateral and equiangular hexagon in a given circle*



From any point in the circumference of the given circle describe passing through its centre, and draw the diameters , and ; draw , , , &c. and the required hexagon is inscribed in the given circle.

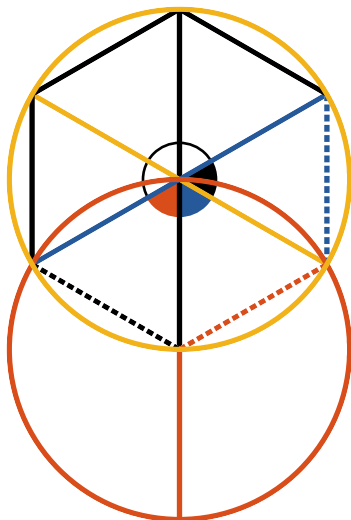
Since passes through the centres of the circles, and are equilateral triangles, hence

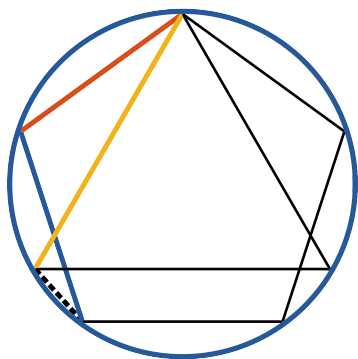
= = one-third of (pr. 1.32) but

= (pr. 1.13);

\therefore = = = one-third of
(pr. 1.32), and the angles vertically opposite to these are all equal to one another (pr. 1.15), and stand on equal arches (pr. 3.26), which are subtended by equal chords (pr. 3.29); and since each of the angles of the hexagon is double of the angle of an equilateral triangle, it is also equiangular.

Q. E. D.





o inscribe a circle in an equilateral and equiangular quindecagon in a given circle.

Let — and — be the sides of an equilateral pentagon inscribed in the given circle, and — the side of an inscribed equilateral triangle.

The arc subtended
by — and — } = $\frac{2}{5} = \frac{6}{15}$ { of the whole
circumference

The arc subtended
by — } = $\frac{1}{3} = \frac{5}{15}$ { of the whole
circumference

$$\text{Their difference} = \frac{1}{15}$$

\therefore the arc subtended by - - - - - = $\frac{1}{15}$ difference of the whole circumference.

Hence if straight lines equal to - - - - - be placed in the circle (pr. 4.1), an equilateral and equiangular quindecagon will be thus inscribed in the circle.

Q. E. D.



Book V

Definitions

I

A less magnitude is said to be an aliquot part or submultiple of a greater magnitude, when the less measures the greater; that is, when the less is contained a certain number of times exactly in the greater.

2

A greater magnitude is said to be a multiple of a less, when the greater is measured by the less; that is, when the greater contains the less a certain number of times exactly.

3

Ratio is the relation which one quantity bears to another of the same kind, with respect to magnitude.

4

Magnitudes are said to have a ratio to one another, when they are of the same kind; and the one which is not the greater can be multiplied so as to exceed the other.

*The other definitions will be given throughout
the book where their aid is first required.*

Axioms

I

Equimultiples or equisubmultiples of the same, or of equal magnitudes, are equal.

If $A = B$, then

twice $A =$ twice B ,

that is, $2A = 2B$;

$3A = 3B$;

$4A = 4B$;

&c. &c.

and $\frac{1}{2}$ of $A = \frac{1}{2}$ of B ;

$\frac{1}{3}$ of $A = \frac{1}{3}$ of B ;

$\frac{1}{4}$ of $A = \frac{1}{4}$ of B ;

&c. &c.

II

A multiple of a greater magnitude is greater than the same multiple of a less.

Let $A > B$,

then $2A > 2B$;

$3A > 3B$;

$4A > 4B$;

&c. &c.

III

That magnitude, of which a multiple is greater than the same multiple of another, is greater than the other.

Let $2A > 2B$,

then $A > B$;

or, let $3A > 3B$,

then $A > B$;

or, let $mA > mB$,

then $A > B$.



If any number of magnitudes be equimultiples of as many others, each of each: what multiple soever any one of the first is of its part, the same multiple shall of the first magnitudes taken together be of all the others taken together.

Let $\square\square\square\square\square$ be the same multiple of \square ,
 that $\blacklozenge\blacklozenge\blacklozenge\blacklozenge\blacklozenge$ is of \blacklozenge ,
 that $\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare$ is of \blacksquare .

Then is evident that

$\left. \begin{array}{c} \square\square\square\square\square \\ \blacklozenge\blacklozenge\blacklozenge\blacklozenge\blacklozenge \\ \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \end{array} \right\}$ is the same multiple of $\left\{ \begin{array}{c} \square \\ \blacklozenge \\ \blacksquare \end{array} \right\}$
 which that $\square\square\square\square\square$ is of \square ;

because there are as many magnitudes

in $\left\{ \begin{array}{c} \square\square\square\square\square \\ \blacklozenge\blacklozenge\blacklozenge\blacklozenge\blacklozenge \\ \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \end{array} \right\} = \left\{ \begin{array}{c} \square \\ \blacklozenge \\ \blacksquare \end{array} \right\}$

as there are in $\square\square\square\square\square = \square$.

The same demonstration holds in any number of magnitudes, which has here been applied to three.

\therefore If any number of magnitudes, &c.



If the first magnitude be the same multiple of the second that the third of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth, then shall the first, together with the fifth, be the same multiple of the second that the third, together with the sixth, is of the fourth.

Let $\bullet \bullet \bullet$, the first, be the same multiple of \bullet , the second, that $\diamond \diamond \diamond$, the third, is of \diamond , the fourth; and let $\bullet \bullet \bullet \bullet$, the fifth, be the same multiple of \bullet , the second, that $\diamond \diamond \diamond \diamond$, the sixth, is of \diamond , the fourth.

Then it is evident, that $\left\{ \begin{array}{c} \bullet \bullet \bullet \\ \bullet \bullet \bullet \bullet \end{array} \right\}$, the first and fifth together, is the same multiple of \bullet , the second, that $\left\{ \begin{array}{c} \diamond \diamond \diamond \\ \diamond \diamond \diamond \diamond \end{array} \right\}$, the third and sixth together, is of the same multiple of \diamond , the fourth; because there are as many magnitudes in $\left\{ \begin{array}{c} \bullet \bullet \bullet \bullet \\ \bullet \bullet \bullet \bullet \end{array} \right\} = \bullet$ as there are in $\left\{ \begin{array}{c} \diamond \diamond \diamond \\ \diamond \diamond \diamond \diamond \end{array} \right\} = \diamond$.

\therefore If the first magnitude, &c.



If the first of four magnitudes be the same multiple of the second that the third is to the fourth, and if any equimultiples whatever of the first and third be taken, those shall be equimultiples; one of the second, and the other of the fourth.

Let $\left\{ \begin{array}{c} \text{■} \\ \text{■} \text{ ■} \\ \text{■} \end{array} \right\}$ be the same multiple of ■

which $\left\{ \begin{array}{c} \text{◆} \text{ ◆} \\ \text{◆} \text{ ◆} \end{array} \right\}$ is of ◆ ;

take $\left\{ \begin{array}{c} \text{■} \text{ ■} \text{ ■} \text{ ■} \\ \text{■} \text{ ■} \text{ ■} \text{ ■} \\ \text{■} \text{ ■} \text{ ■} \text{ ■} \\ \text{■} \text{ ■} \text{ ■} \text{ ■} \end{array} \right\}$ the same multiple of $\left\{ \begin{array}{c} \text{■} \\ \text{■} \text{ ■} \\ \text{■} \end{array} \right\}$,

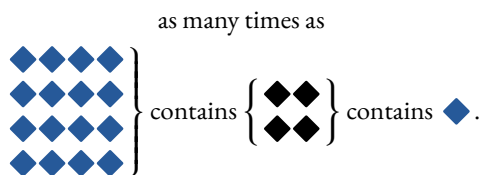
which $\left\{ \begin{array}{c} \text{◆} \text{ ◆} \text{ ◆} \text{ ◆} \\ \text{◆} \text{ ◆} \text{ ◆} \text{ ◆} \\ \text{◆} \text{ ◆} \text{ ◆} \text{ ◆} \\ \text{◆} \text{ ◆} \text{ ◆} \text{ ◆} \end{array} \right\}$ is of $\left\{ \begin{array}{c} \text{◆} \text{ ◆} \\ \text{◆} \text{ ◆} \end{array} \right\}$.

Then it is evident,

that $\left\{ \begin{array}{c} \text{■} \text{ ■} \text{ ■} \text{ ■} \\ \text{■} \text{ ■} \text{ ■} \text{ ■} \\ \text{■} \text{ ■} \text{ ■} \text{ ■} \\ \text{■} \text{ ■} \text{ ■} \text{ ■} \end{array} \right\}$ is the same multiple of ■

which $\left\{ \begin{array}{c} \text{◆} \text{ ◆} \text{ ◆} \text{ ◆} \\ \text{◆} \text{ ◆} \text{ ◆} \text{ ◆} \\ \text{◆} \text{ ◆} \text{ ◆} \text{ ◆} \\ \text{◆} \text{ ◆} \text{ ◆} \text{ ◆} \end{array} \right\}$ is of ◆ ;

$\therefore \left\{ \begin{array}{c} \text{■} \text{ ■} \text{ ■} \text{ ■} \\ \text{■} \text{ ■} \text{ ■} \text{ ■} \\ \text{■} \text{ ■} \text{ ■} \text{ ■} \\ \text{■} \text{ ■} \text{ ■} \text{ ■} \end{array} \right\}$ contains $\left\{ \begin{array}{c} \text{■} \\ \text{■} \text{ ■} \\ \text{■} \end{array} \right\}$ contains ■

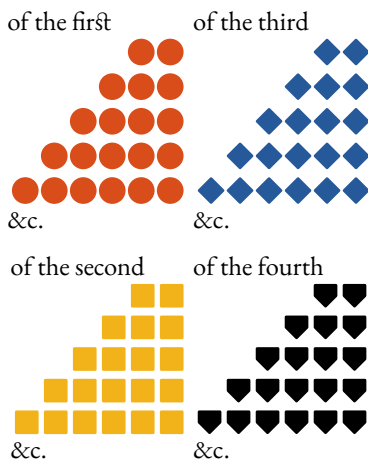


The same reasoning is applicable in all cases.

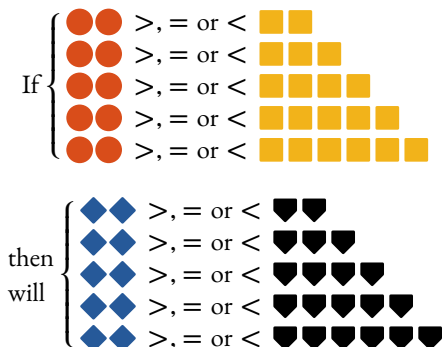
\therefore If the first of four, &c.

Definition V.

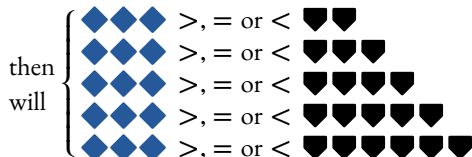
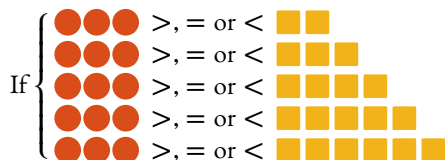
Four magnitudes, \bullet , \blacksquare , \blacklozenge , \blacktriangledown , are said to be proportionals when every equimultiple of the first and third be taken, and every equimultiple of the second and fourth, as,



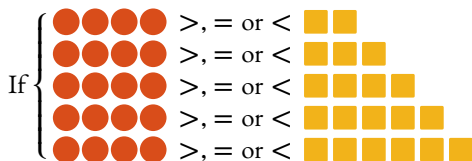
Then taking every pair of equimultiples of the first and third, and every pair of equimultiples of the second and fourth,

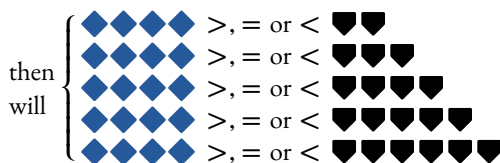


That is, if twice the first be greater, equal, or less than twice the second, twice the third will be greater, equal, or less than twice the fourth; or, if twice the first be greater, equal, or less than three times the second, twice the third will be greater, equal, or less than three times the fourth, and so on, as above expressed.



In other terms, if three times the first be greater, equal, or less than twice the second, three times the third will be greater, equal, or less than twice the fourth; or, if three times the first be greater, equal, or less than three times the second, then will three times the third be greater, equal, or less than three times the fourth; or if three times the first be greater, equal, or less than four times the second, then will three times the third be greater, equal, or less than four times the fourth, and so on. Again,









And so on, with any other equimultiples of the four magnitudes, taken in the same manner.





Euclid expresses this definition as follows :—





The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, and the multiple of the third is also less than that of the fourth; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.





In future we shall express this definition generally, thus:

If M , $>, =$ or $< m$ ,

when M , $>, =$ or $< m$ ,

Then we infer that , the first, has the same ratio to , the second, which , the third, has to  the fourth; expressed in the succeeding demonstrations thus:

 :  ::  :  ;



or thus,  :  =  :  ;



or thus, $\frac{\text{red circle}}{\text{yellow square}} = \frac{\text{blue diamond}}{\text{black heart}}$:




and is read,

“as  is to , so is  to .

And if  :  ::  :  we shall infer if

M  $>, =$ or $< m$ , then will

M  $>, =$ or $< m$ .

That is, if the first be to the second, as the third is to the fourth; then if M times the first be greater than, equal to, or less than m times the second, then shall M times third be greater than, equal to, or less than m times the fourth, in which M and m are not to be considered particular multiples, but every pair of multiples whatever; nor are such marks as , , , &c. to be considered any more than representatives of geometrical magnitudes.

The student should thoroughly understand this definition before proceeding further.



If the first of four magnitudes have the same ratio to the second, which the third has to the fourth, then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth; viz., the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.

Let $\text{yellow circle} : \text{black square} :: \text{red diamond} : \text{blue inverted triangle}$, then $3 \text{ yellow circle} : 2 \text{ black square} :: 3 \text{ red diamond} : 2 \text{ blue inverted triangle}$, every equimultiple of 3 yellow circle and 3 red diamond are equimultiples of yellow circle and red diamond , and every equimultiple of 2 black square and $2 \text{ blue inverted triangle}$, are equimultiples of black square and $\text{blue inverted triangle}$ (pr. 5.3)

That is, M times 3 yellow circle and M times 3 red diamond are equimultiples of yellow circle and red diamond , and m times 2 black square and $m2 \text{ blue inverted triangle}$ are equimultiples of 2 black square and $2 \text{ blue inverted triangle}$; but $\text{yellow circle} : \text{black square} :: \text{red diamond} : \text{blue inverted triangle}$ (hyp.); \therefore if $M3 \text{ yellow circle} <, =, \text{ or } > m2 \text{ black square}$, then $M3 \text{ red diamond} <, =, \text{ or } > m2 \text{ blue inverted triangle}$ (def. 5.5) and therefore $3 \text{ yellow circle} : 2 \text{ black square} :: 3 \text{ red diamond} : 2 \text{ blue inverted triangle}$ (def. 5.5)

The same reasoning holds good if any other equimultiple of the first and third be taken, any other equimultiple of the second and fourth.

\therefore If the first four magnitudes, &c.



If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other, the remainder shall be the same multiple of the remainder, that the whole is of the whole.

$$\text{Let } \begin{array}{c} \circ \\ \circ \circ \end{array} = M' \begin{array}{c} \blacktriangle \\ \blacksquare \end{array}$$

$$\text{and } \begin{array}{c} \square \\ \square \end{array} = M' \blacksquare,$$

$$\therefore \begin{array}{c} \circ \\ \circ \circ \\ \square \end{array} - \begin{array}{c} \square \\ \square \end{array} = M' \begin{array}{c} \blacktriangle \\ \blacksquare \end{array} - M' \blacksquare,$$

$$\therefore \begin{array}{c} \circ \\ \circ \circ \end{array} = M' (\begin{array}{c} \blacktriangle \\ \blacksquare \end{array} - \blacksquare),$$

$$\text{and } \therefore \begin{array}{c} \circ \\ \circ \circ \end{array} = M' \blacktriangle.$$

\therefore If one magnitude, &c.



If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two, the remainders are either equal to these others, or equimultiples of them.

Let $\begin{array}{c} \diamond \\ \diamond \diamond \\ \diamond \end{array} = M' \blacksquare$; and $\square\square = M' \blacktriangle$

then $\begin{array}{c} \diamond \\ \diamond \diamond \\ \diamond \end{array} - m' \blacksquare = M' \blacksquare - m' \blacksquare = (M' - m') \blacksquare$

and $\square\square - m' \blacktriangle = M' \blacktriangle - m' \blacktriangle = (M' - m') \blacktriangle$.

Hence, $(M' - m') \blacksquare$ and $(M' - m') \blacktriangle$ are equimultiples of \blacksquare and \blacktriangle , and equal to \blacksquare and \blacktriangle , when $M' - m' = 1$.

\therefore If two magnitudes be equimultiples, &c.



If the first of the four magnitudes has the same ratio to the second which the third has to the fourth, then if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less.

Let \bullet : \blacksquare :: \blacktriangledown : \blacklozenge ; therefore, by the fifth definition, if $\bullet\bullet > \blacksquare\blacksquare$, then will $\blacktriangledown\blacktriangledown > \blacklozenge\blacklozenge$;

but if $\bullet > \blacksquare$, then
 $\bullet\bullet > \blacksquare\blacksquare$ and $\blacktriangledown\blacktriangledown > \blacklozenge\blacklozenge$,
 and $\therefore \blacktriangledown > \blacklozenge$.

Similarly, if $\bullet =$, or $< \blacksquare$,
 then will $\blacktriangledown =$, or $< \blacklozenge$.

\therefore If the first of four, &c.

Definition XIV

Geometricians make use of the technical term “Invertendo,” by inversion, when there are four proportionals, and it is inferred, that the second is to the first as the fourth to the third.

Let $A : B :: C : D$, then, by “invertendo” it is inferred
 $B : A :: D : C$



If four magnitudes are proportionals, they are proportionals also when taken inversely.

Let $\heartsuit : \spadesuit :: \clubsuit : \diamondsuit$,
then, inversely, $\spadesuit : \heartsuit :: \diamondsuit : \clubsuit$.

If $M\heartsuit < m\spadesuit$, then $M\clubsuit < m\diamondsuit$
by the fifth definition.

Let $M\heartsuit < m\spadesuit$, that is, $m\spadesuit > M\heartsuit$,
 $\therefore M\clubsuit < m\diamondsuit$, or, $m\diamondsuit > M\clubsuit$;

\therefore if $m\spadesuit > M\heartsuit$, then will $m\diamondsuit > M\clubsuit$.





In the same manner it may be shown,
that if $m\spadesuit =$ or $< M\heartsuit$,
then will $m\diamondsuit =$, or $< M\clubsuit$;

and therefore, by the fifth definition, we infer
that $\spadesuit : \heartsuit :: \diamondsuit : \clubsuit$.





\therefore If four magnitudes, &c.









F the first be the same multiple of the second, or the same part of it, that the third is of the fourth; the first is to the second, as the third is to the fourth.

Let , the first, be the same
multiple of , the second,
that , the third, is of , the fourth.

Then  :  ::  : 



take M , m , M , m 

because  is the same multiple of 
that  is of  (according to the hypothesis);



and M  if taken the same multiple of 

that M  is of ,



\therefore (according to the third proposition),



M  is the same multiple of 

that M  is of .

Therefore, if M  be of \bullet a
greater multiple than m \bullet is,
then M  is a greater multiple of \blacktriangle than m \blacktriangle is;


that is, if M  be greater than m \bullet ,
then M  will be greater than m \blacktriangle ;

in the same manner it can be shown, if M 
be equal m \bullet , then M  will be equal m \blacktriangle .

And, generally, if M  $>$, $=$ or $<$ m \bullet
than M  will be $>$, $=$ or $<$ m \blacktriangle ;



\therefore by the fifth definition,



$$\begin{array}{c} \text{blue squares} \\ \text{2x2 grid} \end{array} : \bullet :: \begin{array}{c} \text{yellow diamonds} \\ \text{2x2 grid} \end{array} : \blacktriangle$$

Next, let \bullet be the same part of 

that \blacktriangle is of .

In this case also $\bullet : \begin{array}{c} \text{blue squares} \\ \text{2x2 grid} \end{array} :: \blacktriangle : \begin{array}{c} \text{yellow diamonds} \\ \text{2x2 grid} \end{array}$.

For, because \bullet is the same part
of  that \blacktriangle is of ,

therefore  is the same multiple of 

that  is of .

Therefore, by the preceding case,

 :  ::  :  ;

and \therefore  :  ::  :  ,
by proposition B.

\therefore If the first be the same multiple, &c.



If the first be to the second as the third to the fourth, and if the first be a multiple, or a part of the second; the third is the same multiple, or the same part of the fourth.

Let $\begin{array}{c} \bullet \\ \bullet \bullet \end{array} : \blacksquare :: \begin{array}{c} \color{red}\blacklozenge \color{red}\blacklozenge \\ \color{red}\blacklozenge \color{red}\blacklozenge \end{array} : \color{blue}\blacktriangledown;$

and first, let $\begin{array}{c} \bullet \\ \bullet \bullet \end{array}$ be a multiple \blacksquare ;

$\begin{array}{c} \color{red}\blacklozenge \color{red}\blacklozenge \\ \color{red}\blacklozenge \color{red}\blacklozenge \end{array}$ shall be the same multiple of $\color{blue}\blacktriangledown$.

First	Second	Third	Fourth
$\begin{array}{c} \bullet \\ \bullet \bullet \end{array}$	\blacksquare	$\begin{array}{c} \color{red}\blacklozenge \color{red}\blacklozenge \\ \color{red}\blacklozenge \color{red}\blacklozenge \end{array}$	$\color{blue}\blacktriangledown$
	$\begin{array}{c} \color{red}\cup \\ \color{red}\cup \cup \end{array}$	$\begin{array}{c} \color{black}\cup \color{black}\cup \\ \color{black}\cup \color{black}\cup \end{array}$	

Take $\begin{array}{c} \color{red}\cup \\ \color{red}\cup \cup \end{array} = \begin{array}{c} \bullet \\ \bullet \bullet \end{array}.$

Whatever multiple $\begin{array}{c} \bullet \\ \bullet \bullet \end{array}$ is of \blacksquare

take $\begin{array}{c} \color{black}\cup \color{black}\cup \\ \color{black}\cup \color{black}\cup \end{array}$ the same multiple of $\color{blue}\blacktriangledown$,

then, $\therefore \begin{array}{c} \bullet \\ \bullet \bullet \end{array} : \blacksquare :: \begin{array}{c} \color{red}\blacklozenge \color{red}\blacklozenge \\ \color{red}\blacklozenge \color{red}\blacklozenge \end{array} : \color{blue}\blacktriangledown$

and of the second and fourth,
we have taken equimultiples,

$\begin{array}{c} \bullet \\ \bullet \bullet \end{array}$ and $\begin{array}{c} \color{black}\cup \color{black}\cup \\ \color{black}\cup \color{black}\cup \end{array}$ therefore (pr. 5.4)
 $\begin{array}{c} \bullet \\ \bullet \bullet \end{array} : \begin{array}{c} \color{red}\cup \\ \color{red}\cup \cup \end{array} :: \begin{array}{c} \color{red}\blacklozenge \color{red}\blacklozenge \\ \color{red}\blacklozenge \color{red}\blacklozenge \end{array} : \begin{array}{c} \color{black}\cup \color{black}\cup \\ \color{black}\cup \color{black}\cup \end{array},$ but (const.),

$$\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} = \begin{array}{c} \square \\ \square \\ \square \end{array} \therefore (\text{pr. 5.A}) \quad \begin{array}{c} \blacklozenge \\ \blacklozenge \\ \blacklozenge \end{array} = \begin{array}{c} \circ \\ \circ \\ \circ \end{array}$$

and $\begin{array}{c} \circ \\ \circ \\ \circ \end{array}$ is the same multiple of \blacklozenge

that $\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}$ is of \blacksquare .

Next, let $\blacksquare : \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} :: \blacklozenge : \begin{array}{c} \blacklozenge \\ \blacklozenge \\ \blacklozenge \end{array}$,

and also \blacksquare a part of $\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}$;

then \blacklozenge shall be the same part of $\begin{array}{c} \blacklozenge \\ \blacklozenge \\ \blacklozenge \end{array}$.

Inversely (pr. 5.B), $\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} : \blacksquare :: \begin{array}{c} \blacklozenge \\ \blacklozenge \\ \blacklozenge \end{array} : \blacklozenge$,

but \blacksquare is a part of $\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}$;

that is, $\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}$ is a multiple of \blacksquare ;

\therefore by the preceding case, $\begin{array}{c} \blacklozenge \\ \blacklozenge \\ \blacklozenge \end{array}$ is the same multiple of \blacklozenge

that is, \blacklozenge is the same part of $\begin{array}{c} \blacklozenge \\ \blacklozenge \\ \blacklozenge \end{array}$

that \blacksquare is of $\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}$.

\therefore if the first be to the second, &c.



QUAL magnitudes have the same ratio to the same magnitude, and the same has the same ratio to equal magnitudes.

Let \bullet = \blacklozenge and \blacksquare any other magnitude;
then $\bullet : \blacksquare = \blacklozenge : \blacksquare$ and $\blacksquare : \bullet = \blacksquare : \blacklozenge$.

$$\therefore \bullet = \blacklozenge,$$

$$\therefore M\bullet = M\blacklozenge;$$

\therefore if $M\bullet >, =$ or $< m\blacksquare$, then
 $M\blacklozenge >, =$ or $< m\blacksquare$,
and $\therefore \bullet : \blacksquare = \blacklozenge : \blacksquare$ (def. 5.5).

From the foregoing reasoning it is evident that,
if $m\blacksquare >, =$ or $< M\bullet$, then
 $m\blacksquare >, =$ or $< M\blacklozenge$
 $\therefore \blacksquare : \bullet = \blacksquare : \blacklozenge$ (def. 5.5).

\therefore Equal magnitudes, &c.

Definition VII.

When of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the first is greater than that of the second, but multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth: and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

If, among the equimultiples of four magnitudes, compared in the fifth definition, we should find $\bullet \bullet \bullet \bullet \bullet > \blacksquare \blacksquare \blacksquare \blacksquare$, but $\blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge =$ or $> \heartsuit \heartsuit \heartsuit \heartsuit$, or if we should find any particular multiple M' of the first and third, and a particular multiple m' of the second and fourth, such, that M' times the first is $> m'$ times the second, but M' times the third is not $> m'$ times the fourth, i. e. $=$ or $< m'$ times the fourth; then the first is said to have to the second a greater ratio than the third has to the fourth; or the third has to the fourth, under such circumstances, a less ratio than the first has to the second: although several other equimultiples may tend to show that the four magnitudes are proportionals.

This definition will in future be expressed thus :—

If $M' \heartsuit > m' \square$, but $M' \blacksquare =$ or $< m' \blacklozenge$,
then $\heartsuit : \square > \blacksquare : \blacklozenge$.

In the above general expression, M' and m' are to be considered particular multiples, not like the multiples M and m introduced in the fifth definition, which are in that definition considered to be every pair of multiples that can be taken. It must also be here observed, that \heartsuit , \square , \blacksquare , and the like symbols are to be considered merely the representatives of geometrical magnitudes.

In a partial arithmetical way, this may be set forth as follows:

Let us take the four numbers 8, 7, 10, and 9.




First	Second	Third	Fourth
8	7	10	9
16	14	20	18
24	21	30	27
32	28	40	36
40	35	50	45
48	42	60	54
56	49	70	63
64	56	80	72
72	63	90	81
80	70	100	90
88	77	110	99
96	84	120	108
104	91	130	117
112	98	140	126
&c.	&c.	&c.	&c.





Among the above multiples we find $16 > 14$ and $20 > 18$; that is, twice the first is greater than twice the second, and twice the third is greater than twice the fourth; and $16 < 21$ and $20 < 27$; that is, twice the first is less than three times the second, and twice the third is less than three times the fourth; and among the same multiples we can find $72 > 56$ and $90 > 72$; that is, 9 times the first is greater than 8 times the second, and 9 times the third is greater than 8 times the fourth. Many other equimultiples might be selected, which would tend to show that the numbers 8, 7, 10, and 9 were proportionals, but they are not, for we can find multiple of the first $>$ a multiple of the second, but the same multiple of the third that has been taken of the first not $>$ the same multiple of the fourth which has been taken to the second; for instance, 9 times the first is $>$ 10 times the second, but 9 times the third is not $>$ 10 times the fourth, that is $72 > 70$, but $90 \not> 90$, or 8 times the first we find $>$ 9 times the second, but 8 times the third













is not greater than 9 times the fourth, that is, $64 > 63$, but 80 not > 81 . When any such multiples as these can be found, the first (8) is said to have the second (7) a greater ratio than the third (10) has to the fourth (9), and on the contrary the third (10) is said to have to the fourth (9) a less ratio than the first (8) has to the second (7).















unequal magnitudes the greater has a greater ratio to the same than the less has: and the same magnitude has a greater ratio to the less than it has to the greater.






Let  and  be two unequal magnitudes,
and  any other.



We shall first prove that  which is the greater of the
two unequal magnitudes, has a greater ratio to  than
, the less, has to ; that is, $\text{red square} : \text{blue circle} > \text{yellow square} : \text{blue circle}$;





take M' , m' , M' , m' ;
such, that M'  and M'  shall be each $>$ ;
also take m'  the least multiple of ,
which will make m'  $>$ M'  = M' ;

$\therefore M'$  is $\nless m'$ ,

but M'  is $>$ m' , for,
as m'  is the first multiple which first
becomes $>$ M' , than $(m' - 1)$  or
 m'  -  is $\nless M'$ , and  $\nless M'$ 

$\therefore m'$  -  +  must be $<$ M'  + M' ;

that is, m'  must be $<$ M' ;

$\therefore M'$  is $>$ m' ; but it has been shown above
that M'  is $\nless m'$ , therefore, by the seventh

definition,  has to  a greater ratio than $\text{yellow square} : \text{blue circle}$.

Next we shall prove that \bullet has a greater ratio to \blacksquare , the less, than it has to $\blacksquare^\blacktriangle$, the greater;

$$\text{or, } \bullet : \blacksquare > \bullet : \blacksquare^\blacktriangle.$$

Take $m' \bullet$, $M' \blacksquare$, $m' \bullet$ and $M' \blacksquare^\blacktriangle$, the same as in the first case, such that $M' \blacktriangle$ and $M' \blacksquare$ will be each $> \bullet$, and $m' \bullet$ the least multiple of \bullet , which first becomes greater than $M' \blacksquare = M' \blacksquare$.

$$\begin{aligned} \therefore m' \bullet - \bullet &\text{ is } \nless M' \blacksquare, \\ \text{and } \bullet &\text{ is } \nless M' \blacktriangle; \text{ consequently} \\ m' \bullet - \bullet + \bullet &\text{ is } < M' \blacksquare + M' \blacktriangle; \end{aligned}$$

$$\therefore m' \bullet \text{ is } < M' \blacksquare^\blacktriangle, \text{ and } \therefore \text{ by the seventh definition,}$$

$$\bullet \text{ has to } \blacksquare \text{ a greater ratio than } \bullet \text{ has to } \blacksquare^\blacktriangle.$$

\therefore Of unequal magnitudes, &c.

The contrivance employed in this proposition for finding among the multiples taken, as in the fifth definition, a multiple of the first greater than the multiple of the second, but the same multiple of the third which has been taken of the first, not greater than the same multiple of the fourth which has been taken of the second, may be illustrated numerically as follows:—

The number 9 has a greater ratio to 7 than 8 has to 7: that is, $9 : 7 > 8 : 7$; or $8 + 1 : 7 > 8 : 7$.

The multiple of 1, which first becomes greater than 7, is 8 times, therefore, we may multiply the first and third by 8, 9, 10, or any other greater number; in this case, let us multiply the first and third by 8, and we have $64 + 8$ and 64: again, the first multiple of 7 which becomes greater

than 64 is 10 times; then, by multiplying the second and fourth by 10, we shall have 70 and 70; then, arranging these multiples, we have—

8 times the first	10 times the second	8 times the third	10 times the fourth
$64 + 8$	70	64	70

Consequently $64 + 8$, or 72, is greater than 70, but 64 is not greater than 70, \therefore by the seventh definition 9 has a greater ratio to 7 than 8 has to 7.

The above is merely illustrative of the foregoing demonstration, for this property could be shown of these or other numbers very readily in the following manner; because, if an antecedent contains its consequent a greater number of times than another antecedent contains its consequent, or when a fraction is formed of an antecedent for the numerator, and its consequent for the denominator be greater than another fraction which is formed of another antecedent for the numerator and its consequent for the denominator, the ratio of the first antecedent to its consequent is greater than the ratio of the last antecedent to its consequent.

Thus, the number 9 has a greater ratio to 7, than 8 has to 7, for $\frac{9}{7}$ is greater than $\frac{8}{7}$.

Again, 17 : 19 is a greater ratio than 13 : 15, because $\frac{17}{19} = \frac{17 \times 15}{19 \times 15} = \frac{255}{185}$, and $\frac{13}{15} = \frac{13 \times 19}{15 \times 19} = \frac{247}{185}$, hence it is evident that $\frac{255}{185}$ is greater than $\frac{247}{185}$, $\therefore \frac{17}{19}$ is greater than $\frac{13}{15}$, and, according to what has been above shown, 17 has to 19 a greater ratio than 13 has to 15.

So that the general terms upon which a greater, equal, or less ratio exists are as follows :—

If $\frac{A}{B}$ be greater than $\frac{C}{D}$, A is said to have to B a greater ratio than C has to D ; if $\frac{A}{B}$ be equal to $\frac{C}{D}$, then A has to B the same ratio which C has to D ; and if $\frac{A}{B}$ be less to $\frac{C}{D}$, A is said to have to B a less ratio than C has to D .

The student should understand all up to this proposition perfectly before proceeding further, in order fully to comprehend the following propositions of this book. We therefore strongly recommend the learner to commence again, and read up to this slowly, and carefully reason at each step, as he proceeds, particularly guarding against the mischievous system of depending wholly on the memory. By following these instructions, he will find that the parts which usually present considerable difficulties will present no difficulties whatever, in prosecuting the study of this important book.



AGNITUDES which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

Let $\blacklozenge : \blacksquare :: \bullet : \blacksquare$, then $\blacklozenge = \bullet$.

For, if not, let $\blacklozenge > \bullet$, then will $\blacklozenge : \blacksquare > \bullet : \blacksquare$ (pr. 5.8), which is absurd according to the hypothesis.

$\therefore \blacklozenge$ is $\nless \bullet$.

In the same manner it may be shown, that

\bullet is $\nless \blacklozenge$,

$\therefore \blacklozenge = \bullet$.

Again, let $\blacksquare : \blacklozenge :: \blacksquare : \bullet$,
then will $\blacklozenge = \bullet$.

For (invert.) $\blacklozenge : \blacksquare :: \bullet : \blacksquare$,
therefore, by the first case, $\blacklozenge = \bullet$.

\therefore Magnitudes which have the same ratio, &c.

This may be shown otherwise, as follows :—

Let $A : B = A : C$, then $B = C$, for, as the fraction $\frac{A}{B}$ = the fraction $\frac{A}{C}$, and the numerator of one equal to the numerator of the other, therefore, denominators of these fractions are equal, that is $B = C$.

Again, if $B : A = C : A$, $B = C$. For, as $\frac{B}{A} = \frac{C}{A}$, B must = C .



HAT magnitude which has a greater ratio than another has unto the same magnitude, is the greater of the two; and that magnitude to which the same has a greater ratio than it has unto another magnitude, is the less of the two.

Let $\blacktriangleleft : \blacksquare > \bullet : \blacksquare$, then $\blacktriangleleft > \bullet$.

For if not, let $\blacktriangleleft =$ or $< \bullet$;
 then, $\blacktriangleleft : \blacksquare = \bullet : \blacksquare$ (pr. 5.7) or
 $\blacktriangleleft : \blacksquare < \bullet : \blacksquare$ (pr. 5.8) and (invert.),
 which is absurd according to the hypothesis.

$\therefore \blacktriangleleft$ is \neq or $< \bullet$, and
 $\therefore \blacktriangleleft$ must be $> \bullet$.

Again, let $\blacksquare : \bullet > \blacksquare : \blacktriangleleft$,
 then, $\bullet < \blacktriangleleft$.

For if not, \bullet must be $>$ or $= \blacktriangleleft$,
 then $\blacksquare : \bullet < \blacksquare : \blacktriangleleft$ (pr. 5.8) and (invert.);
 or $\blacksquare : \bullet = \blacksquare : \blacktriangleleft$ (pr. 5.7), which is absurd (hyp.);

$\therefore \bullet$ is \neq or $= \blacktriangleleft$,
 and $\therefore \bullet$ must be $< \blacktriangleleft$.

\therefore That magnitude which has, &c.



RATIOS that are the same to the same ratio, are the same to each other.

Let $\blacklozenge : \blacksquare = \bullet : \blacktriangledown$ and $\bullet : \blacktriangledown = \blacktriangle : \bullet$,
then will $\blacklozenge : \blacksquare = \blacktriangle : \bullet$.

For if $M\blacklozenge >, =$ or $< m\blacksquare$,
then $M\bullet >, =$ or $< m\blacktriangledown$,
and if $M\bullet >, =$ or $< m\blacktriangledown$,
then $M\blacktriangle >, =$ or $< m\bullet$ (def. 5.5);

\therefore if $M\blacklozenge >, =$ or $< m\blacksquare$, $M\blacktriangle >, =$ or $< m\bullet$
and \therefore (def. 5.5) $\blacklozenge : \blacksquare = \blacktriangle : \bullet$.

\therefore Ratios that are the same &c.



*F any number of magnitudes be proportionals,
as one of the antecedents is to its consequent, so
shall all the antecedents taken together be to
all the consequents.*

Let

$$\begin{aligned} \blacksquare : \bullet &= \square : \diamond = \blacklozenge : \blacktriangledown = \bullet : \blacktriangledown = \blacktriangle : \bullet; \\ \text{then will } \blacksquare : \bullet &= \blacksquare + \square + \blacklozenge + \bullet + \blacktriangle : \\ &\quad \bullet + \diamond + \blacktriangledown + \blacktriangledown + \bullet. \end{aligned}$$

For if $M\blacksquare > m\bullet$, then $M\square > m\diamond$,
and $M\blacklozenge > m\blacktriangledown$, $M\bullet > m\blacktriangledown$,
also $M\blacktriangle > m\bullet$ (def. 5.5)

Therefore, if $M\blacksquare > m\bullet$, then will
 $M\blacksquare + M\square + M\blacklozenge + M\bullet + M\blacktriangle$, or
 $M(\blacksquare + \square + \blacklozenge + \bullet + \blacktriangle)$ be greater
than $m\bullet + m\diamond + m\blacktriangledown + m\blacktriangledown + m\bullet$,
or $m(\bullet + \diamond + \blacktriangledown + \blacktriangledown + \bullet)$.

In the same way it may be shown, if M times one of the antecedents be equal to or less than m times one of the consequents, M times all the antecedents taken together, will be equal to or less than m times all the consequents taken together. Therefore, by the fifth definition, as one of the antecedents is to its consequent, so are all the antecedents taken together to all the consequents taken together.

\therefore If any number of magnitudes, &c.



If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall also have to the second a greater ratio than the fifth to the sixth.

Let $\heartsuit : \spadesuit = \blacksquare : \blacklozenge$, but $\blacksquare : \blacklozenge > \diamond : \bullet$,
then $\heartsuit : \spadesuit > \diamond : \bullet$

For, $\because \blacksquare : \blacklozenge > \diamond : \bullet$, there are some multiples
(M' and m') of \blacksquare and \diamond , and of \blacklozenge and \bullet ,
such that $M' \blacksquare > m' \blacklozenge$,
but $M' \diamond \nless m' \bullet$, by the seventh definition.

Let these multiples be taken, and take
the same multiples of \heartsuit and \spadesuit .

\therefore (def. 5.5) if $M' \heartsuit >, =$, or $< m' \spadesuit$;
then will $M' \blacksquare >, =$, or $< m' \blacklozenge$,
but $M' \blacksquare > m' \blacklozenge$ (construction);

$\therefore M' \heartsuit > m' \spadesuit$,
but $M' \diamond$ is $\nless m' \bullet$ (construction);
and therefore by the seventh definition,

$$\heartsuit : \spadesuit > \diamond : \bullet$$

\therefore If the first has to the second, &c.



If the first has the same ratio to the second which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

Let $\heartsuit : \square :: \blacksquare : \spadesuit$, and first suppose
 $\heartsuit > \blacksquare$, then will $\square > \spadesuit$.

For $\heartsuit : \square > \blacksquare : \square$ (pr. 5.8), and by
 the hypothesis $\heartsuit : \square = \blacksquare : \spadesuit$;
 $\therefore \blacksquare : \spadesuit > \blacksquare : \square$ (pr. 5.13),
 $\therefore \spadesuit < \square$ (pr. 5.10), or $\square > \spadesuit$.

Secondly, let $\heartsuit = \blacksquare$, then will $\square = \spadesuit$.

For $\heartsuit : \square = \blacksquare : \square$ (pr. 5.7),
 and $\heartsuit : \square = \blacksquare : \spadesuit$ (hyp.);
 $\therefore \blacksquare : \square = \blacksquare : \spadesuit$ (pr. 5.11),
 and $\therefore \square = \spadesuit$ (pr. 5.9).

Thirdly, if $\heartsuit < \blacksquare$, then will $\square < \spadesuit$;
 $\therefore \blacksquare > \heartsuit$ and $\blacksquare : \spadesuit = \heartsuit : \square$;
 $\therefore \spadesuit > \square$, by the first case,
 that is, $\square < \spadesuit$.

\therefore If the first has the same ratio, &c.



MAGNITUDES *have the same ratio to one another which their equimultiples have.*

Let \bullet and \blacksquare be two magnitudes;
then, $\bullet : \blacksquare :: M' \bullet : M' \blacksquare$.

$$\begin{aligned} \text{For } \bullet : \blacksquare &= \bullet : \blacksquare \\ &= \bullet : \blacksquare \\ &= \bullet : \blacksquare \\ \therefore \bullet : \blacksquare &:: 4 \bullet : 4 \blacksquare. \text{ (pr. 5.12).} \end{aligned}$$

And as the same reasoning is generally applicable, we have

$$\bullet : \blacksquare :: M' \bullet : M' \blacksquare.$$

\therefore Magnitudes have the same ratio, &c.

Definition XIII

The technical term *permutando*, or *alternando*, by permutation or alternately, is used when there are four proportionals, and it is inferred that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second to the fourth: as is shown in the following proposition :—

Let $\text{yellow circle} : \text{black diamond} :: \text{red heart} : \text{blue square}$,
 by “*permutando*” or “*alternando*” it is inferred
 $\text{yellow circle} : \text{red heart} :: \text{black diamond} : \text{blue square}$.

It may be necessary here to remark that the magnitudes yellow circle , black diamond , red heart , blue square , must be homogeneous, that is, of the same nature or similitude of kind; we must therefore, in such cases, compare lines with lines, surfaces with surfaces, solids with solids, &c. Hence the student will readily perceive that a line and a surface, a surface and a solid, or other heterogenous magnitudes, can never stand in the relation of antecedent and consequent.



*F*our magnitudes of the same kind be proportional, they are also proportionals when taken alternately.

Let $\heartsuit : \square :: \blacksquare : \blacklozenge$, then $\heartsuit : \blacksquare :: \square : \blacklozenge$.

For $M\heartsuit : M\square :: \heartsuit : \square$ (pr. 5.15),
and $M\heartsuit : M\square :: \blacksquare : \blacklozenge$ (hyp.) and (pr. 5.11);
also $m\blacksquare : m\blacklozenge :: \blacksquare : \blacklozenge$ (pr. 5.15);

$\therefore M\heartsuit : M\square :: m\blacksquare : m\blacklozenge$ (pr. 5.14),
and \therefore if $M\heartsuit >, =$ or $< m\blacksquare$,
then will $M\square >, =$, or $< m\blacklozenge$ (pr. 5.14);

therefore, by the fifth definition,

$$\heartsuit : \blacksquare :: \square : \blacklozenge$$

\therefore If four magnitudes of the same kind, &c.

Definition XVI

Dividendo, by division, when there are four proportionals, and it is inferred, that the excess of the first above the second is to the second, as the excess of the third above the fourth, is to the fourth.

Let $A : B :: C : D$;
 by “dividendo” it is inferred
 $A - B : B :: C - D : D$.

According to the above, A is supposed to be greater than B , and C greater than D ; if this be not the case, but to have B greater than A , and D greater than C , B and D can be made to stand as antecedents, and A and C as consequents, by “inversion”

$B : A :: D : C$;
 then, by “dividendo,” we infer
 $B - A : A :: D - C : C$



If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately: that is, if two magnitudes together have to one of them the same ratio which two others have to one of these, the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.

Let $\heartsuit + \square : \square :: \blacksquare + \spadesuit : \spadesuit$,
then will $\heartsuit : \square :: \blacksquare : \spadesuit$.

Take $\heartsuit > m\square$ to each add $M\square$,
then we have $M\heartsuit + M\square > m\square + M\square$,
or $M(\heartsuit + \square) > (m + M)\square$;
but $\heartsuit + \square : \square :: \blacksquare + \spadesuit : \spadesuit$ (hyp.),
and $M(\heartsuit + \square) > (m + M)\square$;
 $\therefore M(\blacksquare + \spadesuit) > (m + M)\spadesuit$ (def. 5.5);
 $\therefore M\blacksquare + M\spadesuit > m\spadesuit + M\spadesuit$;
 $\therefore M\blacksquare > m\spadesuit$, by taking $M\spadesuit$ from both sides:
that is, when $M\heartsuit > m\square$, then $M\blacksquare > m\spadesuit$.

In the same manner it may be proved, that if
 $M\heartsuit =$ or $< m\square$, then will $M\blacksquare =$ or $< m\spadesuit$;
and $\therefore \heartsuit : \square :: \blacksquare : \spadesuit$ (def. 5.5)

\therefore If magnitudes taken jointly, &c.

Definition XV

The term componendo, by composition, is used when there are four proportionals; and it is inferred that the first together with the second is to the second as the third together with the fourth is to fourth.

$$\text{Let } A : B :: C : D;$$

then, by term “componendo,” it is inferred that

$$A + B : B :: C + D : D$$

By “inversion” B and D may become the first and the third, and A and C the second and fourth, as

$$B : A :: D : C,$$

then, by “componendo,” we infer that

$$B + A : A :: D + C : C.$$



If magnitudes, taken separately, be proportionals, they shall also be proportionals taken jointly: that is, if the first be to the second as the third is to the fourth, the first and second together shall be to the second as the third and fourth together is to the fourth.

Let $\heartsuit : \square :: \blacksquare : \blacklozenge$,
 then $\heartsuit + \square : \square :: \blacksquare + \blacklozenge : \blacklozenge$;

for if not, let $\heartsuit + \square : \square :: \blacksquare + \bullet : \bullet$,
 supposing $\bullet \neq \blacklozenge$;

$\therefore \heartsuit : \square :: \blacksquare : \bullet$ (pr. 5.17)

but $\heartsuit : \square :: \blacksquare : \blacklozenge$ (hyp.);

$\therefore \blacksquare : \bullet :: \blacksquare : \blacklozenge$ (pr. 5.11);

$\therefore \bullet = \blacklozenge$ (pr. 5.9),

which is contrary to the supposition;

$\therefore \bullet$ is not unequal to \blacklozenge ;

that is $\bullet = \blacklozenge$;

$\therefore \heartsuit + \square : \square :: \blacksquare + \blacklozenge : \blacklozenge$.

\therefore If magnitudes, taken separately, &c.



If a whole magnitude be to a whole, as a magnitude taken from the first, is to a magnitude taken from the other; the remainder shall be to the remainder, as the whole to the whole.

Let $\heartsuit + \square : \blacksquare + \diamond :: \heartsuit : \blacksquare$,
 then will $\square : \diamond :: \heartsuit + \square : \blacksquare + \diamond$.

For $\heartsuit + \square : \heartsuit :: \blacksquare + \diamond : \blacksquare$ (alter.),

$\therefore \square : \heartsuit :: \diamond : \blacksquare$ (divid.),

again $\square : \diamond :: \heartsuit : \blacksquare$ (alter.),

but $\heartsuit + \square : \blacksquare + \diamond :: \heartsuit : \blacksquare$ (hyp.);

therefore $\square : \diamond :: \heartsuit + \square : \blacksquare + \diamond$ (pr. 5.11).

\therefore If a whole magnitude be to a whole, &c.

Definition XVII

The term “convertendo,” by conversion, is made use of by geometricians, when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third is to its excess above the fourth. See the following proposition :—



If four magnitudes be proportionals, they are also proportionals by conversion: that is, the first is to its excess above the second, as the third is to its excess above the fourth.

$$\begin{aligned}
 &\text{Let } \bullet : \circ : \circ :: \blacksquare \blacklozenge : \blacklozenge, \\
 &\text{then shall } \bullet : \circ : \bullet :: \blacksquare \blacklozenge : \blacksquare. \\
 \\
 &\therefore \bullet : \circ : \circ :: \blacksquare \blacklozenge : \blacklozenge; \\
 &\therefore \bullet : \circ : \circ :: \blacksquare : \blacklozenge \text{ (divid.)}, \\
 \\
 &\therefore \circ : \bullet : \bullet :: \blacklozenge : \blacksquare \text{ (inver.)}, \\
 \\
 &\therefore \bullet : \circ : \bullet :: \blacksquare \blacklozenge : \blacksquare \text{ (compo.)}.
 \end{aligned}$$

\therefore If four magnitudes, &c.

Definition XVIII

“Ex æquali” (sc. distantia), or ex æquo, from equality of distance: when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each rank, and it is inferred that the first is to the last of the first rank of magnitudes, as the first is to the last of the others: “of this there are two following kinds, which arise from the different order in which the magnitudes are taken, two and two.”

Definition XIX

“Ex æquali,” from equality. This term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second to the third of the first rank, so is the second to the third of the other; and so in order: and the inference is as mentioned in the preceding definition; whence this is called ordinate proposition. It is demonstrated in pr. 5.22.

Thus, if there be two ranks of magnitudes,

A, B, C, D, E, F , the first rank, and L, M, N, O, P, Q , the second, such that $A : B :: L : M, B : C :: M : N$,

$C : D :: N : O, D : E :: O : P, E : F :: P : Q$;

we infer by the term “ex æquali” that $A : F :: L : Q$

Definition XX

“Ex æquali in proportione perturbatâ seu inordinatâ,” from equality in perturbate, or disorderly proportion. This term is used when the first magnitude is to the second of the first rank as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank; and so on in a cross order: and the inference is in the 18th definition. It is demonstrated in pr. 5.23.

Thus, if there be two ranks of magnitudes,





A, B, C, D, E, F , the first rank, and L, M, N, O, P, Q , the second, such that $A : B :: P : Q, B : C :: O : P$,
 $C : D :: N : O, D : E :: M : N, E : F :: L : M$;

the term “ex æquali in proportione perturbatâ seu inordinatâ” infers that $A : F :: L : Q$



If there be three magnitudes, and other three, which taken two and two, have the same ratio; then, if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.





Let , , , the first three magnitudes,
and , , , be the other three,
such that  :  ::  : ,
and  :  ::  : .

Then, if  >, =, or < ,
then will  >, =, or < .

From the hypothesis, by alternando, we have

 :  ::  : ,
and  :  ::  : 

∴  :  ::  :  (pr. 5.11);

∴ if  >, =, or < ,
then will  >, =, or <  (pr. 5.14).

∴ If there be three magnitudes, &c.



F there be three magnitudes, and other three which have the same ratio, taken two and two, but in a cross order; then if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal, and if less, less.

Let $\color{brown}{\blacklozenge}$, $\color{red}{\blacktriangle}$, $\color{blue}{\blacksquare}$, be the first three magnitudes,
and $\color{blue}{\blacklozenge}$, $\color{red}{\triangle}$, $\color{yellow}{\bullet}$, the other three,
such that $\color{brown}{\blacklozenge} : \color{red}{\blacktriangle} :: \color{red}{\triangle} : \color{yellow}{\bullet}$,
and $\color{red}{\blacktriangle} : \color{blue}{\blacksquare} :: \color{blue}{\blacklozenge} : \color{red}{\triangle}$.

Then if $\color{brown}{\blacklozenge} >, =,$ or $< \color{blue}{\blacksquare}$,
then will $\color{blue}{\blacklozenge} >, =,$ or $< \color{yellow}{\bullet}$.

First, let $\color{brown}{\blacklozenge}$ be $> \color{blue}{\blacksquare}$:
then, because $\color{red}{\blacktriangle}$ is any other magnitude,
 $\color{brown}{\blacklozenge} : \color{red}{\blacktriangle} > \color{blue}{\blacksquare} : \color{red}{\blacktriangle}$ (pr. 5.8);
but $\color{red}{\triangle} : \color{yellow}{\bullet} :: \color{brown}{\blacklozenge} : \color{red}{\blacktriangle}$ (hyp.);
 $\therefore \color{red}{\triangle} : \color{yellow}{\bullet} > \color{blue}{\blacksquare} : \color{red}{\blacktriangle}$ (pr. 5.13);
and $\therefore \color{red}{\blacktriangle} : \color{blue}{\blacksquare} :: \color{blue}{\blacklozenge} : \color{red}{\triangle}$ (hyp.);
 $\therefore \color{blue}{\blacksquare} : \color{red}{\blacktriangle} :: \color{red}{\triangle} : \color{blue}{\blacklozenge}$ (inv.),
and it was shown that $\color{red}{\triangle} : \color{yellow}{\bullet} > \color{blue}{\blacksquare} : \color{red}{\blacktriangle}$,
 $\therefore \color{red}{\triangle} : \color{yellow}{\bullet} > \color{red}{\triangle} : \color{blue}{\blacklozenge}$ (pr. 5.13);
 $\therefore \color{yellow}{\bullet} < \color{blue}{\blacklozenge}$,
that is $\color{blue}{\blacklozenge} > \color{yellow}{\bullet}$.

Secondly, let $\color{brown}{\blacklozenge} = \color{blue}{\blacksquare}$; then shall $\color{blue}{\blacklozenge} = \color{yellow}{\bullet}$.

For $\therefore \color{brown}{\blacklozenge} = \color{blue}{\blacksquare}$,
 $\color{brown}{\blacklozenge} : \color{red}{\blacktriangle} = \color{blue}{\blacksquare} : \color{red}{\blacktriangle}$ (pr. 5.7);
but $\color{brown}{\blacklozenge} : \color{red}{\blacktriangle} = \color{red}{\triangle} : \color{yellow}{\bullet}$ (hyp.),
and $\color{blue}{\blacksquare} : \color{red}{\blacktriangle} = \color{red}{\triangle} : \color{blue}{\blacklozenge}$ (hyp. and inv.),
 $\therefore \color{red}{\triangle} : \color{yellow}{\bullet} = \color{red}{\triangle} : \color{blue}{\blacklozenge}$ (pr. 5.11),
 $\therefore \color{blue}{\blacklozenge} = \color{yellow}{\bullet}$ (pr. 5.9).

Next, let $\color{brown}{\blacklozenge}$ be $< \color{blue}{\blacksquare}$, then $\color{blue}{\blacklozenge}$ shall be $< \color{yellow}{\bullet}$;
for $\color{blue}{\blacksquare} > \color{brown}{\blacklozenge}$,

and it has been shown that $\blacksquare : \blacktriangle = \triangle : \blacklozenge$,

and $\blacktriangle : \blacktriangledown = \bullet : \triangle$;

\therefore by the first case \bullet is $>$ \blacklozenge ,







that is, $\blacklozenge < \bullet$.

\therefore If there be three, &c.























F there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last of the same.

N.B.— This is usually cited by the words “*ex æquali*,” or “*ex æquo*.”











First, let there be magnitudes , , ,
and as many others , , ,
such that









 :  ::  : ,
and  :  ::  : ,
then shall  :  ::  : .





Let these magnitudes, as well as any equimultiples whatever of the antecedents and consequents of the ratios, stand as follows :—





, , , , , ,
and
 M , m , N , M , m , N ,
∴  :  ::  : ,
∴ M  : m  :: M  : m  (pr. 5.4).





For the same reason





m  : N  :: m  : N ;
and because there are three magnitudes
 M , m , N ,
and other three, M , m , N ,
which, taken two and two, have the same ratio;





∴ if M  $>$, $=$ or $<$ N 
then will M  $>$, $=$ or $<$ N , by (pr. 5.20),
and ∴  :  ::  :  (def. 5.5).





Next, let there be four magnitudes, , , , ,











and other four, , , , ,





which, taken two and two, have the same ratio,
that is to say,  :  ::  : ,





 :  ::  : ,

and  :  ::  : ,

then shall  :  ::  :  ;

for, ∵ , , , are three magnitudes,
and , , , other three,
which, taken two and two, have the same ratio;
therefore, by the foregoing case  :  ::  : ,

but  :  ::  :  ;







therefore again, by the first case,  :  ::  :  ;
and so on, whatever the number of magnitudes be.

∴ If there be any number, &c.















































F there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last of the same.







N.B.— This is usually cited by the words “*ex æquali in portione perturbatâ*,” or “*ex æquo perturbato*.”

First, let there be three magnitudes , , ,
and other three, , , ,
which, taken two and two in a
cross order, have the same ratio;

that is,  :  ::  : ,
and  :  ::  : ,
then shall  :  ::  : .

Let these magnitudes and their respective equimultiples be arranged as follows :—

, , , , , ,
 M , M , m , M , m , m ,
then  :  :: M  : M  (pr. 5.15);
and for the same reason
 :  :: m  : m ;
but  :  ::  :  (hyp.),
 $\therefore M$  : M  ::  :  (pr. 5.11);
and \therefore  :  ::  :  (hyp.),
 $\therefore M$  : m  :: M  : m  (pr. 5.4);
then, because there are three magnitudes,
 M , M , m ,
and other three M , m , m ,
which, taken two and two in a
cross order, have the same ratio;
therefore, if M  $>$, $=$, or $<$ m ,

then will M  $>$, $=$, or $<$ m  (pr. 5.21),
and \therefore  :  ::  :  (def. 5.5).

Next, let there be four magnitudes,

, , , ,

and other four,

, , , ,

which, when taken two and two in
a cross order, have the same ratio;

namely,  :  ::  : ,

 :  ::  : ,





and  :  ::  : ,

then shall  :  ::  : .

For, \therefore , ,  are three magnitudes,





and , , , other three,

which, taken two and two in a
cross order have the same ratio,

therefore, by the first case,  :  ::  : ,

but  :  ::  : ,

therefore again, by the first case,

 :  ::  : .

and so on, whatever be the number of such magnitudes.

\therefore If there be any number, &c.



F the first has to the second the same ratio which the third has to the fourth, and the fifth to the second the same which the sixth has to the fourth, the first and fifth together shall have to the second the same ratio which the third and sixth together have to the fourth.

First Second Third Fourth



Fifth

Sixth



Let $\heartsuit : \cup :: \blacksquare : \blacklozenge$,

and $\heartsuit : \cup :: \bullet : \blacklozenge$,

then $\heartsuit + \heartsuit : \cup :: \blacksquare + \bullet : \blacklozenge$.

For $\heartsuit : \cup :: \bullet : \blacklozenge$ (hyp.),

and $\cup : \heartsuit :: \blacklozenge : \blacksquare$ (hyp.) and (invert.),

$\therefore \heartsuit : \heartsuit :: \bullet : \blacksquare$ (pr. 5.22);

and, because these magnitudes are proportionals,
they are proportionals when taken jointly,

$\therefore \heartsuit + \heartsuit : \heartsuit :: \bullet + \blacksquare : \bullet$ (pr. 5.18),

but $\heartsuit : \cup :: \bullet : \blacklozenge$ (hyp.),

$\therefore \heartsuit + \heartsuit : \cup :: \bullet + \blacksquare : \blacklozenge$ (pr. 5.22)

\therefore If the first, &c.



four magnitudes of the same kind are proportional, the greatest and least of them together are greater than the other two together.

Let four magnitudes, $\heartsuit + \square$, $\blacksquare + \diamond$, \square , and \diamond ,
of the same kind, be proportionals, that is to say,

$$\heartsuit + \square : \blacksquare + \diamond :: \square : \diamond,$$

and let $\heartsuit + \square$ be the greatest of the four, and
consequently by pr. 5.A and pr. 5.14, \diamond is the least;
then will $\heartsuit + \square + \diamond$ be $> \blacksquare + \diamond + \square$;

$$\therefore \heartsuit + \square : \blacksquare + \diamond :: \square : \diamond,$$

$$\therefore \heartsuit : \blacksquare :: \heartsuit + \square : \blacksquare + \diamond \text{ (pr. 5.19),}$$

$$\text{but } \heartsuit + \square > \blacksquare + \diamond \text{ (hyp.),}$$

$$\therefore \heartsuit > \blacksquare \text{ (pr. 5.A);}$$

to each of these add $\square + \diamond$,

$$\therefore \heartsuit + \square + \diamond > \blacksquare + \square + \diamond.$$

\therefore If four magnitudes, &c.

Definition X

When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

For example, if A, B, C , be continued proportionals, that is, $A : B :: B : C$, A is said to have to C the duplicate ratio of $A : B$;

$$\text{or } \frac{A}{C} = \text{the square of } \frac{A}{B}.$$

This property will be more readily seen of the quantities ar^2, ar, a , for $ar^2 : ar :: ar : a$;

$$\text{and } \frac{ar^2}{a} = r^2 = \text{the square of } \frac{ar^2}{ar} = r,$$

or of a, ar, ar^2 ;

$$\text{for } \frac{a}{ar^2} = \frac{1}{r^2} = \text{the square of } \frac{a}{ar} = \frac{1}{r}.$$

Definition XI

When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which is has to the second; and so on, quadruplicate, &c. increasing the denomination still by unity, in any number of proportionals.

For example, let A, B, C, D , be four continued proportionals, that is, $A : B :: B : C :: C : D$; A is said to have to D , the triplicate ratio of A to B ;

$$\text{or } \frac{A}{D} = \text{the cube of } \frac{A}{B}.$$

This definition will be better understood, and applied to a greater number of magnitudes than four that are continued proportionals, as follows :—

Let ar^3, ar^2, ar, a , be four magnitudes in continued proportion, that is, $ar^3 : ar^2 :: ar^2 : ar :: ar : a$,

$$\text{then } \frac{ar^3}{a} = r^3 = \text{the cube of } \frac{ar^3}{ar^2} = r.$$

Or, let $ar^5, ar^4, ar^3, ar^2, ar, a$, be six magnitudes in proportion, that is

$$ar^5 : ar^4 :: ar^4 : ar^3 :: ar^3 : ar^2 :: ar^2 : ar :: ar : a,$$

$$\text{then the ratio } \frac{ar^5}{a} = r^5 = \text{the fifth power of } \frac{ar^5}{ar^4} = r$$

Or, let a, ar, ar^2, ar^3, ar^4 , be five magnitudes in continued proportion; then $\frac{a}{ar^4} = \frac{1}{r^4} =$

$$\text{the fourth power of } \frac{a}{ar} = \frac{1}{r}.$$

Definition A

To know a compound ratio :—

When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth; and so on, unto the last magnitude.

For example, if A, B, C, D , be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B , and of the ratio of B to C , and of the ratio of C to D ; or, the ratio of A to D is said to be compounded of the ratios of A to B , B to C , and C to D .

And if A has to B the same ratio which E has to F , and B to C the same ratio that G has to H , and C to D the same that K has to L ; then by this definition, A is said to have to D the ratio compounded of ratios which are the same with the ratios of E to F , G to H , and K to L . And the same thing is to be saying, A has to D the ratio compounded of the ratios of E to F , G to H , and K to L .

In like manner, the same things being supposed; if M has to N the same ratio which A has to D , then for shortness sake, M is said to have to N the ratio compounded of the ratios E to F , G to H , and K to L .

This definition may be better understood from an arithmetical or algebraical illustration; for, in fact, a ratio compounded of several other ratios, is nothing more than a ratio which has for its antecedent the continued product of all antecedents of the ratios compounded, and for its consequent the continued product of all the consequents of the ratios compounded.

A	B	C	D
E	F	G	H
K	L		
M	N		

Thus, the ratio compounded of the ratios of

$$2 : 3, 4 : 7, 6 : 11, 2 : 5,$$

is the ratio of $2 \times 4 \times 6 \times 2 : 3 \times 7 \times 11 \times 5$,

or the ratio of 96 : 1155, or 32 : 385.

And of the magnitudes A, B, C, D, E, F , of the same kind, $A : F$ is the ratio compounded of the ratios of

$$A : B, B : C, C : D, D : E, E : F;$$

for $A \times B \times C \times D \times E : B \times C \times D \times E \times F$,

$$\text{or } \frac{A \times B \times C \times D \times E}{B \times C \times D \times E \times F} = \frac{A}{F}, \text{ or the ratio of } A : F.$$



RATIOS which are compounded of the same ratios are same to one another.

Let $A : B :: F : G$,
 $B : C :: G : H$,
 $C : D :: H : K$,
 and $D : E :: K : L$,

Then the ratio which is compounded by the ratios of $A : B$, $B : C$, $C : D$, $D : E$, or the ratio of $A : E$, is the same as the ratio compounded of the ratios $F : G$, $G : H$, $H : K$, $K : L$, or the ratio of $F : L$.

A	B	C	D	E
F	G	H	K	L

$$\begin{aligned} \text{For } \frac{A}{B} &= \frac{F}{G}, \\ \frac{B}{C} &= \frac{G}{H}, \\ \frac{C}{D} &= \frac{H}{K}, \\ \text{and } \frac{D}{E} &= \frac{K}{L}; \\ \therefore \frac{A \times B \times C \times D}{B \times C \times D \times E} &= \frac{F \times G \times H \times K}{G \times H \times K \times L}, \\ \text{and } \therefore \frac{A}{E} &= \frac{F}{L}, \end{aligned}$$

or the ratio of $A : E$ is the same as the ratio $F : L$.

The same may be demonstrated of any number of ratios so circumstanced.

Next, let $A : B :: K : L$,
 $B : C :: H : K$,
 $C : D :: G : H$,
 $D : E :: F : G$,

Then the ratio which is compounded of the ratios of $A : B$, $B : C$, $C : D$, $D : E$, or the ratio of $A : E$, is

the same as the ratio compounded of the ratios of $K : L$,
 $H : K$, $G : H$, $F : G$, or the ratio of $F : L$

$$\text{For } \frac{A}{B} = \frac{K}{L},$$

$$\frac{B}{C} = \frac{H}{K},$$

$$\frac{C}{D} = \frac{G}{H},$$

$$\text{and } \frac{D}{E} = \frac{F}{G};$$

$$\therefore \frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{K \times H \times G \times F}{L \times K \times H \times G},$$

$$\text{and } \therefore \frac{A}{E} = \frac{F}{L},$$

or the ratio of $A : E$ is the same as the ratio $F : L$.

\therefore Ratios which are compounded, &c.



F several ratios be the same to several ratios, each to each, the ratio which is compounded of ratios which are the same to the first ratios, each to each, shall be the same to the ratio compounded of ratios which are the same to the other ratios, each to each.

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>P</i>	<i>Q</i>	<i>R</i>	<i>S</i>	<i>T</i>
<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>V</i>	<i>W</i>	<i>X</i>	<i>Y</i>	<i>Z</i>

If $A : B :: a : b$ and $A : B :: P : Q$ and $a : b :: V : W$
 $C : D :: c : d$ $C : D :: Q : R$ $c : d :: W : X$
 $E : F :: e : f$ $E : F :: R : S$ $e : f :: X : Y$
 and $G : H :: g : h$ $G : H :: S : T$ $f : h :: Y : Z$

then $P : T = V : Z$.

$$\begin{aligned} \text{For } \frac{P}{Q} &= \frac{A}{B} = \frac{a}{b} = \frac{V}{W} \\ \frac{Q}{R} &= \frac{C}{D} = \frac{c}{d} = \frac{W}{X} \\ \frac{R}{S} &= \frac{E}{F} = \frac{e}{f} = \frac{X}{Y} \\ \frac{S}{T} &= \frac{G}{H} = \frac{g}{h} = \frac{Y}{Z} \end{aligned}$$

$$\text{and } \therefore \frac{P \times Q \times R \times S}{Q \times R \times S \times T} = \frac{V \times W \times X \times Y}{W \times X \times Y \times Z},$$

$$\text{and } \therefore \frac{P}{T} = \frac{V}{Z},$$

$$\text{or } P : T = V : Z.$$

\therefore If several ratios, &c.



If a ratio which is compounded of several ratios be the same to a ratio which is compounded of several other ratios; and if one of the first ratios, or the ratio which is compounded of several of them, be the same to one of the last ratios, or to the ratio which is compounded of several of them; then the remaining ratio of the first, or, if there be more than one, the ratio compounded of the remaining ratios, shall be the same to the remaining ratios, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio, compounded of these remaining ratios.

$A : B, B : C, C : D, D : E, E : F, F : G, G : H,$ $P : Q, Q : R, R : S, S : T, T : X$

Let $A : B, B : C, C : D, D : E, E : F, F : G, G : H,$ be the first ratios, and $P : Q, Q : R, R : S, S : T, T : X,$ the other ratios; also, let $A : H,$ which is compounded of the first ratios, be the same as the ratio of $P : X,$ which is the ratio compounded of the other ratios; and, let the ratio of $A : E,$ which is compounded of the ratios of $A : B, B : C, C : D, D : E,$ be the same as the ratio of $P : R,$ which is compounded of the ratios $P : Q, Q : R.$

Then the ratio which is compounded of the remaining first ratios, that is, the ratio compounded of the ratios $E : F, F : G, G : H,$ that is, the ratio of $E : H,$ shall be the same as the ratio of $R : X,$ which is compounded of the ratios of $R : S, S : T, T : X,$ the remaining other ratios.

$$\begin{aligned}
 & \text{Because} \\
 & \frac{A \times B \times C \times D \times E \times F \times G}{B \times C \times D \times E \times F \times G \times H} = \frac{P \times Q \times R \times S \times T}{Q \times R \times S \times T \times X}, \\
 \text{or } & \frac{A \times B \times C \times D}{B \times C \times D \times E} \times \frac{E \times F \times G}{F \times G \times H} = \frac{P \times Q}{Q \times R} \times \frac{R \times S \times T}{S \times T \times X}, \\
 & \text{and } \frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{P \times Q}{Q \times R} \\
 & \therefore \frac{E \times F \times G}{F \times G \times H} = \frac{R \times S \times T}{S \times T \times X},
 \end{aligned}$$

$$\therefore \frac{E}{H} = \frac{R}{X},$$

$$\therefore E : H = R : X.$$

\therefore If a ratio which, &c.

$d : e, e : f, f : g$, which are the same as the ratio of $A : B$,
 $C : D, E : F, G : H, K : L, M : N$, each to each.

Also, $O : P = b : k$,

$Q : R = k : l$,

$S : T = l : m$,

$V : W = m : n$,

$X : Y = n : p$,

Then will the ratio of $b : p$ be the ratio compounded of the ratios of $b : k, k : l, l : m, m : n, n : p$, which are the same as the ratios of $O : P, Q : R, S : T, V : W, X : Y$, each to each.

\therefore by the hypothesis $a : g = b : p$.

Also, let the ratio which is compounded of the ratios of $A : B, C : D$, two of the first ratios (or the ratios of $a : c$ for $A : B = a : b$, and $C : D = b : c$), be the same as the ratio of $a : d$, which is compounded of the ratios of $a : b, b : c, c : d$, which are the same as the ratios of $O : P, Q : R, S : T$, three of the other ratios.

And let the ratios of $b : s$, which is compounded of the ratios of $b : k, k : m, m : n, n : s$, which are the same as the remaining first ratios, namely, $E : F, G : H, K : L, M : N$; also, let the ratio of $e : g$, be that which is compounded of the ratios $e : f, f : g$, which are the same, each to each, to the remaining other ratios, namely $V : W, X : Y$. Then the ratios of $b : s$ shall be the same as the ratio of $e : g$; or $b : s = e : g$.

$$\text{For } \frac{A \times C \times E \times G \times K \times M}{B \times D \times F \times H \times L \times N} = \frac{a \times b \times c \times d \times e \times f}{b \times c \times d \times e \times f \times g},$$

$$\text{and } \frac{O \times Q \times S \times V \times X}{P \times R \times T \times W \times Y} = \frac{b \times k \times l \times m \times n}{k \times l \times m \times n \times p},$$

by the composition of the ratios;

$$\therefore \frac{a \times b \times c \times d \times e \times f}{b \times c \times d \times e \times f \times g} = \frac{b \times k \times l \times m \times n}{k \times l \times m \times n \times p} \text{ (hyp.)},$$

$$\text{or } \frac{a \times b}{b \times c} \times \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{b \times k \times l}{k \times l \times m} \times \frac{m \times n}{n \times p},$$

$$\begin{aligned} \text{but } \frac{a \times b}{b \times c} &= \frac{A \times C}{B \times D} = \frac{O \times Q \times S}{P \times R \times T} = \\ \frac{a \times b \times c}{b \times c \times d} &= \frac{b \times k \times l}{k \times l \times m}; \end{aligned}$$

$$\therefore \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{m \times n}{n \times p}.$$

$$\text{And } \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{b \times k \times l \times m \times n}{k \times l \times m \times n \times p} \text{ (hyp.)},$$

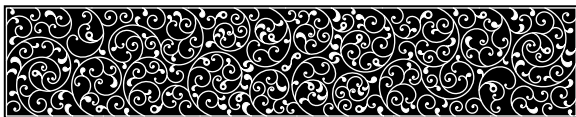
$$\text{and } \frac{m \times n}{n \times p} = \frac{e \times f}{f \times g} \text{ (hyp.)},$$

$$\therefore \frac{b \times k \times l \times m \times n}{k \times l \times m \times n \times p} = \frac{ef}{fg},$$

$$\therefore \frac{b}{s} = \frac{e}{g},$$

$$\therefore b : s = e : g.$$

\therefore If there be any number, &c.



Book VI

Definitions

I

Rectilinear figures are said to be similar, when they have their several angles equal, each to each, and the sides about the equal angles proportional.

2

Two sides of one figure are said to be reciprocally proportional to two sides of another figure when one of the sides of the first is to the second, as the remaining side of the second is to the remaining side of the first.

3

A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less.





4




The altitude of any figure is the straight line drawn from its vertex perpendicular to its base, or the base produced.











TRIANGLES and parallelograms having the same altitude are to one another as their bases.





Let the triangles  and  have a common vertex, and their bases  and  in the same straight line.

Produce  both ways, take successively on  produced lines equal to it; and on  produced lines successively equal to it; and draw lines from the common vertex to their extremities.

The triangles  thus formed are all equal to one another, since their bases are equal. (pr. 1.38)

\therefore  and its base are respectively equimultiples of  and the base .

In like manner  and its base are respectively equimultiples of  and the base .

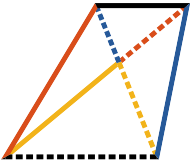
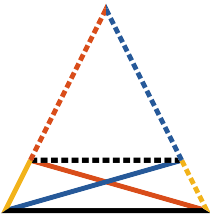
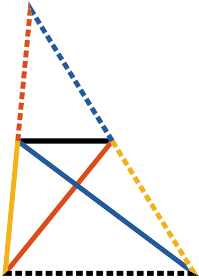
\therefore if m or 6 times  $>$, $=$ or $<$ n or 5 times  then m or 6 times  $>$, $=$ or $<$ n or 5 times , m and n stand for every multiple taken as in the fifth definition of the Fifth Book. Although we have only shown that

this property exists when m equal 6, and n equal 5, yet it is evident that the property holds good for every multiple value that may be given to m , and to n .



Parallelograms having the same altitude are the doubles of the triangles, on their bases, and are proportional to them (Part I), and hence their doubles, the parallelograms, are as their bases (pr. 5.15).

Q. E. D.



IF a straight line ——— be drawn parallel to any side of a triangle, it shall cut the other sides, or those sides produced, into proportional segments


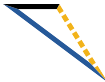
And if any straight line ——— divide the sides of a triangle, or those sides produced, into proportional segments, it is parallel to the remaining side

Part I.

Let ——— \parallel , then shall

$$\text{yellow} : \text{red} :: \text{orange} : \text{blue}.$$

Draw ——— and ———,

and  =  (pr. 1.37);

$$\therefore \text{yellow} : \text{red} :: \text{orange} : \text{blue} \quad (\text{pr. 5.7});$$

but  :  :: yellow : red (pr. 6.1),





$$\therefore \text{yellow} : \text{red} :: \text{orange} : \text{blue} \quad (\text{pr. 5.11}).$$





Part II.

Let  :  ::  : .

then  || .

Let the same construction remain,


\therefore  :  ::  :  } (pr. 6.1);

and  :  ::  :  }

but  :  ::  :  (hyp.)

\therefore  :  ::  :  (pr. 5.11)

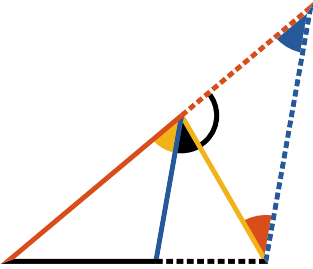
\therefore  =  (pr. 5.9);

but they are on the same base ,

and at the same side of it, and

\therefore  ||  (pr. 1.39).

Q. E. D.



RIGHT line (—) bisecting the angle of a triangle, divides the opposite side into segments (—, - - - - -) proportional to the conterminous sides (—, —).

And if a straight line (—) drawn from any angle of a triangle divide the opposite side (— - - - -) into segments (—, - - - - -) proportional to the conterminous sides (—, —), it bisects the angle.

Part I.

Draw — - - - - || —, to meet — - - - -;
then,  =  (pr. I.29).

$\therefore \blacktriangle = \triangle$; but $\blacktriangle = \triangle$, $\therefore \triangle = \triangle$,

$\therefore \text{— - - - -} = \text{—}$ (pr. I.6);

and $\therefore \text{—} \parallel \text{— - - - -}$,

$\text{— - - - -} : \text{—} :: \text{— - - - -} : \text{—}$ (pr. 6.2);

but $\text{— - - - -} = \text{—}$;

$\therefore \text{—} : \text{—} :: \text{— - - - -} : \text{—}$ (pr. 5.7).



Part II.



Let the same construction remain,

and  :  ::  :  (pr. 6.2);

but  :  ::  :  (hyp.)





\therefore  :  ::  :  (pr. 5.11).




and \therefore  =  (pr. 5.9),

and \therefore  =  (pr. 1.5);

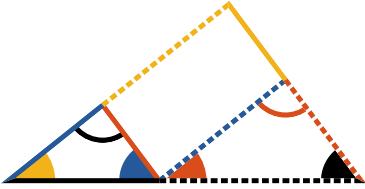
but since  \parallel ;  = ,



and  =  (pr. 1.29);









\therefore  = , and  = ,

and \therefore  bisects  .

Q. E. D.







In equiangular triangles ( and ) the sides about the equal angles are proportional, and the sides which are opposite to the equal angles are homologous.

Let the equiangular triangles be so placed that two sides ,  opposite to equal angles  and  may be conterminous, and in the same straight line; and that the triangles lying at the same side of that straight line; and that the triangles lying at the same side of that straight line, may have the equal angles not conterminous, i. e.  opposite to , and  to .



Draw  and .

Then, \therefore  = ,  \parallel  (pr. 1.28);

and for a like reason  \parallel ,

\therefore   is a parallelogram.

But  :  $::$  :  (pr. 6.2);

and since  =  (pr. 1.34),

 :  $::$  : ;

and by alternation

 :  $::$  :  (pr. 5.16).

In like manner it may be shown, that

 :  $::$  : ;

and by alternation, that

 :  $::$  : ;

but it has been already proved that

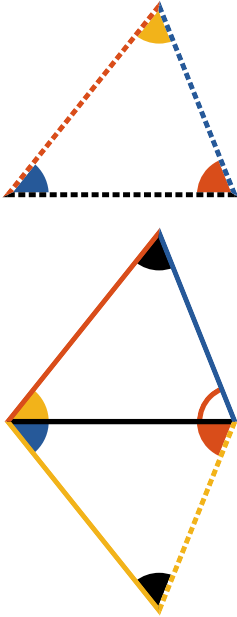
 :  $::$  : 

and therefore, ex æquali,

 :  ::  :  (pr. 5.22),

therefore the sides about the equal angles
are proportional, and those which are
opposite to the equal angles are homologous.

Q. E. D.



IF two triangles have their sides proportional
 (———— : ———— :: ———— : ————)
 and (———— : ———— :: ———— :
 ————) they are equiangular, and the equal
 angles are subtended by the homologous sides.





From the extremities of ———— ,
 draw ———— and ———— , making
 $\blacktriangleleft = \triangleleft$, $\blacktriangleright = \triangleright$ (pr. I.23);
 and consequently $\blacktriangle = \triangle$ (pr. I.32),
 and since the triangles are equiangular,
 ———— : ———— :: ———— : ———— (pr. 6.4);
 but ———— : ———— :: ———— : ———— (hyp.);
 \therefore ———— : ———— :: ———— : ———— ,
 and consequently ———— = ————
 (pr. 5.9). In like manner it may be shown that
 ———— = ———— .

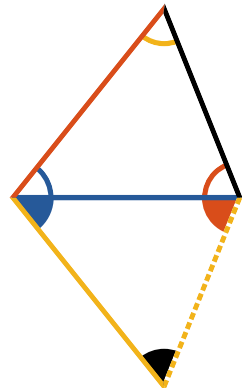
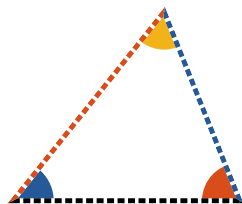
Therefore, the two triangles having a common
 base ———— , and their sides equal, have
 also equal angles opposite to equal sides, i. e.


$\triangleleft = \blacktriangleleft$ and $\triangleright = \blacktriangleright$ (pr. I.8).
 but $\blacktriangleleft = \triangleleft$ (const.) and $\therefore \triangleleft = \triangleleft$;
 for the same reason $\triangleright = \blacktriangleright$,
 and consequently $\blacktriangle = \triangle$ (pr. I.32);
 and therefore the triangles are equiangular,
 and it is evident that the homologous
 sides subtended by the equal angles.

Q. E. D.



F two triangles ( and ) have one angle () of the one, equal to one angle () of the other, and the sides about the equal angles proportional, the triangles shall be equiangular, and have those angles equal which the homologous sides subtend.




From the extremities of , one

of the sides of , about ,

draw  and ,





making  = ,

and  = .

then  =  (pr. I.32),

and two triangles being equiangular,

 :  ::  :  (pr. 6.4);

but  :  ::  :  (hyp.)

\therefore  :  ::  :  (pr. 5.11),

and consequently  =  (pr. 5.9);

\therefore  =  in every respect (pr. I.4).

But  =  (const.),

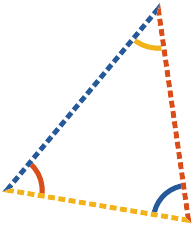
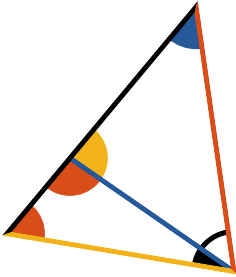
and \therefore  = .

and since also  = ,

 =  (pr. I.32);

and \therefore  and  are equiangular, with their equal angles opposite to homologous sides.

Q. E. D.



IF two triangles (and) have one angle in each equal (equal to), the sides about two other angles (: :: :), and each of the remaining angles (and) either less or not less than a right angle, the triangles are equiangular, and those angles are equal about which the sides are proportional.

First let it be assumed that the angles and are each less than a right angle: then if it be supposed that and contained by the proportional sides, are not equal, let be greater, and make = .

$$\therefore \text{blue angle} = \text{yellow angle (hyp.) and black angle} = \text{blue angle (const.)}$$

$$\therefore \text{yellow angle} = \text{red angle (pr. 1.32);}$$

$$\therefore \text{red side} : \text{blue side} :: \text{dotted red side} : \text{dotted yellow side (pr. 6.4),}$$

but $\text{red side} : \text{yellow side} :: \text{dotted red side} : \text{dotted yellow side (hyp.)}$

$$\therefore \text{red side} : \text{blue side} :: \text{red side} : \text{yellow side};$$

$$\therefore \text{blue side} = \text{yellow side (pr. 5.9),}$$



$$\text{and } \therefore \text{red angle} = \text{yellow angle (pr. 1.5).}$$


But is less than a right angle (hyp.)



\therefore is less than a right angle;

and \therefore must be greater than a right angle

(pr. 1.13), but it has been proven = and therefore less than a right angle, which is absurd. \therefore and are not unequal;





\therefore they are equal, and since  =  (hyp.)



\therefore  =  (pr. 1.32), and
therefore the triangles are equiangular.

But if  and  be assumed to be each not less than a right angle, it may be proved as before, that the triangles are equiangular, and have the sides about equal the angles proportional (pr. 6.4).

Q. E. D.









IN a right angled triangle (, if a perpendicular () be drawn from the right angle to the opposite side, the triangles (, ) on each side of it are similar to the whole triangle and to each other.

$$\therefore \text{  =  (ax. XI),$$

and  common to  and 

$$\triangle = \text{  (pr. I.32);$$

\therefore  and  are equiangular; and consequently have their sides about the equal angles proportional (pr. 6.4), and are therefore similar (def. 6.1).

In like manner it may be proved that  is similar to ; but  has been shown to be similar to 

\therefore  and  are similar to the whole and to each other.

Q. E. D.



FROM a given straight line (———) to cut off any required part.

From either extremity of the given line draw ——— making any angle with ———; and produce ——— till the whole produced line ——— contains — as often as — contains the required part.

Draw ———, and draw ——— || ———.

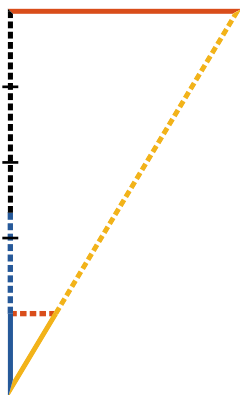
—— is the required part of ———.

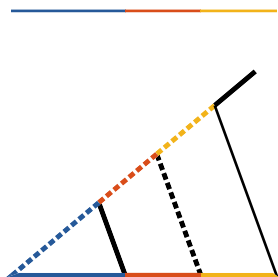
For since ——— || ———
 ——— : ——— :: ——— : ——— (pr. 6.2),
 and by composition (pr. 5.18);
 ——— : ——— :: ——— : ———;

but ——— contains — as often as — contains the required part (const.);

∴ — is the required part.

Q. E. D.





○ *divide a given straight line* (————)
similarly to a given divided line (————).

From either extremity of the given line
 ———— draw ———— making any angle;

take ————, ———— and ———— equal to ————,
 ———— and ———— respectively (pr. 1.2);

draw ————, and draw ———— and ———— \parallel to it.

Since { ———— } are \parallel ,

——— : ———— :: ———— : ———— (pr. 6.2),

or ———— : ———— :: ———— : ———— (const.),

and ———— : ———— :: ———— : ———— (pr. 6.2),

——— : ———— :: ———— : ———— (const.),

and \therefore the given line ———— is
 divided similarly to ————.

Q. E. D.



o find a third proportional to two given straight lines (— and —).

At either extremity of the given line —

draw making an angle;

take = —, and draw —;

make = —,

and draw \parallel —; (pr. 1.31)

— is the third proportional to — and —.

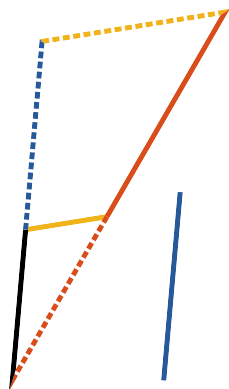
For since — \parallel ,

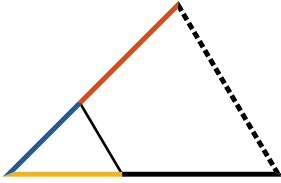
\therefore — : :: : — (pr. 6.2);

but = = — (const.);













\therefore — : — :: — : — (pr. 5.7).





Q. E. D.





o *find a fourth proportional to three given lines*
 $\left\{ \begin{array}{l} \text{blue dotted line} \\ \text{red dotted line} \\ \text{yellow dotted line} \end{array} \right\}.$







Draw  and
 making any angle;
 take  = ,
 and  = ,
 also  = ,
 draw ,
 and  \parallel  (pr. 1.31);
 is the fourth proportional.


On account of the parallels,
 :  ::  :  (pr. 6.2);
 but $\left\{ \begin{array}{l} \text{blue dotted line} \\ \text{red dotted line} \\ \text{yellow dotted line} \end{array} \right\} = \left\{ \begin{array}{l} \text{blue segment} \\ \text{red segment} \\ \text{yellow segment} \end{array} \right\}$ (const.);
 $\therefore \text{blue dotted line} : \text{red dotted line} :: \text{yellow dotted line} : \text{black segment}$ (pr. 5.7).

Q. E. D.




Find a mean proportional between two given straight lines {   and   }.



Draw any straight line ,
make  =  and  = ;
bisect ; and from the point
of bisection as a centre, and half line as a


radius, describe a semicircle ,



draw  \perp :

 is the mean proportional required.

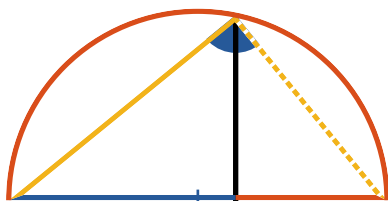
Draw  and .

Since  is a right angle (pr. 3.31),
and  is \perp from it upon the opposite side,

\therefore  is a mean proportional



between  and  (pr. 6.8),

and \therefore between  and  (const.).











Q. E. D.




QUAL parallelograms  and , which have one angle in each equal, have the sides about the equal angles reciprocally proportional

$$(\text{red line} : \text{black line} :: \text{yellow line} : \text{blue line})$$

And parallelograms which have one angle in each equal, and the sides about them reciprocally proportional, are equal.

Let  and ; and  and , be so placed that   and   may be continued right lines. It is evident that they may assume this position. (pr. I.13, pr. I.14, pr. I.15)

Complete .

Since  = ;

$$\therefore \text{yellow parallelogram} : \text{red parallelogram} :: \text{blue parallelogram} : \text{red parallelogram} \quad (\text{pr. 5.7})$$

$$\therefore \text{red line} : \text{black line} :: \text{yellow line} : \text{blue line} \quad (\text{pr. 6.1})$$

The same construction remaining:







$$\text{red line} : \text{black line} :: \left\{ \begin{array}{l} \text{yellow line} : \text{red line} \quad (\text{pr. 6.1}) \\ \text{yellow line} : \text{blue line} \quad (\text{hyp.}) \\ \text{blue line} : \text{red line} \quad (\text{pr. 6.1}) \end{array} \right.$$

$$\therefore \text{yellow parallelogram} : \text{red parallelogram} :: \text{blue parallelogram} : \text{red parallelogram} \quad (\text{pr. 5.11})$$

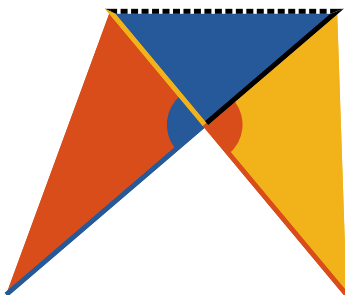
$$\text{and } \therefore \text{yellow parallelogram} = \text{blue parallelogram} \quad (\text{pr. 5.9}).$$

Q. E. D.









QUAL triangles, which have one angle in each
 equal ( = ), have the sides about the
 equal angles reciprocally proportional
 ( :  ::  : )

And two triangles which have an angle of the one equal to
 an angle of the other, and the sides about the equal angles
 reciprocally proportional, are equal.



I.

Let the triangles be so placed that the equal angles 
 and  may be vertically opposite, that is to say, so that
 and  may be in the same straight line.
 Whence also  and  must be in the same
 straight line (pr. 1.14).

Draw , then

$$\text{---} : \text{---} :: \text{---} : \text{---} \quad (\text{pr. 6.1})$$

$$:: \text{---} : \text{---} \quad (\text{pr. 5.7})$$

$$:: \text{---} : \text{---} \quad (\text{pr. 6.1})$$

$$\therefore \text{---} : \text{---} :: \text{---} : \text{---} \quad (\text{pr. 5.11}).$$

II.

Let the same construction remain, and

$$\triangle ABC : \triangle DEF :: \text{line } GH : \text{line } IJ \quad (\text{pr. 6.1})$$

$$\text{and } \text{line } KL : \text{line } MN :: \triangle PQR : \triangle DEF \quad (\text{pr. 6.1})$$

$$\text{but } \text{line } ST : \text{line } UV :: \text{line } KL : \text{line } MN, (\text{hyp.})$$

$$\therefore \triangle ABC : \triangle DEF :: \triangle PQR : \triangle DEF \quad (\text{pr. 5.11});$$

$$\therefore \triangle ABC = \triangle PQR \quad (\text{pr. 5.9}).$$

Q. E. D.





F four straight lines be proportional (————— : ————— :: ————— : —————) the rectangle (————— × —————) contained by the extremes, is equal to the rectangle (————— × —————) contained by the means.

And if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportional.

Part I.

From the extremities of ————— and ————— draw ————— and ————— \perp to them and = ————— and ————— respectively:

complete the parallelograms  and .

and since,

$$\begin{aligned} & \text{—————} : \text{—————} :: \text{—————} : \text{—————} \text{ (hyp.)} \\ \therefore & \text{—————} : \text{—————} :: \text{—————} : \text{—————} \text{ (const.)} \end{aligned}$$

$$\therefore \text{  =  \text{ (pr. 6.14),}$$

that is, the rectangle contained by the extremes, equal to the rectangle contained by the means.



Part II.

Let the same construction remain;

$$\therefore \text{dotted line} = \text{solid line}, \text{orange square} = \text{yellow square}$$

$$\text{and orange line} = \text{dotted line}.$$

$$\therefore \text{yellow line} : \text{blue line} :: \text{orange line} : \text{solid line} \quad (\text{pr. 6.14})$$

$$\text{But orange line} = \text{dotted line},$$

$$\text{and solid line} = \text{dotted line} \quad (\text{const.})$$

$$\therefore \text{yellow line} : \text{blue line} :: \text{dotted line} : \text{dotted line} \quad (\text{pr. 5.7}).$$

Q. E. D.



F three straight lines be proportional (— : — :: — : —) the rectangle under the extremes is equal to the square of the mean.

And if the rectangle under the extremes be equal to the square of the mean, the three straight lines are proportional.

Part I.

Assume — = — ,
 and since — : — :: — : — ,
 then — : — :: — : — ,
 \therefore — \times — = — \times —
 (pr. 6.16).

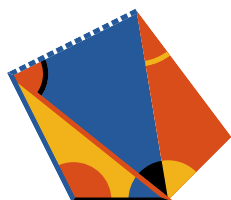
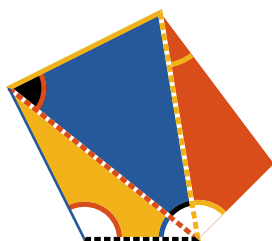
But — = — ,
 \therefore — \times — = —
 — \times — or = —²;
 therefore, if the three straight lines are
 proportional, the rectangle contained by the
 extremes is equal to the square of the mean.

Part II.


Assume — = — ,
 then — \times — = — \times — ,
 \therefore — : — :: — : — (pr. 6.16),
 and — : — :: — : — .

Q. E. D.





IN a given straight line (—) to construct a rectilinear figure similar to a given one





() and similarly placed.

Resolve the given figure into triangles by drawing the lines  and .





At the extremities of — make

 =  and  = ;

again at the extremities of — make

 =  and  = ;

in like manner make

 =  and  = .

Then  = .



It is evident from the construction and (pr. 1.32) that the figures are equiangular; and since the

triangles  and  are equiangular;

then by (pr. 6.4),

— :  ::  : 

and  :  ::  : 

Again, because  and  are equiangular,

 :  ::  : 

∴ ex æquali,

 :  ::  :  (pr. 5.22)

In like manner it may be shown that the remaining sides of the two figures are proportional.

\therefore by (pr. 6.1)



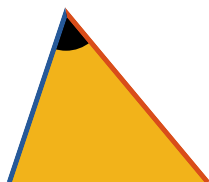
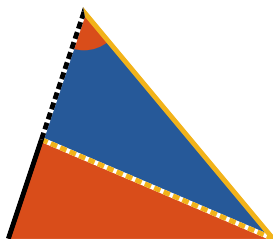
is similar to













and similarly

situated; and on the given line .

Q. E. D.



IMILAR triangles ( and ) are to one another in the duplicate ratio of their homologous sides.

Let  and  be equal angles, and  and  homologous sides of the similar triangles  and  and on  the greater of these lines take  a third proportional, so that

$$\text{---} : \text{---} :: \text{---} : \text{---};$$




draw .

$$\text{---} : \text{---} :: \text{---} : \text{---} \quad (\text{pr. 6.4});$$

$$\therefore \text{---} : \text{---} :: \text{---} : \text{---} \quad (\text{pr. 5.16}),$$

$$\text{but } \text{---} : \text{---} :: \text{---} : \text{---} \quad (\text{const.}),$$

$$\therefore \text{---} : \text{---} :: \text{---} : \text{---}$$



consequently  =  for they have the sides about the equal angles  and

 reciprocally proportional (pr. 6.15);

$$\therefore \text{---} : \text{---} :: \text{---} : \text{---} \quad (\text{pr. 5.7});$$

$$\text{but } \text{---} : \text{---} :: \text{---} : \text{---} \quad (\text{pr. 6.1}),$$

$$\therefore \text{---} : \text{---} :: \text{---} : \text{---},$$

that is to say, the triangles are to one another in the duplicate ratio of their homologous sides  and  (def. 5.11).

Q. E. D.



SIMILAR polygons may be divided into the same number of similar triangles, each similar pair of which are proportional to the polygons; and the polygons are to each other in the duplicate ratio of their homologous sides.

Draw and , and and , resolving the polygons into triangles. Then because the polygons are similar, = , and : :: : ,

\therefore and are similar,
and = (pr. 6.6);

but = because they are angles of similar polygons; therefore the remainders and are equal;

hence : :: : ,

on account of the similar triangles,

and : :: : ,

on account of the similar polygons,

\therefore : :: : ,









ex æquali (pr. 5.22), and as these proportional sides contain equal angles, the triangles















and are similar (pr. 6.6).

In like manner it may be shown that the triangles and are similar.









But is to in the duplicate ratio of to (pr. 6.19),



and  is to  in like manner, in
 the duplicate ratio of  to ;
 \therefore  :  ::  :  (pr. 5.11).




Again  is to  in the duplicate ratio of
 to , and  is to 
 in the duplicate ratio of  to .
 :  ::  : 
 \therefore  :  ;







and as one of the antecedents is to one of the consequents,
 so is the sum of all the antecedents to the sum of all the
 consequents; that is to say, the similar triangles have to
 one another the same ratio as the polygons (pr. 5.12).

But  is to  in the duplicate
 ratio of  to ;
 \therefore  is to  in the
 duplicate ratio of  to .

Q. E. D.

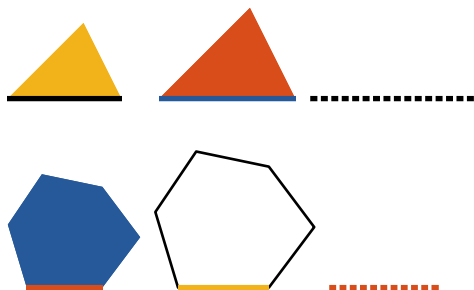


RECTILINEAR figures ( and )
which are similar to the same figure ()
are similar also to each other.

Since  and  are similar, they are equiangular, and have the sides about the equal angles proportional (def. 6.1); and since the figures  and  are also similar, they are equiangular, and have the sides about the equal angles proportional; therefore  and  are also equiangular, and have the sides about the equal angles proportional (pr. 5.11), and are therefore similar.

Q. E. D.





F four straight lines be proportional
 (— : — :: — : —
 —), the similar rectilinear
 figures similarly described on them
 are also proportional.

And if four similar rectilinear figures, similarly described on four straight lines, be proportional, the straight lines are also proportional.

Part I.

Take ——— a third proportional to
 — and —, and ——— a third
 proportional to — and — (pr. 6.11);

since — : — :: — : — (hyp.)
 — : ——— :: — : ——— (const.)

∴ ex æquali,

— : ——— :: — : ———;

but — : — :: — : ——— (pr. 6.20),

and — : — :: — : ———;

∴ — : — :: — : — (pr. 5.11).

Part II.

Let the same construction remain;







\therefore ——— : ——— :: ——— : ——— (const.),

and \therefore ——— : ——— :: ——— : ——— (pr. 5.11).

Q. E. D.





QUIANGULAR *parallelograms* ( and ) are to one another in a ratio compounded of the ratios of their sides.

Let two of the sides  and  about the equal angles be placed so that they may form one straight line.

Since  +  = ,

and  =  (hyp.),







 +  = ,

and \therefore  and 
form one straight line (pr. I.14);

complete  .




Since  :  ::  :  (pr. 6.1),

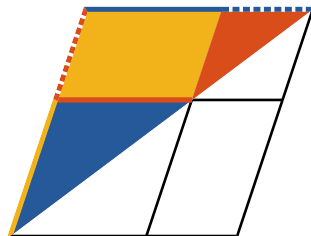
and  :  ::  :  (pr. 6.1),





 has to  a ratio compounded of the ratios
of  to , and of  to .

Q. E. D.





In any parallelogram () the parallelograms ( and ) which are about the diagonal are similar to the whole, and to each other.





As  and  have a common angle they are equiangular;
but \because  \parallel 




 and  are similar (pr. 6.4),
 \therefore  $:$  $::$  $:$ ;

and the remaining opposite sides are equal to those,

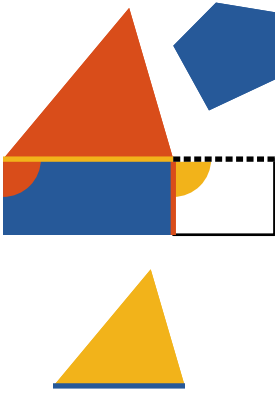
\therefore  and  have the sides about the equal angles proportional, and are therefore similar.

In the same manner it can be demonstrated that the



parallelograms  and  are similar.
Since, therefore, each of the parallelograms

 and  is similar to ,
they are similar to each other.




Q. E. D.








o describe a rectilinear figure, which shall be similar to a given rectilinear figure



(, and equal to another ().


Upon  describe  = ,

and upon  describe  = ,

and having  =  (pr. I.45),
and then  and will lie in
the same straight line (pr. I.29, pr. I.14).

Between  and find a
mean proportional  (pr. 6.13),

and upon  describe , similar

to , and similarly situated.



Then  = .

For since  and  are similar, and
 :  ::  : (const.),

 :  ::  : (pr. 6.20);

but  :  ::  :  (pr. 6.1);

\therefore  :  ::  :  (pr. 5.11);

but  =  (const.),

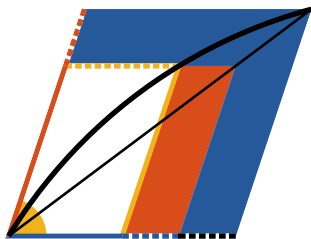
and \therefore  =  (pr. 5.14);

and  =  (const.);

consequently,  which is

similar to  is also = .

Q. E. D.







F similar and similarly posited parallelograms

( and ) have a common angle, they are about the same diagonal.

For, if possible, let  be the diagonal of





and draw  \parallel  (pr. 1.31).




Since  and  are about the same diagonal , and have  common, they are similar (pr. 6.24);

\therefore  :  ::  : ;

but  :  ::  :  (hyp.),

\therefore  :  ::  : ,






and \therefore  =  (pr. 5.9), which is absurd.

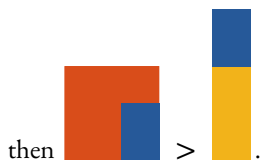
\therefore  is not the diagonal of 
in the same manner it can be demonstrated
that no other line is except .

Q. E. D.



Of all the rectangles contained by the segments of a given straight line, the greatest is the square which is described on half the line.

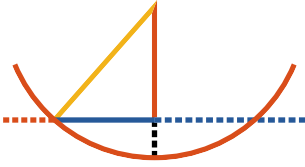
Let  be the given line,
 and  unequal segments,
 and  and  equal segments;



For it has been demonstrated already (pr. 2.5), that the square of half the line is equal to the rectangle contained by any unequal segments together with the square of the part intermediate between the middle point and the point of unequal section. The square described on half the line exceeds therefore the rectangle contained by any unequal segments of the line.

Q. E. D.





○ divide a given straight line (.....) so that the rectangle contained by its segments may be equal to a given area, not exceeding the square of half the line.

Let the given area be =².

Bisect , or make =;

and if² =²,

problem is solved.

But if² ≠²,

then must > (hyp.).

Draw ⊥ =;

make = or;

with as radius describe a circle cutting the given line; draw

Then × +² =²
(pr. 2.5) =².

But² =² +² (pr. 1.47);

∴ × +² =² +²,

from both, take²,

and × =².

But = (const.),

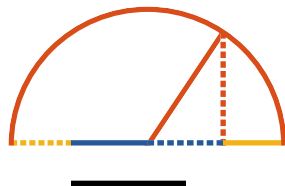
and ∴ is so divided

that × =².

Q. E. D.



Q^o produce a given straight line (— ·····), so that the rectangle contained by the segments between the extremities of the given line and the point to which it is produced, may be equal to a given area, i. e. equal to the square on ———.



Make — = ·····,
and draw ····· ⊥ ····· = ———;
draw ———;
and with the radius ———, describe a
circle meeting —···· produced.

Then —···· × — + ·····² = —····²
(pr. 2.6) = ———².

But ———² = ·····² + ·····² (pr. 1.47)

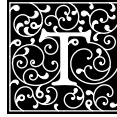
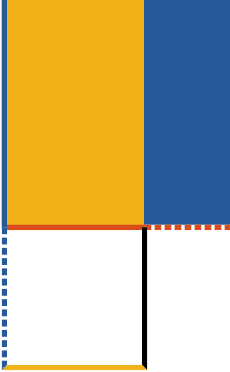
∴ —···· × — + ·····² = ·····² + ·····²,
from both take ·····²,

and ∴ —···· × — = ·····²


but ····· = ———,

∴ ·····² = the given area.

Q. E. D.





Q cut a given finite straight line (— ···) in extreme and mean ratio.

On — ··· describe the square  (pr. I.46);






and produce , so that

$$\text{— ···} \times \text{···} = \text{— ···}^2 \text{ (pr. 6.29);}$$


take  = ,



and draw  \parallel ,




meeting  \parallel  (pr. I.31).




Then  =  \times , and is \therefore =   ;

and if from both these equals be


taken the common part ,

, which is the square of ,

will be = , which is =  \times ;

that is  2 =  \times ;


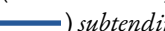
\therefore  :  $::$  : ,

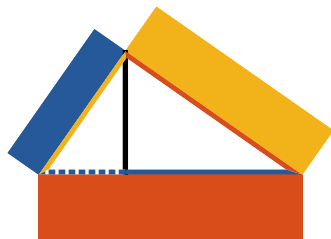
and  is divided in extreme and mean ratio (def. 6.3).



Q. E. D.



F any similar rectilinear figures be similarly described on the sides of a right angled triangle

() , the figure described on the side () subtending the right angle is equal to the sum of the figures on the other sides.



From the right angle draw 
perpendicular to ;
then

$$\text{---} : \text{---} :: \text{---} : \text{---}$$

(pr. 6.8).

$$\therefore \text{---} : \text{---} :: \text{---} : \text{---}$$

(pr. 6.20).

but $\text{---} : \text{---} :: \text{---} : \text{---}$
(pr. 6.20).

Hence $\text{---} + \text{---} : \text{---} ::$







$\text{---} + \text{---} : \text{---};$









but $\text{---} + \text{---} = \text{---};$















and $\therefore \text{---} + \text{---} = \text{---}.$

Q. E. D.





IN equal circles ( , ), angles, whether at the centre or circumference, are in the same ratio to one another as the arcs on which they stand ( :  ::  : ); so also are sectors.

Take in the circumference of  any number of arcs  ,  , &c. each =  , and also in the circumference of  take any number of arcs  ,  , &c. each =  , draw the radii to the extremities of the equal arcs.

The since the arcs  ,  ,  , &c. are all equal, the angles  ,  ,  , &c. are also equal (pr. 3.27); \therefore  is the same multiple of  which the arc  is of  ; and in the same manner  is the same multiple of  , which the arc  is of the arc .





Then it is evident (pr. 3.27),

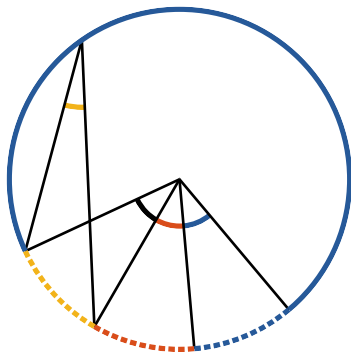
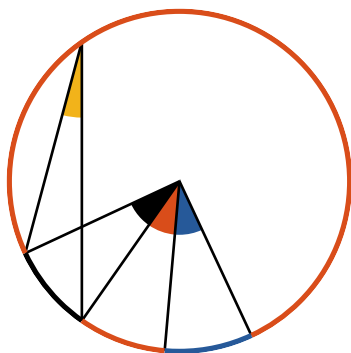
if  (or if m times )

$>$, $=$, $<$  (or n times )

then  (or m times )

$>$, $=$, $<$  (or n times );

\therefore  :  ::  :  (def. 5.5), or the angles at the centre are as the arcs on which they stand; but the angles at the circumference being halves of the angles at



the centre (pr. 3.20) are in the same ratio (pr. 5.15), and therefore are as the arcs on which they stand.

It is evident, that sectors in equal circles, and on equal arcs are equal (pr. 1.4, 3.24, 3.27, and def. 3.9). Hence, if the sectors be substituted for the angles in the above demonstration, the second part of the proportion will be established, that is, in equal circles the sectors have the same ratio to one another as the arcs on which they stand.

Q. E. D.



F the right line (— · — · — ·) bisecting an external

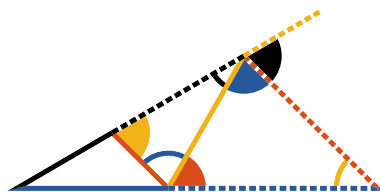
angle



of the triangle




meet the opposite side (—) produced, that whole produced side (— · — · — ·), and its external segment (— · — · — ·) will be proportional to the sides (— · — · — · and —), which contain the angle adjacent to the external bisected angle.



For if — be drawn \parallel — · — · — ·,

then  = , (pr. 1.29);

= , (hyp.),

= , (pr. 1.29);

and \therefore — · — · — · = —, (pr. 1.6),

and — · — · — · : — :: — · — · — · : — · — · — · (pr. 5.7);

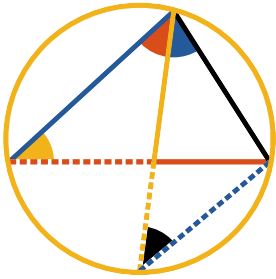
But also, — · — · — · : — · — · — · :: — · — · — · : — · — · — ·

(pr. 6.2);









and therefore

— · — · — · : — · — · — · :: — · — · — · : — (pr. 5.11).



Q. E. D.





IF an angle of a triangle be bisected by a straight line, which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line which bisects the angle.

Let  be drawn, making  =  ;
then shall  ×  =  ×  + ².

About  describe  (pr. 4.5),
produce  to meet the circle, and draw .

Since  =  (hyp.),

and  =  (pr. 3.21),

∴  and  are equiangular (pr. 1.32);

∴  :  ::  :  (pr. 6.4);

∴  ×  =  ×  (pr. 6.16)
=  ×  + ² (pr. 2.3);

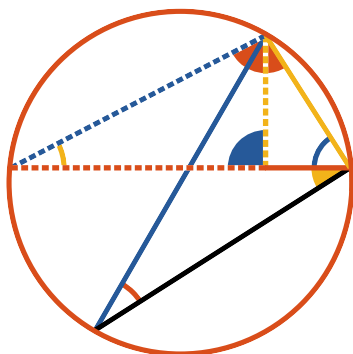
but  ×  =  ×  (pr. 3.35);


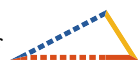





∴  ×  =  ×  + ²


Q. E. D.





From any angle of a triangle a straight line be drawn perpendicular to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.





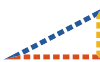

From  of 
draw  \perp ;
then shall  \times  =  \times
the diameter of the described circle.

Describe  (pr. 4.5), draw its





diameter , and draw ;

then \therefore  =  (const. and pr. 3.31);

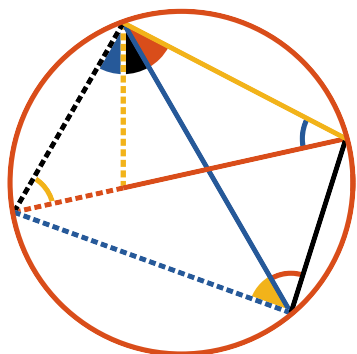
and  =  (pr. 3.21);

\therefore  is equiangular to  (pr. 6.4);

\therefore  :  ::  : ;


and \therefore  \times  =  \times 
(pr. 6.16).

Q. E. D.



THE rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

Let  be any quadrilateral

figure inscribed in ;

and draw  and ;





then  \times  =



 \times  +  \times .

Make  =  (pr. 1.23),

\therefore  = ; and  =  (pr. 3.21);





\therefore  :  ::  :  (pr. 6.4);

and \therefore  \times  =  \times 
(pr. 6.16);





again, \therefore  =  (const.),

and  =  (pr. 3.21);

\therefore  :  ::  :  (pr. 6.4);

and \therefore  \times  =  \times 
(pr. 6.16);

but, from above,

 \times  =  \times ;

\therefore  \times  =

 \times  +  \times  (pr. 2.1).

Q. E. D.